THE FIRST HILBERT COEFFICIENTS OF PARAMETERS

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Abstract. The conjecture of Wolmer Vasconcelos [13] on the vanishing of the first Hilbert coefficient $\mathrm{e}_Q^1(A)$ is solved affirmatively, where Q is a parameter ideal in a commutative Noetherian local ring A. Basic properties of the rings for which $\mathrm{e}_Q^1(A)$ vanishes are derived. The invariance of $\mathrm{e}_Q^1(A)$ for parameter ideals Q and its relationship to Buchsbaum rings are studied.

Key Words: commutative algebra, Cohen-Macaulay local ring, Buchsbaum local ring, Hilbert coefficient.

2000 *Mathematics Subject Classification*: Primary 13D40; Secondary 13H15

1. Introduction

This is based on [1, 5] a joint work with L. Ghezzi, J. Hong, T. T. Phuong, and W. V. Vasconcelos.

Let A be a commutative Noetherian local ring with the maximal ideal $\mathfrak{m}$ and the Krull dimension $d = \dim A > 0$. Let $\ell_A(M)$ denote, for an A-module M, the length of M. Then, for each $\mathfrak{m}$-primary ideal I in A, we have integers $\{\mathrm{e}_I^i(A)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_I^d(A)$$

holds true for all integers $n \gg 0$, which we call the Hilbert coefficients of A with respect to I. We say that A is unmixed, if $\dim \widehat{A}/\mathfrak{p} = d$ for every $\mathfrak{p} \in \operatorname{Ass} \widehat{A}$, where $\widehat{A}$ denotes the $\mathfrak{m}$-adic completion of A.

With this notation Wolmer V. Vasconcelos posed, exploring the vanishing of the first Hilbert coefficient $\mathrm{e}_Q^1(A)$ for parameter ideals Q, in his lecture at the conference in Yokohama of March, 2008 the following conjecture.

Conjecture 1 ([13]). *Assume that A is unmixed. Then A is a Cohen-Macaulay local ring, once $\mathrm{e}_Q^1(A) = 0$ for some parameter ideal Q of A.*

In Section 2 of the present paper we shall settle Conjecture 1 affirmatively. Here we should note that Conjecture 1 is already solved partially by [2] and [7]. Let us call those local rings A with $\mathrm{e}_Q^1(A) = 0$ for some parameter ideals Q *Vasconcelos*. In Section 3 we shall explore basic properties of Vasconcelos rings. In Section 4 we will study the problem of when $\mathrm{e}_Q^1(A)$ is constant and independent of the choice of parameter ideals Q in A. We shall show that A is a Buchsbaum ring, if A is unmixed and $\mathrm{e}_Q^1(A)$ is constant (Theorem 12).

The detailed version of this paper has been submitted for publication elsewhere.

In what follows, unless otherwise specified, let A denote a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let $\{\mathrm{H}^i_{\mathfrak{m}}(*)\}_{i\in\mathbb{Z}}$ be the local cohomology functors of A with respect to the maximal ideal $\mathfrak{m}$.

Let $\mathrm{Assh}\, A = \{\mathfrak{p} \in \mathrm{Ass}\, A \mid \dim A/\mathfrak{p} = d\}$ and let $(0) = \bigcap_{\mathfrak{p}\in\mathrm{Ass}\, A} \mathrm{I}(\mathfrak{p})$ be a primary decomposition of (0) in A with $\mathfrak{p}$-primary ideals $\mathrm{I}(\mathfrak{p})$ in A. We put

$$\mathrm{U}_A(0) = \bigcap_{\mathfrak{p}\in\mathrm{Assh}\, A} \mathrm{I}(\mathfrak{p})$$

and call it the unmixed component of (0) in A.

2. Proof of the conjecture of Vasconcelos

The purpose of this section is to prove the following, which settles Conjecture 1 affirmatively. One of the main results of this paper is the following.

Theorem 2. *Let A be unmixed. Then the following four conditions are equivalent.*

1. *A is a Cohen-Macaulay local ring.*
2. *$\mathrm{e}^1_I(A) \geq 0$ for every $\mathfrak{m}$-primary ideal I in A .*
3. *$\mathrm{e}^1_Q(A) \geq 0$ for some parameter ideal Q in A.*
4. *$\mathrm{e}^1_Q(A) = 0$ for some parameter ideal Q in A.*

In our proof of Theorem 2 the following facts are the key. See [3, Section 3] for the proof.

