ARTINIAN RINGS WITH INDECOMPOSABLE RIGHT MODULES UNIFORM

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Abstract. It is well known that any indecomposable module over a generalized uniserial ring is uniserial, therefore it is local as well as uniform. This motivated Tachikawa (1959) to study rings satisfying the following conditions. A ring R is said to satisfy condition (*) if it is artinian and every finitely generated indecomposable right R-module is local. A ring R is said to satisfy condition (**), if it is artinian and every finitely generated indecomposable right R-module is uniform. He had given a characterisation of condition (**). If a ring R satisfies (*), it admits a finitely generated injective co-generator. Consider any artinian ring R such that *mod*-R admits a finitely generated injective co-generator M, Let $Q = End(M)$ acting on left. By Tachikawa, every finitely generated indecomposable right R-module is local if and only if every finitely generated indecomposable left Q-module is uniform. In the present note, we give a characterisation of condition (**) in terms of the structure of the right ideals of the given ring. The approach in the present paper is quite different from that followed by Tachikawa. Let M be a uniform module of finte composition length, $D = End(soc(M))$ and D' the subdivision ring of D consisting of those $\sigma \in D$, which have some extensions in $End(M)$. Then the pair (D, D') is called *division ring pair associate* (in short *drpa*) of M. An outline of the proof the following result is given. A ring R with Jacobson radical J satisfies (**) if and only if it satisfies the following conditions: (1) R is a both sided artinian, right serial ring; (2) for any three indecomposable idempotents e, f, $g \in R$ with eJ, fJ, gJ non-zero the following hold: (i) If (D, D') is the *drpa* of $\frac{eR}{eJ^2}$, then the left dimension and the right dimensions of D over D' both are less than or equal to 2; (ii) if e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2}$, then $eJ^2 = 0$ or $fJ^2 = 0$; (iii) if e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2} \cong \frac{gJ}{gJ^2}$, then g is isomorphic to e or f; (iv) if $\frac{eR}{eJ^2}$ is not quasi-injective, then $eJ^2 = 0$ and $\frac{eJ}{eJ^2} \ncong \frac{fJ}{fJ^2}$, whenever e is not isomorphic to f. First step in the proof is to develop some techniques of construction of indecomposable modules which may be uniform or may not be uniform. There after a theorem involving lifting of an isomorphisms between simple homomorphic images of two finitely generated uniform modules is established, which is used to give the proof of the main theorem.

Key Words: Right serial rings, uniserial modules, quasi-injective, quasi-projective modules.

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Introduction

We consider the following conditions on a ring R. (**) R is a both sided artinian ring such that every finitely generated indecomposable right R-module is uniform. And its

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dual condition (*) R is both sided artinian such that every finitely generated indecomposable right module is local. These conditions have been studied by Tachikawa [7]. In [7, Theorem 5.3], a characterization of a ring satisfying (**) on the left is given. Here we discuss another approach to the study of rings satisfying these conditions and give a characterization of rings satisfying (**) in terms of the structure of its right ideals. The main purpose is to outline the proof of the main theorem, therefore. The main steps in the proof of the main theorem are given detail, but are stated without proof. In the process we also determine the structure of indecomposable modules over a ring satisfying (**). Throughout R is an artinian ring. In Section 1, some concepts and results proved in [5] are collected, in particular the concept of division ring pair associate of a uniform module of finite composition length is given in Definition 1.2. In Section 2, the ring of endomorphisms of a finite direct sum of uniform modules of finite composition lengths is investigated. These results can be of independent interest. The study of condition (**) is started in Section 3. To start with a lifting property of isomorphism between simple homomorphic images of uniform modules over a ring R satisfying (**) is proved. If an artinian ring R satisfies this lifting property, then R is said to satisfy condition weak (**). In Proposition 3.6, it is proved that any ring satisfying weak (**) is right serial. The concept of a critical uniserial submodule of a uniform module over an artinian ring is given in Definition 3.3. In Proposition 3.4, it is proved that if a ring R satisfies weak (**), then a uniform right R-module is either uniserial or its critical uniserial submodule is simple. In Proposition 3.9 and Theorem 3.10, some properties and relations between indecomposable summand of R_R, where R satisfies weak (**) are proved, which form a basis for giving a characterization of rings satisfying (**). In Section 4, a condition (***) motivated by results in Section 3, is introduced, which is satisfied by any ring satisfying (**). The structure of indecomposable modules over a ring R satisfying (**) is given in Theorems 4.6 and 4.7. The main result is given in Theorem 4.12. The whole paper depends on various constructions of indecomposable, non-uniform modules.

1. Preliminaries

All the modules considered here are unitary right modules, unless otherwise stated. For any ring R, its Jacobson radical is denoted by $J(R)$ (or simply by J). For any module M, $E(M)$, $End(M), d(M)$, $J(M)$ denote its injective hull, ring of endomorphisms, composition length, radical of M respectively. By a summand of a module M, we shall mean a summand other than 0, M. If a module $M = A \oplus B$, the resulting projection of M on A will be sometime denoted by π_A. The symbols $A \leqslant B$ $(A < B)$ will mean that A is a submodule of a module B (A is a submodule of a module B, but $A \neq B$). A non-zero element x of a module M_R is called a *local* (*uniform*) element, if xR is a local (uniform) module. A ring R is said to be *artinian*, if it is right artinian as well as left artinian. Let S, T be two simple modules over a ring R. Then T is called a *predecessor* of S and S is called a *successor* of T, if there exists a uniserial module A_R such that $d(A) = 2$, and for the maximal submodule B of A , $S \cong B$, $T \cong \frac{A}{B}$. A module M is said to be *uniserial*, if the family of its submodules is linearly ordered under inclusion. If a ring R is such that R_R is a finite direct sum of uniserial modules, then R is called a *right serial* *ring.* An artinian ring that is right and left serial is called a *generalized uniserial ring.* For various concepts on rings and modules one may consult [1] or [8].

The following is a modification of [5, Lemma 2.1]. For this see also [7, Lemma 1.3, Proposition 2.3].

Lemma 1.1. *Let A, B be two uniform modules over a right artinian ring R, and S be the simple submodule of A. Let there exist a monomorphism $\sigma : S \to B$, $L = \{(a, -\sigma(a)) : a \in S\}$ and $M = \frac{A \times B}{L}$. If $\overline{(x, y)}R$ is a simple submodule of M other than $T = \{\overline{(s, 0)} : s \in S\}$, then $f : xR \to yR$, $f(xr) = yr$, $r \in R$ defines a homomorphism extending $-\sigma$. Further M is uniform if and only if there is no module C_R with $S < C \leqslant A$ for which there exists a homomorphism $f : C \to B$ extending σ.*

Definition 1.2. Let A_R be a uniform module of finite composition length, $S = soc(A)$, $D = End(S)$, and D' be the division subring of D consisting of those $\sigma \in D$ that can be extended to some endomorphisms of A. Then the pair (D, D') is called the *division ring pair associate* (in short the *drpa*) of A.