Proposition 3 ([3]). *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A \geq 2$, possessing the canonical module K_A. Assume that $\dim A/\mathfrak{p} = d$ for every $\mathfrak{p} \in \mathrm{Ass}A \setminus \{\mathfrak{m}\}$. Then the following assertions hold true.*

1. *The local cohomology module $\mathrm{H}^1_{\mathfrak{m}}(A)$ is finitely generated.*
2. *The set $\mathcal{F} = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \dim A_{\mathfrak{p}} > \mathrm{depth}\, A_{\mathfrak{p}} = 1\}$ is finite.*
3. *Suppose that the residue class field $k = A/\mathfrak{m}$ of A is infinite and let I be an $\mathfrak{m}$-primary ideal in A. Then one can choose an element $a \in I \setminus \mathfrak{m}I$ so that a is superficial for I and $\dim A/\mathfrak{p} = d - 1$ for every $\mathfrak{p} \in \mathrm{Ass}_A A/aA \setminus \{\mathfrak{m}\}$.*

Proof of Theorem 2. The implication (1) $\Rightarrow$ (2) is due to [8]. The implication (1) $\Rightarrow$ (4) is well known, and (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) are trivial. Thus we have only to check the implication (3) $\Rightarrow$ (1). Let $Q = (a_1, a_2, \cdots, a_d)$ with a system $a_1, a_2, \cdots, a_d$ of parameters in A. Enlarging the residue class field $A/\mathfrak{m}$ of A and passing to the $\mathfrak{m}$-adic completion of A, we may assume that the field $A/\mathfrak{m}$ is infinite and that A is complete. The assertion is obvious in the case where $d \leq 2$. Recall that for any Noetherian local ring $(A, \mathfrak{m})$ of dimension one, we have $\mathrm{e}^1_Q(A) = -\ell_A(\mathrm{H}^0_{\mathfrak{m}}(A))$; see [4, Lemma 2.4 (1)], and the two-dimensional case is readily deduced from this fact via the reduction modulo some superficial element $x = a_1$ of Q; see [4, Lemma 2.2] and notice that x is A-regular.

We may assume that $d \geq 3$ and that our assertion holds true for $d-1$. Then we are able to choose, thanks to Proposition 3 (3), the element $x = a_1$ so that x is a superficial element of the parameter ideal Q and (the ring A/xA is *not necessarily* unmixed but) the unmixed component $U = \mathrm{U}_B(0)$ of (0) in $B = A/xA$ has finite length, whence $U = \mathrm{H}^0_{\mathfrak{m}}(B)$. Then the $d-1$ dimensional local ring B/U is Cohen-Macaulay by the hypothesis of induction on d, because

$$\mathrm{e}^1_{Q\cdot(B/U)}(B/U) = \mathrm{e}^1_{QB}(B) = \mathrm{e}^1_Q(A) \geq 0$$

(cf. [4, Lemma 2.2]). Hence $\mathrm{H}^i_{\mathfrak{m}}(B) = (0)$ for all $i \neq 0, d-1$. We now look at the long exact sequence

$$\begin{array}{rcl} \cdots & \to & \mathrm{H}^1_{\mathfrak{m}}(A) \xrightarrow{x} \mathrm{H}^1_{\mathfrak{m}}(A) \to \mathrm{H}^1_{\mathfrak{m}}(B) \to \\ \cdots \to \mathrm{H}^{i-1}_{\mathfrak{m}}(B) & \to & \mathrm{H}^i_{\mathfrak{m}}(A) \xrightarrow{x} \mathrm{H}^i_{\mathfrak{m}}(A) \to \cdots \\ \cdots \to \mathrm{H}^{d-2}_{\mathfrak{m}}(B) & \to & \mathrm{H}^{d-1}_{\mathfrak{m}}(A) \xrightarrow{x} \mathrm{H}^{d-1}_{\mathfrak{m}}(A) \to \cdots \end{array}$$

of local cohomology modules, derived from the short exact sequence

$$0 \to A \xrightarrow{x} A \to B \to 0$$

of A-modules. We then have $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $2 \leq i \leq d-1$, since $\mathrm{H}^i_{\mathfrak{m}}(B) = (0)$ for all $1 \leq i \leq d-2$, while $\mathrm{H}^1_{\mathfrak{m}}(A) = x\mathrm{H}^1_{\mathfrak{m}}(A)$, since $\mathrm{H}^1_{\mathfrak{m}}(B) = (0)$. Consequently $\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$, because the A-module $\mathrm{H}^1_{\mathfrak{m}}(A)$ is finitely generated by Proposition 2 (1). Thus A is a Cohen-Macaulay ring. □

Let us give one consequence of Theorem 2.

Corollary 4 ([7])**.** *We have $\mathrm{e}^1_Q(A) \leq 0$ for every parameter ideals Q in A.*

3. Vasconcelos rings

The purpose of this section is to develop a theory of Vasconcelos rings. Let us begin with the definition.

Definition 5. We say that A is a *Vasconcelos* ring, if either $d = 0$, or $d > 0$ and $\mathrm{e}^1_Q(A) = 0$ for some parameter ideal Q in A.

Here is a basic characterization of Vasconcelos rings.

Theorem 6. *Suppose that $d = \dim A > 0$. Then the following four conditions are equivalent.*

(1) *A is a Vasconcelos ring.*
(2) *$\mathrm{e}^1_Q(A) = 0$ for every parameter ideal Q in A.*
(3) *$\widehat{A}/\mathrm{U}_{\widehat{A}}(0)$ is a Cohen-Macaulay ring and $\dim_{\widehat{A}} \mathrm{U}_{\widehat{A}}(0) \leq d-2$, where $\mathrm{U}_{\widehat{A}}(0)$ denotes the unmixed component of (0) in the $\mathfrak{m}$-adic completion $\widehat{A}$ of A.*
(4) *The $\mathfrak{m}$-adic completion $\widehat{A}$ of A contains an ideal $I \neq \widehat{A}$ such that $\widehat{A}/I$ is a Cohen-Macaulay ring and $\dim_{\widehat{A}} I \leq d-2$.*

When this is the case, $\widehat{A}$ is a Vasconcelos ring, $\mathrm{H}^{d-1}_{\mathfrak{m}}(A) = (0)$, and the canonical module $\mathrm{K}_{\widehat{A}}$ of $\widehat{A}$ is a Cohen-Macaulay $\widehat{A}$-module.

Proof. See [1, Theorem 3.3]. □

Notice that condition (3) of Theorem 6 is free from parameters. Therefore, since $\mathrm{e}^1_Q(A) = 0$ for some parameter ideal, then $\mathrm{e}^1_Q(A) = 0$ for every parameter ideals Q in A. This is what the theorem says.

In the rest of this section, let us give some consequences of Theorem 6.

Corollary 7. *Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let Q be a parameter ideal in A. Assume that $\mathrm{e}_Q^i(A) = 0$ for all $1 \leq i \leq d$. Then A is a Cohen-Macaulay ring.*

Suppose that $d > 0$ and let Q be a parameter ideal in A. We denote by $R = \mathcal{R}(Q)$ (resp. $G = \mathrm{G}(Q)$) the Rees algebra (resp. the associated graded ring) of Q. Hence

$$R = A[Qt] \quad \text{and} \quad G = \mathcal{R}'(Q)/t^{-1}\mathcal{R}'(Q),$$

where t is an indeterminate over A and $\mathcal{R}'(Q) = A[Qt, t^{-1}]$. Let $\mathfrak{M} = \mathfrak{m}R + R_+$ be the graded maximal ideal in R. With this notation we have the following.

Corollary 8. *The following assertions hold true.*

(1) *A is a Vasconcelos ring if and only if $G_{\mathfrak{M}}$ is a Vasconcelos ring.*
(2) *Suppose that A is a homomorphic image of a Cohen-Macaulay ring. Then $R_{\mathfrak{M}}$ is a Vasconcelos ring, if A is a Vasconcelos ring.*

Thus Vasconcelos rings enjoy very nice properties.

4. Buchsbaumness in local rings possessing constant first Hilbert coefficients of parameters

In this section we study the problem of when $\mathrm{e}_Q^1(A)$ is constant and independent of the choice of parameter ideals Q in A.

Here let us briefly recall the definition of Buchsbaum local rings. The readers may consult the monumental book [11] of J. Stückrad and W. Vogel for a detailed theory, some of which we shall note here for the use in this paper.