For any subdivision ring D' of a division ring D, $[D, D']_l$ ($[D, D']_r$) will denote the dimesion of D as a left (right) vector space over D'.

Lemma 1.3. *Let A_R be a uniserial quasi-projective module with $d(A) = 2$ and $S = soc(A)$. Let (D, D') be the drpa of A.*

(i) [5, Lemma 2.2]. *Let ω_1 $(= I)$, ω_2, ..., ω_n any n non-zero members of the D. Then $M = \frac{A^{(n)}}{L}$, where $L = \{(\omega_1 x_1, \omega_2 x_2, .., \omega_n x_n) : x_i \in S, \Sigma_i x_i = 0\}$ is uniform if and only if $\omega_1^{-1}, \omega_2^{-1}, ..., \omega_n^{-1}$ are right linearly independent over D'.*

(ii) [5, Lemmas 2.3, 2.4]. *Let $E = E(A)$, λ_i be automorphisms of E for $1 \le i \le n$, with $\lambda_1 = I$, where n is some positive integer. Let $K = A_1 + A_2 + ... + A_n$, where each $A_i = \lambda_i(A)$, and let $\omega_i = \lambda_i \mid S$. Then $A_j \not\subseteq \Sigma_{i \neq j} A_i$ for any j if and only if $\omega_1, \omega_2, ..., \omega_n$ are right linearly independent over D'. Further, if $A_j \not\subseteq \Sigma_{i \neq j} A_i$ for every j, for $A = A_1$, $\frac{A^{(n)}}{L} \cong K$, where $L = \{(\omega_1^{-1} x_1, \omega_2^{-1} x_2, ..., \omega_n^{-1} x_n) : x_i \in S,\ \Sigma_i x_i = 0\}$, and this isomorphism is induced by the epimorphism $\lambda : A^{(n)} \to K$, $\lambda(a_1, a_2, ..., a_n) = \lambda_1(a_1) + \lambda_2(a_2) + ... + \lambda_n(a_n)$.*

Lemma 1.4. *Let A_R be a uniserial module with $d(A) = 2$, $S = soc(A)$ and $E = E(A)$.*

(i) *If λ is an automorphism of E such that $\lambda \mid S$ is identity on S, then $\lambda(A) = A$.*

(ii) *If two automorphisms σ, η of E are equal on S, then $\sigma(A) = \eta(A)$.*

Proof. (i) Suppose $A \neq \lambda(A)$. Then $S = A \cap \lambda(A)$. Let (D, D') be the *drpa* of A. Consider the mapping $\mu : A \times A \to A + \lambda(A)$, $\mu(a, b) = a + \lambda(b)$. Here $a + \lambda(b) = 0$ gives $a \in S$, therefore $ker\, \mu$ is $L = \{(a, -\lambda^{-1}(a)) : a \in S\}$. Therefore $M = \frac{A \times A}{L}$ is uniform, and by (1.3), I , ω $(= \lambda \mid S)$ are right linearly independent over D', which is a contradiction. Hence $\lambda(A) = A$.

(ii) is immediate from (i). □

The following is from Lemmas 2.6 and 2.7 in [5]

Lemma 1.5. *Let K_R be a non-simple uniform module of finite composition length, $S = soc(K)$ and (D, D') drpa of K. Let ω_1 $(= I)$, $\omega_2, ..., \omega_n$ be any n non-zero members of D,* *and* $L = \{(\omega_1 x, \omega_2 x, ..., \omega_n x) : x \in S\}$. *Then* L *is not contained in a summand of* $K^{(n)}$ *if and only if* $\omega_1,\ \omega_2, ..., \omega_n$ *are left linearly independent over* D'. *If* L *is not contained in a summand of* $K^{(n)}$ *and* K *is quasi-projective then* $M = \frac{K^{(n)}}{L}$ *is indecomposable; if in addition* $n > 2$, *then* M *is not uniform.*

2. Endomorphism rings

Theorem 2.1. *Let* $A_1,\ A_2,\ ....,\ A_n$ *be any finitely many uniform right modules of finite composition lengths, over a ring* R, $M = A_1 \oplus\ A_2 \oplus ... \oplus A_n$ *and* $K = End(M)$. *Then* $J(K)$ *is the set of all those* $n \times n$*-matrices* $[\sigma_{ij}]$, *where no* $\sigma_{ij} : A_j \to A_i$ *is an isomorphism.*

Proof. Let A, B, C be any three non-zero uniform right modules of finite composition lengths, over R. Let σ, $\eta : A \to B$ be two homomorphisms, which are not isomorphisms. If one of σ, η is a monomorphism, then $d(A) < d(B)$, therefore $\sigma + \eta$ is not an isomorphism. If neither of σ, η is a monomorphism, then both of them are zero on the $soc(A)$, therefore again $\sigma + \eta$ is not an isomorphism. After this it can be seen that the set N of all $[\sigma_{ij}] \in K$, in which no entry is an isomorphism, is an ideal of K.

Now suppose that $d(A) \leq d(B)$. Let $\lambda : B \to C$ be a homomorphism which is not an isomorphism, but $d(C) \leq d(B)$. As λ is not a monomorphism, it is zero on $soc(B)$. Let $0 \neq L \leqslant A$ and $\sigma : A \to B$ be a homomorphism. If $\sigma(L) \neq 0$, then $soc(B) \leqslant \sigma(L)$, $d(\lambda\sigma(L)) < d(\sigma(L))$. Therefore $d(\lambda\sigma(L)) < d(L)$. Let $\mu : C \to A$ be an homomorphism which is not an isomorphism, and $d(C) \geq d(A)$. Then μ is zero on $soc(C)$. Therefore for any non-zero submodule L of C, $d(\mu(L)) < d(L)$, $d(\sigma\mu(L)) < d(L)$. If we take an admissible product of a sequence of up to $n+1$ entries of $[\sigma_{ij}] \in N$, it results in some homomorphisms $\eta_{ji} : A_i \to A_j$, $\eta_{kj} : A_j \to A_k$ where the situation is similar to the one discussed above in the sense that either $d(A_i) \leq d(A_j) \geq d(A_k)$ or $d(A_i) \geq d(A_j) \leq d(A_k)$. Using this we find that each member of N is nilpotent. Hence $N \subseteq J(K)$. If a $[\sigma_{ij}] \in J(K)$, it can be easily seen that no entry in $[\sigma_{ij}]$ is an isomorphism. Hence $N = J(K)$. □