We say that our local ring A is Buchsbaum, if the difference

$$\ell_A(A/Q) - \mathrm{e}_Q^0(A)$$

is independent of the choice of parameter ideals Q in A and is an invariant of A, which we denote by $\mathbb{I}(A)$. As is well-known, A is a Buchsbaum ring if and only if every system $a_1, a_2, \cdots, a_d$ of parameters in A forms a d-sequence in the sense of C. Huneke ([6]). When A is a Buchsbaum local ring, one has

$$\mathfrak{m}{\cdot}\mathrm{H}_{\mathfrak{m}}^i(A) = (0)$$

for all $i \neq d$, whence the local cohomology modules $\{\mathrm{H}_{\mathfrak{m}}^i(A)\}_{i \neq d}$ are finite-dimensional vector spaces over the field $A/\mathfrak{m}$, and the equality

$$\mathbb{I}(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}_{\mathfrak{m}}^i(A))$$

holds true.

We say that A is a generalized Cohen-Macaulay local ring, if all the local cohomology modules $\{\mathrm{H}_{\mathfrak{m}}^i(A)\}_{i \neq d}$ are finitely generated. Hence every Cohen-Macaulay local ring is Buchsbaum with $\mathbb{I}(A) = 0$ and Buchsbaum local rings are generalized Cohen-Macaulay. A given Noetherian local ring A with $d = \dim A > 0$ is a generalized Cohen-Macaulay local ring if and only if

$$\mathbb{I}(A) := \sup_Q \{\ell_A(A/Q) - \mathrm{e}_Q^0(A)\} < \infty,$$

where Q runs through parameter ideals in A ([10]). When this is the case, one has

$$\mathbb{I}(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}_\mathfrak{m}^i(A)).$$

Suppose that A is a generalized Cohen-Macaulay local ring and let Q be a parameter ideal in A. Then Q is called standard, if

$$\mathbb{I}(A) = \ell_A(A/Q) - \mathrm{e}_Q^0(A).$$

This condition is equivalent to saying that Q is generated by a system $a_1, a_2, \cdots, a_d$ of parameters which forms a strong d-sequence in any order ([10]).

Let

$$\Lambda = \Lambda(A) = \{\mathrm{e}_Q^1(A) \mid Q \text{ be a parameter ideal in } A\ \}.$$

Then we can ask the following questions.

Question 9. When is Λ a finite set or a singleton?

For example, our characterization of Vasconcelos rings says that $0 \in \Lambda$ if and only if $\Lambda = \{0\}$.

Let us summarize what is known about the questions, where we put $h^i(A) = \ell_A(\mathrm{H}_\mathfrak{m}^i(A))$ for each $i \in \mathbb{Z}$.

Proposition 10 ([4, 9])**.** *Suppose that A is a generalized Cohen-Macaulay local ring and $d \geq 2$. Let Q be a parameter ideal in A. Then we have the following.*

(1) $\mathrm{e}_Q^1(A) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$.

(2) *We have $\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$, if Q is standard.*

Thanks to Proposition 10 (1) and Corollary 4, if A is a generalized Cohen-Macaulay ring then we have

$$0 \geq \mathrm{e}_Q^1(A) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$$

for every parameter ideal Q in A. Hence Λ is finite. If A is a Buchsbaum ring then, since all parameter ideals in A are standard, we have

$$\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$$

for every parameter ideal Q in A. Thus, we have

$$\Lambda = \left\{-\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)\right\},$$

so that Λ is a singleton. It is natural to guess the converse is also true.

Our answer is the following.

Theorem 11. *Suppose that $d \geq 2$ and A is unmixed. Assume that Λ is a finite set and put $\ell = -\min \Lambda$. Then $\mathfrak{m}^\ell \mathrm{H}_\mathfrak{m}^i(A) = (0)$ for every $i \neq d$. Hence $\mathrm{H}_\mathfrak{m}^i(A)$ is a finitely generated A-module for every $i \neq d$, so that A is a generalized Cohen-Macaulay local ring.*

The main result of this section is stated as follows.

Theorem 12. *Suppose that $d = \dim A \geq 2$ and A is unmixed. Then the following two conditions are equivalent.*

(1) *A is a Buchsbaum local ring.*
(2) *The first Hilbert coefficients $\mathrm{e}_Q^1(A)$ of A are constant and independent of the choice of parameter ideals Q in A.*

When this is the case, one has the equality

$$\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$$

for every parameter ideal Q in A.

Thus Buchsbaum rings are characterized in terms of consistency of the first Hilbert coefficients of parameters. This is a new characterization of Buchsbaum rings.

The following result is a key for the proof of Theorem 12.

Theorem 13. *Suppose that A is a generalized Cohen-Macaulay local ring with $d = \dim A \geq 2$ and $\operatorname{depth} A > 0$. Let Q be a parameter ideal in A. Then the following two conditions are equivalent.*

(1) *Q is a standard parameter ideal in A.*
(2) $\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$.

In our proof of Theorem 13 we need the following result. Let

$$\mathrm{U}(a) = \bigcup_{n \geq 0} [(a) :_A \mathfrak{m}^n]$$

for each $a \in A$.

Proposition 14. *Suppose that A is a generalized Cohen-Macaulay local ring with $d = \dim A \geq 3$ and $\operatorname{depth} A > 0$. Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Assume that $(a_1, a_d)\mathrm{H}_\mathfrak{m}^1(A) = (0)$ and that the parameter ideal $(a_1, a_2, \cdots, a_{d-1}){\cdot}[A/\mathrm{U}(a_d)]$ is standard in the generalized Cohen-Macaulay local ring $A/\mathrm{U}(a_d)$. Then*

$$\mathrm{U}(a_1) \cap Q = (a_1).$$

Proof. Since $\mathrm{U}(a_1) \cap Q = (a_1) + [\mathrm{U}(a_1) \cap (a_2, a_3, \cdots, a_d)]$, we have only to show

$$\mathrm{U}(a_1) \cap (a_2, a_3, \cdots, a_d) \subseteq (a_1).$$

Let $x \in \mathrm{U}(a_1) \cap (a_2, a_3, \cdots, a_d)$ and put $\overline{A} = A/\mathrm{U}(a_d)$. Let $\overline{x}$ and $\overline{a_i}$ respectively denote the images of x and a_i in $\overline{A}$. Then we have

$$\overline{x} \in \mathrm{U}(\overline{a_1}) \cap (\overline{a_2}, \overline{a_3}, \cdots, \overline{a_{d-1}}) \subseteq (\overline{a_1}),$$

because $\mathrm{U}(\overline{a_1}) = (\overline{a_1}) :_{\overline{A}} \overline{a_2}$ and $\overline{a_2}, \overline{a_3}, \cdots, \overline{a_{d-1}}$ forms a d-sequence in $\overline{A}$ (recall that by our assumption $(\overline{a_2}, \overline{a_3}, \cdots, \overline{a_{d-1}})$ is a standard parameter ideal in the generalized Cohen-Macaulay local ring $\overline{A}$). Hence

$$x \in [(a_1) + \mathrm{U}(a_d)] \cap \mathrm{U}(a_1) = (a_1) + [\mathrm{U}(a_1) \cap \mathrm{U}(a_d)].$$

Let $x = y + z$ with $y \in (a_1)$ and $z \in \mathrm{U}(a_1) \cap \mathrm{U}(a_d)$. We will show that $z \in (a_1)$.

Since $a_1\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$ and a_1 is A-regular, we have

$$\mathrm{H}^1_{\mathfrak{m}}(A) \cong \mathrm{H}^0_{\mathfrak{m}}(A/(a_1)) = \mathrm{U}(a_1)/(a_1),$$

whence $a_d\mathrm{U}(a_1) \subseteq (a_1)$, because $a_d\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$ by our assumption. By the same argument applied to a_d we get $a_1\mathrm{U}(a_d) \subseteq (a_d)$. Hence $a_1 z \in (a_d)$ and $a_d z \in (a_1)$. Let us now write

$$a_1 z = a_d u \quad \text{and} \quad a_d z = a_1 v \quad \text{with} \quad u, v \in A.$$

Then, since $a_1 a_d z = a_d^2 u = a_1^2 v$, we have $u \in \mathrm{U}(a_1^2)$. Notice that

$$\mathrm{H}^1_{\mathfrak{m}}(A) \cong \mathrm{H}^0_{\mathfrak{m}}(A/(a_1^2)) = \mathrm{U}(a_1^2)/(a_1^2),$$

since $a_1^2\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$ and a_1^2 is A-regular. Therefore $a_d\mathrm{U}(a_1^2) \subseteq (a_1^2)$, because $a_d\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$. Hence $a_1 a_d z = a_d{\cdot}a_d u \in (a_1^2 a_d)$, so that $z \in (a_1)$. Thus $x = y + z \in (a_1)$, as is claimed. $\square$

To prove Theorem 13 we also need the following lemma.