Theorem 2.2. *Let* $A_1,\ A_2,\ ....,\ A_n$ *be any finitely many uniform right modules of finite composition lengths, over a ring* R, *such that they have isomorphic socles. Let* $C_1,\ C_2,\ ...,\ C_t$ *be the isomorphism classes of* $A_1,\ A_2,\ ....,\ A_n$. *arranged in such a way that for any* $i < t$, *if some* $A_k \in C_i$ *and* $A_l \in C_{i+1}$, *then* $d(A_k) \leq d(A_l)$. *Let* $A_1,\ A_2,\ ....,\ A_n$ *be re-indexed such that if an* $A_k \in C_i$ *and an* $A_l \in C_{i+1}$, *then* $k < l$. *Let* $S = soc(A_i)$ *for* $1 \leq i \leq n$, $\omega_1 (= I),\ \omega_2,\ ....,\ \omega_n$ *be any* n *non-zero members of* $D = End(S_R)$. *Let* $0 \neq x_1 \in S$ *and* $x_i = \omega_i x_1$ *for* $1 \leq i \leq n$. *Then* $x = (x_1, x_2, ..., x_n) \in M = A_1 \times A_2 \times .... \times A_n$ *is contained in a summand of* M *if and only if for some* $1 < j \leq n$, *there exist homomorphisms* $\eta_{jk} : A_k \to A_j$ *for* $1 \leq k < j$ *such that* $\omega_j = \mu_{j1}\omega_1 + \mu_{j2}\omega_2 + ... + \mu_{jj-1}\omega_{j-1}$, *where each* $\mu_{jk} = \eta_{jk} \mid S$.

Proof. For $1 \leq j \leq t$, let B_j be the direct sum of those A_i's that are in C_j. Suppose the cardinality of C_j is k_j. Now $M = B_1 \oplus B_2 \oplus .... \oplus B_t$. Any $\sigma \in T = End(M)$ can be represented as a block matrix $[H_{ij}]$, where each H_{ij} is a $k_i \times k_j$-matrix representing an R-homomorphism from $B_j \to B_i$. Now x is contained in a summand of M if and only if there exists a non-zero idempotent $\sigma \in T$, satisfying $\sigma y = 0$, where y is the transpose of the row matrix $[x_1, x_2, ..., x_n]$. Consider any $j > i$. For any $A_k \in C_i$, $A_l \in C_j$, as

$d(A_k) \leq d(A_l)$, any homomorphism $\theta_{kl} : A_l \to A_k$ is zero on $soc(A_l)$. That means that the effect of H_{ij} on the corresponding block in y is zero. Thus the lower triangular block matrix $\eta = [G_{ij}]$ such that $G_{ij} = H_{ij}$ for $i \geq j$, and $G_{ij} = 0$ for $j > i$ has same effect on y as of σ on y. But $\eta \equiv \sigma \ (mod\ J(T))$. Suppose no entry of any G_{ii} is an isomorphism, then the matrix of the diagonal block of η is in $J(T)$, from which it follows that η is nilpotent. Consequently σ is nilpotent, which is a contradiction. Hence there exists smallest positive integer k such that some entry of G_{kk} is an isomorphism. Write $x = [z_1, z_2, ..., z_t]$, where each z_i is a block with k_i entries and let u_i be the transpose of z_i . As $\sigma y = 0$, $\sum_{i=1}^{k} H_{ki} z_i = 0$. Now H_{kk} has an entry, say σ_{rs} which is an isomorphism. For this, we choose s to be largest with respect to the fixed r. At the same time write $\sigma = [\sigma_{ij}]$ where each $\sigma_{ij} : A_j \to A_i$. Let $\sigma_{rs}^{-1} e_{sr}$ be the $n \times n$ matrix whose (s, r)-th entry is σ_{rs}^{-1} and its other entries are zero. Then $(\sigma_{rs}^{-1} e_{sr})\sigma y = 0$. This gives $\sum_{i=1}^{s} \lambda_{si} x_i = 0$, for some $\lambda_{si} : A_i \to A_s$, $\lambda_{ss} = I$, the identity map on A_s. As $x_i = \omega_i x_1$, we get $\omega_s = \sum_{i=1}^{s-1} \mu_{si} \omega_i$, where each $\mu_{si} = -\lambda_{si} \mid S$.

Conversely, let $\omega_s = \sum_{i=1}^{s-1} \mu_{si} \omega_i$ for some $s > 1$, such that each $-\mu_{si}$ is the restriction to S of some homomorphism $\eta_{si} : A_i \to A_s$. Let $\psi = [\psi_{ij}]$, where $\psi_{ij} = 0$ for $i \neq s$, $\psi_{si} = -\mu_{si}$ for $1 \leq i \leq s-1, \psi_{ss} = I$, $\psi_{sj} = 0$ for $j > s$. Then ψ is a non-zero idempotent such that $\psi y = 0$. □

3. Condition weak (**)

We start with the following condition. (**) R is an artinian ring such that every finitely generated indecomposable right R-module is uniform.

Following is a lifting property for condition (**).

Theorem 3.1. *Let R be a ring satisfying (**). Let M, N be two finitely generated uniform right R-modules. If for some maximal submodules M', N' of M, N respectively, there exists an isomorphism $\sigma : \frac{M}{M'} \to \frac{N}{N'}$, then σ or σ^{-1} can be lifted to a homomorphism η from M to N or from N to M respectively.*

Proof. Let $T = \{(a, b)\colon\ a \in M,\ b \in N,\ \sigma(\overline{a}) = \overline{b}\}$. Then T is a submodule of $M \times N$ containing $M' \times N'$ such that if an $(a, b) \in T$ with $a \notin M'$, then $T = (a, b)R + M' \times N'$. Therefore $M' \times N'$ is maximal in T and T is maximal in $M \times N$. For the projections $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$, $\pi_1(T) = M$, $\pi_2(T) = N$. As $d(soc(T)) = 2$, $T = C \oplus D$ for some uniform submodules C, D. Let $0 \neq s \in soc(M)$. Then $(s, 0) = c + u$, $c \in soc(C)$, $u \in soc(D)$.

We take $d(M) \geqslant d(N)$, $c = (s_1, s_2)$, $u = (u_1, u_2)$ for some s_1, $u_1 \in M$, u_2, $s_2 \in N$. Now $c \neq 0$ or $u \neq 0$. Suppose $c \neq 0$. Then M' embeds in C under π_C. therefore $d(C) = d(M)$ or $d(C) = d(M) - 1$. Suppose $d(C) = d(M)$. Then $d(D) = d(N) - 1 < d(C)$. Now Suppose $s_1 = 0$, then C embeds in N under π_2, therefore $d(C) = d(N)$, $\pi_2(C) = N$. Thus there exists $(a', b') \in C$ with $b' \notin N'$. Now $\sigma^{-1}(\overline{b'}) = \overline{a'}$. Let $y \in N$. As $\pi_2 \mid C$ is an isomorphism, there exists unique $(x, y) \in C$, with $x \in M$. We get a homomorphism $\eta : N \to M$ for which $\eta(y) = x$, this homomorphism lifts σ^{-1}. Suppose $s_1 \neq 0$. Then

C is isomorphic to M under π_1. Let $x \in M$. Then there exists unique $y \in N$ such that $(x, y) \in C$. This gives a homomorphism $\eta : M \to N$ for which $\eta(x) = y$, which lifts σ. We shall be using similar arguments for some other situations.

Now suppose $d(C) = d(M) - 1$. Then $d(D) = d(N)$. Suppose $u = 0$, then $soc(C) = soc(M)$, $soc(D) \neq soc(M)$, and $D \cong \pi_2(D) = N$; as before, we get a homomorphism $\eta : N \to M$ lifting σ^{-1}. Now suppose $u \neq 0$. Suppose $u_2 \neq 0$. Then $D \cong \pi_2(D) = N$. We get a lifting of σ^{-1}. Suppose $u_2 = 0$. Then $s_2 = 0$, $u_1 \neq 0$ as $(s, 0) = c + u$. Then $C \cap D = 0$, gives $c = 0$, $C \cong \pi_2(C) = N$. This gives a lifting of σ^{-1}. □

The above result is a partial dual of [6, Proposition 2.2]. It is not known, whether the converse of the above result holds

The above theorem motivates the following condition.

Definition 3.2. A ring R is said to satisfy condition weak (**) if it is artinian and it has the following property: Let M, N be any two finitely generated, uniform right R-modules and M', N' be any maximal submodules of M, N respectively. If there exists an isomorphism $\sigma : \frac{M}{M'} \to \frac{N}{N'}$, then there exists a homomorphism η from M to N or from N to M, lifting σ or σ^{-1} respectively.

Suppose A, B are two non-simple uniserial modules over a ring R satisfying weak (**), such that $d(A) = d(B)$, and there exists an isomorphism $\sigma : \frac{A}{AJ} \to \frac{B}{BJ}$. By the definition, we can fix σ such that it has a lifting $\eta : A \to B$. Then $\eta(A) \nsubseteq BJ$, therefore η is an isomorphism. Thus A, B are isomorphic.

Similar arguments shows that if A_R, B_R are two uniform modules such that $d(A) = d(B)$, $\frac{A}{soc(A)}$, $\frac{B}{soc(B)}$ are semi-simple and some simple module embeds in both $\frac{A}{soc(A)}$, $\frac{B}{soc(B)}$. Then $A \cong B$. Using this result, one can easily prove the following. Let M_R be a uniform module, and $\frac{M}{soc(M)}$ be semi-simple, then either $\frac{M}{soc(M)}$ is homogeneous or it has only two homogeneous component, and each of them is simple, in other words, either $\frac{M}{soc(M)}$ is homogeneous or $d(M) = 3$.

Let R be any right artinian ring, A_R a uniform modules. If k is a positive integer and $soc^k(A)$ is uniserial, then for any $x \in A \backslash soc^k(M)$. $soc^k(M) < xR$. If k is maximal such that $soc^{k-1}(M) < soc^k(M) \neq soc^{k+1}(M)$, then $\frac{soc^{k+1}(M)}{soc^k(M)}$ is not simple. This motivates the following.

Definition 3.3. Let R be any right artinian ring, and K_R a non-zero uniform module. Then a uniserial submodule N of K is called the *critical uniserial submodule* of K, if for some $k > 0$, $N = soc^k(K)$, but $\frac{soc^{k+1}(K)}{soc^k(K)}$ is not simple, whenever it is non-zero.

The critical uniserial submodule of a uniform module over a right artinian ring is uniquely determined.

Proposition 3.4. *Let R be a ring satisfying weak (**), K_R a uniform module. and N its critical uniserial submodule . If $N = soc^k(K)$ and $\frac{soc^{k+1}(K)}{N}$ is non-zero , then $k = 1$. The module K is either uniserial or the critical uniserial submodule of K is simple.*

Proof. Suppose $k \geqslant 2$. As $\frac{soc^{k+1}(K)}{N}$ is not simple, we get two uniserial submodules U, V of E such that $U \cap V = N$, $d(V) = k+1 = d(U)$. Let $M = U + V$. Let S be the simple submodule of N and $B = \frac{V}{S}$, Clearly, B is uniserial. Now $\frac{M}{U} \cong \frac{B}{N}$, where $\overline{N} = \frac{N}{S}$.Therefore there exists a homomorphism $\sigma : M \to B$ such that $\sigma(U) \subseteq \overline{N}$ and $\sigma(V)+\overline{N} = B$, or $\sigma : B \to M$ such that $\sigma(\overline{N}) \subseteq U$, $\sigma(B) \nsubseteq U$.

Suppose $\sigma : M \to B$. Let $L = ker\,\sigma$. Then $\frac{M}{L}$ is uniserial. But $\frac{M}{N}$ is not uniform and $d(\frac{M}{N}) = 2$. Thus $L \nsubseteq N$, therefore $N < L$ and $d(\frac{M}{L}) = 1$. Thus $d(\sigma(M)) = 1$. But $\sigma(V) \nsubseteq \overline{N}$, gives $d(\sigma(M)) \geqslant 2$, which is a contradiction. Thus $\sigma : B \to M$. Then $\sigma(B) \nsubseteq U$ gives $N < \sigma(B)$, $d(\sigma(B)) \geq k+1$. But $d(\sigma(B)) \leqslant d(B) = k$, which gives a contradiction. Hence $k = 1$. Now the last part is immediate. □

Lemma 3.5. *Let R be a ring satisfying weak (**) and A_R, B_R be two uniserial modules with $d(B) \leqslant d(A)$. Then A is B-projective. Any uniserial right R-module is quasi-projective. If A is quasi-injective, then any homomorphic image of A is quasi-projective.*

Proof. Let $\sigma : A \to \frac{B}{C}$ be a non-zero homomorphism. Without loss of generality, we take σ an epimorphism. Let $d(\frac{B}{C}) = n$. We apply induction on n. Let $ker\,\sigma = L$. If $n = 1$, then L is maximal in A. As R satisfies weak (**), there exists an epimorphism $\eta : A \to B$, lifting $\overline{\sigma} : \frac{A}{L} \to \frac{B}{C}$. Then η lifts σ. Hence the result holds for $n = 1$.