Lemma 15 ([1, Lemma 4.5])**.** *Suppose that A is a generalized Cohen-Macaulay local ring with $d = \dim A \geq 2$ and $\operatorname{depth} A > 0$. Let Q be a parameter ideal in A and assume that $\mathrm{e}^1_Q(A) = -\sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(A)$. Then $Q\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $1 \leq i \leq d-1$.*

We are now in a position to prove Theorem 13.

Proof of Theorem 13. Enlarging the residue class field $A/\mathfrak{m}$ of A if necessary, we may assume that the field $A/\mathfrak{m}$ is infinite. Let $Q = (a_1, a_2, \cdots, a_d)$, where each a_j is superficial for the ideal Q. Recall that $Q\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $1 \leq i \leq d-1$ by Lemma 15. Hence Q is standard, if $d = 2$ ([12, Corollary 3.7]).

Assume that $d \geq 3$ and that our assertion holds true for $d-1$. Let $B = A/(a_j)$ with $1 \leq j \leq d$ and put $\overline{A} = B/\mathrm{H}^0_{\mathfrak{m}}(B)$ $(= A/\mathrm{U}(a_j))$. Then $\mathrm{H}^i_{\mathfrak{m}}(\overline{A}) \cong \mathrm{H}^i_{\mathfrak{m}}(B)$ for all $i \geq 1$. On the other hand, since $a_j\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for $1 \leq i \leq d-1$ and a_j is A-regular, we get for each $0 \leq i \leq d-2$ the short exact sequence

$$0 \to \mathrm{H}^i_{\mathfrak{m}}(A) \to \mathrm{H}^i_{\mathfrak{m}}(B) \to \mathrm{H}^{i+1}_{\mathfrak{m}}(A) \to 0$$

of local cohomology modules. Consequently we get $\mathbb{I}(A) = \mathbb{I}(B)$ and

$$\begin{aligned}
\mathrm{e}^1_Q(A) = \mathrm{e}^1_{QB}(B) &= \mathrm{e}^1_{Q\overline{A}}(\overline{A}) \\
&\geq -\sum_{i=1}^{d-2}\binom{d-3}{i-1}h^i(\overline{A}) \\
&= -\sum_{i=1}^{d-2}\binom{d-3}{i-1}h^i(B) \\
&= -\sum_{i=1}^{d-2}\binom{d-3}{i-1}[h^i(A) + h^{i+1}(A)] \\
&= -\sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(A) \\
&= \mathrm{e}^1_Q(A).
\end{aligned}$$

Hence the equality

$$\mathrm{e}^1_{Q\overline{A}}(\overline{A}) = -\sum_{i=1}^{d-2} \binom{d-3}{i-1} h^i(\overline{A})$$

holds true for the parameter ideal $Q\overline{A}$ in the generalized Cohen-Macaulay local ring $\overline{A}$. Thus the hypothesis of induction on d yields that $Q{\cdot}[A/\mathrm{U}(a_j)]$ is a standard parameter ideal in $A/\mathrm{U}(a_j)$ for every $1 \leq j \leq d$. Therefore $\mathrm{U}(a_1) \cap Q = (a_1)$ by Proposition 14, so that $Q{\cdot}[A/(a_1)]$ is a standard parameter ideal in $A/(a_1)$ ([12, Corollary 2.3]), since $Q{\cdot}[A/\mathrm{U}(a_1)]$ is a standard parameter ideal for the local ring $A/\mathrm{U}(a_1)$. Thus Q is a standard parameter ideal in A ([12, Corollary 2.1]), since $\mathbb{I}(A) = \mathbb{I}(A/(a_1))$. ☐

We are now ready to prove Theorem 12.

Proof of Theorem 12. We have only to show the implication (2) ⇒ (1). Since $\sharp\Lambda = 1$, by Theorem 11, A is a generalized Cohen-Macaulay local ring, so that

$$\Lambda = \left\{ -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A) \right\}$$

by Proposition 10 (2). Hence by Theorem 13 every parameter ideal Q is standard in A, because $\mathrm{e}^1_Q(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$, so that A is a Buchsbaum local ring. ☐

Unless A is unmixed, Theorem 12 is no more true, even if $\mathrm{e}^1_Q(A) = 0$ for every parameter ideal Q in A (cf. [1, Theorem 2.7]). Let us note one example.