For some $k \geqslant 1$, let the result hold for $n = k$. Suppose $n = k+1$. We get $C < C' < B$ with $d(\frac{C'}{C}) = 1$. Let $\pi : \frac{B}{C} \to \frac{B}{C'}$ be the natural mapping. Then $\sigma' = \pi\sigma : A \to \frac{B}{C'}$ is an epimorphism. Then $L = ker\,\sigma$, $L' = ker\,\sigma'$ are such that $d(\frac{A}{L}) = k+1$, $d(\frac{A}{L'}) = k$, therefore $d(\frac{L'}{L}) = 1$. By the induction hypothesis, σ' lifts to a homomorphism $\beta : A \to B$. If β lifts σ, we finish. Otherwise we get induced non-zero mapping $\sigma - \overline{\beta} : A \to \frac{B}{C}$with $Im(\sigma - \overline{\beta}) = \frac{C'}{C}$, where $\overline{\beta} : A \to \frac{B}{C}$ is induced by β. As the result holds for $n = 1$, we get a homomorphism $\mu : A \to B$ lifting $\sigma - \overline{\beta}$.Then $\eta = \beta + \mu$ lifts σ. Hence A is B-projective. It also follows that A is quasi-projective, After this the second part is obvious. □

Proposition 3.6. *Let R be a ring satisfying weak (**), then R is right serial and any local right R-module is uniserial.*

Proof. Let e be an indecomposable idempotent in R. It is enough to show that $\frac{eR}{eJ^2}$is uniserial. Therefore we take $J^2 = 0$. Suppose $eJ \neq 0$, then $eJ = A \oplus B$, where A, B are right ideals with A a minimal right ideal. Now $M = \frac{eR}{B}$ is uniserial, and by (3.4) it is quasi-projective. Let $T = ann(M)$. Any quasi-projective module H over an artinian ring Q is projective as a $\frac{Q}{ann(H)}$-module [3]. Thus M is a projective $\frac{R}{T}$-module. Now eR embeds in a finite direct sum of uniserial modules , each of composition length two and a homomorphic image of eR, by (3.3), these uniserial modules are isomorphic, therefore $T = ann(eR)$, $eT = 0$. Consequently $M \cong eR$, eR is uniserial. Hence R is right serial. The last part is obvious. □

Let e, f, g be three non-isomorphic, indecomposable idempotents in a ring R satisfying (**), such that eJ, fJ, gJ are non-zero and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2} \cong \frac{gJ}{gJ^2}$. Then for the simple module $S = \frac{eJ}{eJ^2}$, $E = E(S)$, $\frac{soc^2(E)}{S}$ has more than two homogeneous components, which is a contradiction.

The following result will lead us to the structure of indecomposable modules over rings satisfying (**). This is also a dual of [6, Lemma 2.7].

Proposition 3.7. *Let R be a ring satisfying weak (**), M_R a uniform R-module of finite composition length and $S = soc(M)$. Then $\frac{M}{S}$ has no uniform submodule which is not uniserial. If R satisfies (**), then $\frac{M}{S}$ is a direct sum of uniserial modules.*

Proof. Let $L \leqslant M$ be such that $\frac{L}{S}$ is a non-zero uniform module. Let $\frac{T}{S} = soc(\frac{L}{S})$. Then T in uniserial and $d(T) = 2$. Let K be a submodule of L not contained in T. Then $T < K$. Thus the critical uniserial submodule of L is not simple. By (3.6), L is uniserial. The second part is immediate from the definition of condition (**). □

Theorem 3.8. (i) *Let R be a local ring satisfying (**) and $J = J(R)$. Then either $J^2 = 0$ or R is both sided serial.*

(ii) *Let R be an indecomposable ring satisfying (**), for which there exists a simple module S_R as its own successor. If $E = E(S)$ is such that $\frac{soc^2(E)}{S}$ is non-zero and homogeneous, then $J^2 = 0$ and R is a full matrix ring over a local ring.*

(iii) *Let R be a ring satisfying (**). If $e \in R$ is an indecomposable idempotent such that $\frac{eR}{eJ^2}$ is not quasi-injective, then $eJ^2 = 0$.*

Proof. R is both sided serial iff $\frac{R}{J^2}$ is right self-injective. Suppose $\frac{R}{J^2}$ is not right self-injective and $J^2 \neq 0$. We take $J^3 = 0$. Now $A_R = \frac{R}{J^2}$ is not quasi-injective, therefore its injective hull over $\frac{R}{J^2}$ is not uniserial. Let $E = E(R_R)$. Set $S = J^2$, $\overline{E} = \frac{E}{S}$. Let $\sigma : \frac{J}{J^2} \to \overline{E}$ be a non-zero homomorphism. It induces homomorphism $\sigma' : J \to \overline{E}$. Then $(range\, \sigma') = \frac{L}{S}$ with $d(L) = 2$. As J is a projective $\frac{R}{J^2}$-module, σ' lifts a homomorphism $\eta : J \to E$. However, E is injective, therefore η extends to a homomorphism $\lambda : R \to E$. We get induced map $\overline{\lambda} : \frac{R}{J^2} \to \overline{E}$. This proves that $\overline{E}$ is an injective $\frac{R}{J^2}$-module. Hence $\overline{E}$ is a direct sum of uniform modules, none of which is uniserial. This contradicts (3.7).

(ii) The hypothesis gives that S is its only predecessor. Now $S \cong \frac{eR}{eJ}$, then all composition factors of eR are isomorphic and $fRe = 0$ for any indecomposable idempotent f not isomorphic to e. Therefore R is a matrix ring over a local ring R', which by (i) is such that $J(R')^2 = 0$.

(iii) Suppose $eJ^2 \neq 0$. Set $M = \frac{eR}{eJ^3}$, and $S = \frac{eJ}{eJ^2}$ As A is not quasi-injective, there exists an $\omega \in End(S)$ which cannot be extended in $End(A)$. As $\frac{eJ}{eJ^3}$ is quasi-projective, ω lifts to a $\mu \in End(\frac{eJ}{eJ^3})$. Set $\sigma = \mu \mid \frac{eJ^2}{eJ^3}$, $N = \frac{M \times M}{L}$, where $L = \{(x, -\sigma x) : x \in \frac{eJ^2}{eJ^3}\}$. As μ is an extension of σ, N is not uniform, so it has a summand. Therefore there exists an extension $\lambda \in End(M)$ of σ. Then λ is not an extension of μ for otherwise, we get an extension of ω in $End(\frac{eR}{eJ^2})$. Set $\lambda_1 = \lambda \mid \frac{eJ}{eJ^3}$. Then $(\lambda_1 - \mu)\frac{eJ}{eJ^3} = \frac{eJ^2}{eJ^3}$, which proves that the successor of S is also S. By (ii) $J^2 = 0$. This proves the result. □

Proposition 3.9. *Let R be a ring satisfying weak (**), e, f two non-isomorphic indecomposable idempotents such that $eJ^2 \neq 0 \neq fJ^2$ and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2}$. Then there exists an indecomposable right R-module of finite composition length, that is neither uniform nor local. If R satisfies (**), then $eJ^2 = 0$ or $fJ^2 = 0$.*

Proof. Let $S = \frac{eJ^2}{eJ^3}$, $E = E(S)$. The hypothesis gives two submodules A, B of E such that $A \cong \frac{eR}{eJ^3}$, $B \cong \frac{fR}{fJ^3}$, $d(A \cap B) = 2$. As the critical uniserial submodule of E is S, there exists a uniserial submodule L of E such that $d(L) = 2$, $L \cap K = S$, where $K = A \cap B$. Set $M = L \oplus A \oplus B$.