Example 16. Let R be a regular local ring with the maximal ideal n and $d = \dim R \geq 3$. Let $X_1, X_2, \cdots, X_d$ be a regular system of parameters of R. We put $\mathfrak{p} = (X_1, X_2, \cdots, X_{d-1})$ and $D = R/\mathfrak{p}$. Then D is a DVR. Let $A = R \ltimes D$ denote the idealization of D over R. Then A is a Noetherian local ring with the maximal ideal $\mathfrak{m} = \mathrm{n} \times D$ and $\dim A = d$. Let Q be a parameter ideal in A and put $\mathfrak{q} = \varphi(Q)$, where $\varphi : A \to R, \varphi(a,x) = a$ denotes the projection map. We then have

$$\begin{aligned} \ell_A(A/Q^{n+1}) &= \ell_R(R/\mathfrak{q}^{n+1}) + \ell_D(D/\mathfrak{q}^{n+1}D) \\ &= \ell_R(R/\mathfrak{q}){\cdot}\binom{n+d}{d} + \ell_D(D/\mathfrak{q}D){\cdot}\binom{n+1}{1} \\ &= \mathrm{e}^0_{\mathfrak{q}}(R)\binom{n+d}{d} + \mathrm{e}^0_{\mathfrak{q}D}(D)\binom{n+1}{1} \end{aligned}$$

for all integers $n \geq 0$, so that $\mathrm{e}^0_Q(A) = \mathrm{e}^0_{\mathfrak{q}}(R), \mathrm{e}^{d-1}_Q(A) = (-1)^{d-1}\mathrm{e}^0_{\mathfrak{q}D}(D)$, and $\mathrm{e}^i_Q(A) = 0$ if $i \neq 0, d-1$. Hence $\mathrm{e}^1_Q(A)$ is constant but A is not even a generalized Cohen-Macaulay local ring, because $\mathrm{H}^1_{\mathfrak{m}}(A)$ ($\cong \mathrm{H}^1_{\mathrm{n}}(D)$) is not a finitely generated A-module. The local ring A is not unmixed, although $\operatorname{depth} A = 1$.

We close this paper with a characterization of Noetherian local rings A possessing $\sharp\Lambda = 1$. Let us note the following.

Proposition 17 ([1, Proposition 4.7])**.** *Suppose that $d = \dim A \geq 2$ and let U be the unmixed component of the ideal* (0) *in A. Assume that there exists an integer $t \geq 0$*

such that $\mathrm{e}_Q^1(A) = -t$ *for every parameter ideal* Q *in* A. *Then* $\dim_A U \leq d-2$ *and* $\mathrm{e}_{\mathfrak{q}}^1(A/U) = -t$ *for every parameter ideal* $\mathfrak{q}$ *in* A/U.

The goal of this paper is the following.

Theorem 18. *Suppose that* $d = \dim A \geq 2$. *Then the following two conditions are equivalent.*

(1) $\sharp\Lambda = 1$.

(2) *Let* $U = \mathrm{U}_{\widehat{A}}(0)$ *be the unmixed component of the ideal* (0) *in the* $\mathfrak{m}$*-adic completion* $\widehat{A}$ *of* A. *Then* $\dim_{\widehat{A}} U \leq d-2$ *and* $\widehat{A}/U$ *is a Buchsbaum local ring.*

When this is the case, one has the equality

$$\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(\widehat{A}/U)$$

for every parameter ideal Q *in* A.

Proof. (1) $\Rightarrow$ (2) For every parameter ideal $\mathfrak{q}$ of $\widehat{A}$ we have $\mathfrak{q} = (\mathfrak{q}\cap A)\widehat{A}$, so that $\mathfrak{q}\cap A$ is a parameter ideal in A. Hence $\Lambda(\widehat{A}) = \Lambda$ and so the implication follows from Theorem 12 and Proposition 17.

(2) $\Rightarrow$ (1) Since $\dim_{\widehat{A}} U \leq d-2$ and $\widehat{A}/U$ is a Buchsbaum local ring, we get $\#\Lambda(\widehat{A}) = 1$ by [1, Lemma 2.4 (c)], whence $\#\Lambda = 1$.

See Proposition 10 (2) and 17 for the last assertion. $\square$

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