Case 1. $\frac{L}{S} \ncong \frac{K}{S}$. Then there is no monomorphism from any of L, A, B into the other. Let $0 \neq u \in S$. Then $T = (u, u, u)R$ is a simple submodule of M, which by (2.2) is not contained in any summand of M. We prove that $\overline{M} = \frac{M}{T}$ is indecomposable. Suppose otherwise, then $M = C + N$ for some C, $N < M$ such that $T < C$, $T < N$ and $T = C \cap N$. As $d(\overline{M}) = 7$, we take $d(C) \leq 4$. No summand of C contains T, and no summand of C is contained in MJ.

Subcase 1. $d(C) = 2$. Then C is uniserial, thus $C \subseteq L \oplus AJ \oplus BJ$. Then $\pi_L(C) = L$, for otherwise, $C \subset MJ$. Thus C is a summand of M, which is a contradiction.

Subcase 2. $d(C) = 3$. If C is uniform, then $\pi_A(C) = A$ or $\pi_B(C) = B$, therefore C is a summand of M, which is a contradiction. It follows that C is not uniform, $C \subseteq L \oplus AJ \oplus BJ$ and $\pi_L(C) = L$. However, R is right serial, therefore L is a projective $\frac{R}{J^2}$-module. As $L \oplus AJ \oplus BJ$ is an $\frac{R}{J^2}$-module, we get that C has a simple summand, which is a contradiction.

Subcase 3. $d(C) = 4$. As M is an $\frac{R}{J^3}$-module, A, B are projective $\frac{R}{J^3}$-module, C cannot project on A or B, for otherwise C will have a simple summand. Thus $C \subseteq L \oplus AJ \oplus BJ$. Then $C = C_1 \oplus C_2$ with $d(C_1) = 2$, $\pi_L(C_1) = L$, $C_2 = C \cap (AL \oplus BL) \subseteq MJ$, which is a contradiction.

Case 2. $\frac{L}{S} \cong \frac{K}{S}$. Then L and K are not quasi-injective, but they are isomorphic. So there exists an $\omega \in End(S)$ that cannot be extended to a homomorphism from L into K. For a fixed $0 \neq u \in S$, $T = (u, \omega u, \omega u)R$ is a simple submodule of M, which is not contained in any summand of M. Now follow the arguments as in Case 1.

This proves that $\overline{M}$ is indecomposable. Clearly $\overline{M}$ is neither uniform nor local. After this the last part is obvious. □

Theorem 3.10. *Let R be a ring satisfying weak* (**).

(i) *If there exists a uniserial module A_R such that $d(A) = 2$, and its drpa (D, D') satisfies $[D : D']_r > 2$, then there exists an indecomposable, non-uniform, non-local right R-module of finite composition length.*

(ii) *If R satisfies* (**), *the drpa (D, D') of a uniserial module A_R with $d(A) = 2$ satisfies $[D : D']_r \leq 2$.*

Proof. (i) Let $E = E(A)$, $S = soc(A)$. We get ω_1 $(= I)$, ω_2, $\omega_3 \in D$, which are right linearly independent over D'. Let $\lambda_1(= I)$, λ_2, λ_3 be extensions of ω_1, ω_2, ω_3 respectively in $End(E)$ Let $A_i = \lambda_i(A)$. Set $B_1 = A_1$, $B_2 = B_1$, $B_3 = A_1 + A_2$. Then B_1, B_2, B_3 are of composition lengths 2, 2, 3 respectively. Fix an $x_1 \neq 0$ in S. Let $M = B_1 \oplus B_2 \oplus B_3$ and $\pi_i : M \to B_i$ be the associated projections. Let $x_i = \omega_i x_1$, then $T = (x_1, x_2, x_3)R$ is a simple submodule of M. Suppose T is contained in a summand of M. By (2.2), we have following possibilities. (1) $\omega_2 = \eta_{21}\omega_1$ for some homomorphism $\eta_{21} : B_1 \to B_2$. Then η_{21} is an automorphism of A. If $\lambda \in End(E)$ is an extension of η_{21}, Then $\mu = \lambda\lambda_1$ is an extension of ω_2, therefore by (1.5), $\mu(A_1) = A_2$. But $\lambda\lambda_1(A_1) = A_1$, which is a contradiction. (2) $\omega_3 = \eta_{31}\omega_1 + \eta_{32}\omega_2$ for some homomorphisms $\eta_{31} : B_1 \to B_3$, $\eta_{32} : B_2 \to B_3$. Let λ, $\mu \in End(E)$ be extensions of η_{31}, η_{32} respectively. Then $\rho = \lambda\lambda_1 + \mu\lambda_2$ is an extension of ω_3, therefore $\rho(A) = A_3$. However $(\lambda\lambda_1 + \mu\lambda_2)(A) \subseteq B_3$ and by (1.3)(ii) $A_3 \nsubseteq B_3$, which is a contradiction. Hence T is not contained in any summand of M.

We now prove that $\overline{M} = \frac{M}{T}$ is indecomposable. Now $d(\overline{M}) = 6$. Suppose $\overline{M}$ has a summand. We get a summand $\overline{C}$ with $d(\overline{C}) \leq 3$. Now $\overline{C} = \frac{C}{T}$ for some $C < M$ containing T. For some $N < M$, $M = C + N$, $T = C \cap N$. As $soc(M)$ is small in M, C has no semi-simple summand. In particular, T is not a summand of C. Indeed no summand of C contains T. As $soc^2(E)$ is a module over $\frac{R}{J^2}$, we take $J^2 = 0$. In that case every uniserial module of composition length 2 is projective. Let $x = (x_1, x_2, x_3)$

Case 1. $d(\overline{C}) = 1$. Then $d(C) = 2$, and C is uniserial, $x \in C$ has projection $x_1 \neq 0$ in B_1. Therefore C projects onto B_1. Thus C is a summand of M, which is a contradiction.

Case 2. $d(\overline{C}) = 2$. Then $d(C) = 3$. The projection of C in B_3 is non-zero, as x has non-zero projection in B_3. If C projects onto B_3, then $C \cong B_3$, therefore C is a summand of M, which is a contradiction. If the image of C in B_3 has composition length 2, then this image being projective, gives that C has a simple summand, which is also a contradiction. Suppose Image of C in B_3 is simple, then $C = T \oplus (C \cap (B_1 + B_2))$, which is also a contradiction.

Case 3. $d(\overline{C}) = 3$. Then $d(C) = 4$. If C projects onto $B_1 \oplus B_2$, then C is a summand of M, which is a contradiction. So $C \cap B_3 \neq 0$. We are left with the situation in which we also have $N \cap B_3 \neq 0$. In this case $C \cap N$ contains $T + soc(B_3)$, which is a contradiction.

Hence $\overline{M}$ is indecomposable. Clearly $\overline{M}$ is neither uniform nor local.

Now (ii) is immediate from (i). □

4. Condition (***)

Definition 4.1. A ring R is said to satisfy condition (***) if R is artinian, right serial, and for any three indecomposable idempotents e, f, $g \in R$ with eJ, fJ, gJ non-zero, the following hold.

(i) The *drpa* (D, D') of $A = \frac{eR}{eJ^2}$ is such that $[D : D']_r \leq 2$, $[D : D']_l \leq 2$.

(ii) If e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2}$, then $eJ^2 = 0$ or $fJ^2 = 0$.

(iii) If e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2} \cong \frac{gJ}{eJ^2}$, then g is isomorphic to e or f.

(iv) If $A = \frac{eR}{eJ^2}$ is not quasi-injective, then $eJ^2 = 0$ and $\frac{eJ}{eJ^2} \ncong \frac{fJ}{fJ^2}$ whenever e is not isomorphic to f.

Suppose R is a ring satisfying (**). By (3.6) R is right serial. By (1.5) and (3.10) condition (i) in (4.1) is satisfied by R. By (3.9) condition (ii) in (4.1) holds. Condition (iii) in (4'1) follows from remarks following (3.6). Condition (iv) from remarks after (3.2). We are going to prove that conditions (**) and (***) are equivalent.

Suppose R is a ring satisfying (***). Let S_R be a simple module, $E = E(S)$ and M any submodule of E. Suppose M is not uniserial and N is its critical uniserial submodule. Then for some $k > 0$, $N = soc^k(M)$, $d(\frac{soc^{k+1}(M)}{N}) > 1$. Let $G = \frac{soc^{k+1}(M)}{NJ}$ is uniform, $\frac{G}{soc(G)} \cong \frac{soc^{k+1}(M)}{N}$. By using conditions (iii) and (iv) in (4.1), we see that $d(\frac{G}{soc(G)}) = 2$, $G = C + H$ for some uniserial submodules C, H such that $d(C) = 2 = d(H)$ and C or H is projective. Let A, B in $soc^{k+1}(M)$ of C, H respectively, then they are uniserial, $d(A) = d(B) = k + 1$. Now $A \cong C$ or $B \cong H$, therefore $k = 1$. Thus any uniform R-module is either uniserial or its critical uniserial submodule is simple. In the later case $soc^2(M) = A+B = soc^2(E)$, where A, B are uniserial submodules such that $d(A) = 2 = d(B)$ and A or B is projective; in case $\frac{soc^2(M)}{soc(M)}$ is not homogeneous, A, B are uniquely determined and both are quasi-injective. In case $\frac{soc^2(M)}{soc(M)}$ is homogeneous, $A \cong B$ and they are projective.

Proposition 4.2. *Let R be a ring satisfying (***), S_R a simple module, and $E = E(S)$ be such that it is not uniserial and $soc^2(E) = A+B$, where A, B are uniserial submodules with $S = A \cap B$ and $d(A) = d(B)$.*

(a) *Let H, K are two uniserial submodules of E such that $H \nsubseteq K$, $K \nsubseteq H$, $A \nsubseteq H \cap K$ and $S < H \cap K$. Then A is not projective, B is projective, $B \subseteq H \cap K$, uniserial submodules of E containing A is linearly ordered under inclusion and there exists a unique uniserial submodule of E of maximum composition length that contains A. Let $H' \leqslant H$ such that $H'J = H \cap K$, then there exists a homomorphism from H' onto A. If a uniserial submodule G of E is such that $d(G) = 2$, $S = C \cap G$, for some projective uniserial submodule C with $d(C) = 2$, then the family of those uniserial submodules of E that contain G is linearly ordered under inclusion.*

(b) *If E is not uniserial and every uniserial submodule of E of composition length 2 is projective, then E is a sum of two uniserial modules whose intersection is S; in particular this holds if $\frac{soc^2(E)}{S}$ is homogeneous. In addition, if $\frac{soc^2(E)}{S}$ is homogeneous, then E is a sum of two isomorphic uniserial submodules whose intersection is S.*

Proof. Now $soc^2(E) = A + B$ for uniserial submodules A, B such that $d(A) = 2 = d(B)$, $S = A \cap B$.

(a) Suppose $S < H \cap K$ and $A \nsubseteq H \cap K$. Then $soc^2(H + K) = soc^2(E)$. We consider any uniserial submodule $xR \leq soc^2(M)$ such that $xR \nsubseteq H \cap K$. There exist $H' \leqslant H$, $K' \leqslant K$ such that $H'J = H \cap K = K'J$. For some indecomposable idempotent $e \in R$, we can take $x = xe$. For some $u \in H'$, $v \in K'$, $ue = u \notin H \cap K$, $ve = v \notin H \cap K$, we have $x = u + v$. We get epimorphism $\sigma : H' \to xR$. If xR is projective, we get $d(H')$ $= 2$, which is a contradiction. Hence xR is not projective, so xR is quasi-injective. In particular, A is not projective. Then B is projective, therefore $B \subseteq H \cap K$. The second part is now obvious.

(b) Suppose every uniserial submodule of $soc^2(E)$ of composition length 2 is projective, this property holds in case $\frac{E}{S}$ is homogeneous. Now $soc^2(E) = A + B$ for some uniserial submodules A, B with $d(A) = d(B)$, $S = A \cap B$. By using (a), we get uniquely determined uniserial submodules H, K of E of maximum composition lengths such that $A \subseteq H$, $B \subseteq K$.

Case 1. $\frac{soc^2(E)}{S}$ is homogeneous. Then there exists a $\lambda \in End(E)$ such that $\lambda(A) = B$. Then $B \subseteq K \cap \lambda(H)$, therefore $\lambda(H) \subseteq K$, and $d(H) \leq d(K)$. We get $d(H) = d(K)$ and $H \cong K$. Set $M = H + K$. Suppose $E \neq M$. Then there exists a uniserial submodule L of E such that $L \nsubseteq M$. Then for $C = soc^2(E) \cap L$, $d(C) = 2$. For the *drpa* (D, D') of A, $[D : D']_r = 2$. There exists a $\sigma \in D$ which has extension $\mu \in End(E)$ such that $\mu(A)$ $= C$. By considering μ^{-1},we get $d(L) \leq d(H)$. Let $\omega = \lambda \mid S$ and $\omega' = \mu \mid S$. Then ω' $= \alpha + \omega\beta$ for some α, $\beta \in D'$. Let η_1, η_2 be extensions in $End(E)$ of α, β respectively. Then $\mu' = \eta_1 + \lambda\eta_2 \in End(E)$ is an extension of ω'. By using (1.5), we get $L \subseteq \mu(H) = \mu'(H) \subseteq H + K$, which is a contradiction. Hence $E = H + K$.

Case 2. $\frac{soc^2(E)}{S}$ is not homogeneous, Then any uniserial submodule L of E with $d(L) \geq 2$ contains A or B, therefore by (a) $L \subseteq H$ or $L \subseteq K$. Hence $E = H + K$. $\square$

Lemma 4.3. *Let R be a ring satisfying (***), and S_R a simple module such that $E = E(S)$ is not uniserial, but $\frac{E}{S}$ is homogeneous. If S is its own predecessor, then R is matrix ring over a local ring and $J^2 = 0$.*

Its proof is similar to that of (3.8). $\square$

If R is an artinian ring which is right serial ring, and A_R is a uniserial, projective module, then any uniserial module B_R containing A is projective.

Lemma 4.4. *Let a ring R satisfy (***). If uniserial module A_R is not quasi-injective, then it is projective.*

Proof. Set $B = soc^2(A)$ and $E = E(A)$. Suppose A is not projective, then B is not projective, therefor B is quasi-injective. Thus $\frac{soc^2(E)}{S}$ is not homogeneous and $soc^2(E) = B + C$ for some uniserial submodule C with $d(C) = 2$. Then C is projective. Now there exists a $\sigma \in End(E)$ for which $\sigma(A) \nsubseteq A$. If $B \subseteq \sigma(A)$, then by (4.2), C is not projective, which is a contradiction. Thus $C \subseteq \sigma(A)$, $\sigma(A)$ is projective. This gives $B \cong \sigma(B) = C$, therefore B is projective. Hence A is projective. $\square$

Theorem 4.5. *Let R be a ring satisfying (***) and E_R an indecomposable injective module that is not uniserial. Let $soc^2(E) = A + B$, where A, B are uniserial submodules of E with $d(A) = 2 = d(B)$, $soc(E) = A \cap B$. If P, Q are uniserial submodules of E of maximum composition lengths containing A, B respectively, then $E = P + Q$.*

Proof. Set $S = soc(E)$. If $\frac{soc^2(E)}{S}$ is homogeneous, the result follows from (4.2)(b). Suppose $\frac{soc^2(E)}{S}$ is not homogeneous. Then A, B are uniquely determined, and one of them say A is projective. Then Q is uniquely determined and it is quasi-injective. Let K be any uniserial submodule of E with $d(K) > 2$. If $B \subseteq K$, then $K \subseteq Q$. Suppose $B \nsubseteq K$. Then $A \subset K$. Every submodule of P containing A is projective. Therefore no two composition factors of $\frac{P}{S}$ are isomorphic. Also, by (4.4) $\frac{P}{S}$ is quasi-injective. Suppose $K \nsubseteq P$, then set $F = K \cap P$. Let K', P' be the submodules of K, P respectively, such that $K'J = F = Q'J$. We have epimorphism $\sigma : K' \to B$, which extends to a homomorphism $\eta : K \to Q$ with $ker\, \eta = FJ$. Thus $\frac{K}{FJ}$ embeds in Q. Similarly $\frac{P}{FJ}$ also embeds in Q. But $d(\frac{K}{FJ}) \leq d(\frac{P}{FJ})$. Hence $\frac{P}{FJ}$ is $\frac{K}{FJ}$-injective. We have a monomorphism $\lambda : \frac{K}{FJ} \to \frac{P}{FJ}$, which is identity on $\frac{F}{FJ}$. As K is projective, we get a monomorphism $\mu : K \to P$ lifting λ. If μ is identity on F, then $P + K = P \oplus W$ for some $W \leqslant P + K$, which is a contradiction. Thus μ is not identity on F. Let $\mu_1 = \mu \mid F$. As every submodule of K containing A is projective, no two composition factors of $\frac{K}{S}$ are isomorphic, therefore $(\mu_1 - I)F = S$, $\overline{\mu} : \frac{K}{S} \to \frac{P}{S}$ is identity on $\frac{F}{S}$, which gives $\frac{P}{S} + \frac{K}{S} = \frac{P}{S} \oplus \frac{V}{S}$ for some uniserial submodule V containing S. As $soc(\frac{V}{S}) \cong \frac{B}{S}$, we get $V \subseteq Q$. Hence $K \subseteq P + Q$, which proves that $E = P + Q$. $\square$

By using (4.2) and (4.5), one can prove the following.

Theorem 4.6. *Let R be a ring satisfying (***), S_R a simple module and $E = E(S)$ not a uniserial module. Let A, B be any two uniserial submodule of E such that $d(A) = d(B)$ $= 2$ and $soc(E) = A + B$.*

(i) *If P is a uniserial submodule of E maximal with respect to containing A, then it is of maximum composition length among the uniserial submodules containing A.*

(ii) *If P, Q are any two uniserial submodules of E which are maximal with respect to containing A, B respectively, then $E = P + Q$.*

The above theorem gives the following.

Theorem 4.7. *Let R be a ring satisfying (***) and M_R a uniform module which is not uniserial and $E = E(M)$. Then $\frac{M}{soc(M)}$ is a direct sum of two uniserial submodules, there exist uniserial submodules P, Q of E such that $soc(M) = P \cap Q$, $E = P + Q$ and $M = G + H$, where $G = P \cap M$, $H = Q \cap M$. If $k = min\{d(G), d(H)\}$, then $soc^i(M) = soc^i(E)$ for $1 \leq i \leq k$.*

Lemma 4.8. (i) *Let R be a right artinian ring, K_R a quasi-projective uniserial module such that $d(K) = 3$, K is not homogeneous, KJ is homogeneous and $\frac{K}{KJ^2}$, KJ are quasi-injective. Then any endomorphism σ of $soc(K)$ is uniquely extendable to an endomorphism of K.*

(ii) *Let R be a ring satisfying (***), S_R a simple module and $E = E(S)$ not a uniserial module. Let A, B be any two uniserial submodule of E such that $d(A) = d(B) = 2$ and $soc(E) = A + B$. Let H be a uniserial submodule of E such that $d(H) \geq 3$. Then any endomorphism σ of $soc(H)$ can be extended to an endomorphism of H.*

Proof. (i) Let $0 \neq \sigma \in D = End(soc(K))$. As KJ is quasi-injective, there exists a $\sigma' \in End(KJ)$ extending σ. As $\frac{K}{KJ^2}$ is quasi-injective, and K is quasi-projective, there exists $\eta \in End(K)$ such that $\overline{\eta} \in End(\frac{K}{KJ^2})$ induced by η is an extension of $\overline{\sigma'} \in End(\frac{KJ}{KJ^2})$ induced by σ'. Let $\lambda = \eta \mid KJ$. Then $\overline{\lambda} - \overline{\sigma'} = 0$ gives $(\lambda - \sigma')KJ \subseteq soc(K)$. Hence λ is an extension of σ. That λ is uniquely determined by σ follows from the hypothesis that K is not homogeneous, but KJ is homogeneous.

(ii) Let L, M be uniserial submodules of maximum composition lengths containing A, B respectively. Then $E = L + M$.

We take $A = soc^2(E) \cap H$ and by using (4.6), we also take $H \subseteq L$. Let $\sigma \neq 0$. If B is projective, then L is uniquely determined, therefore the result holds. Suppose B is not projective. Then A is quasi-injective as well projective. Therefore there exists $\eta \in End(E)$ such that it extends σ and $A = \eta(A)$. If $\eta(H) \subseteq H$, we finish. Suppose $\eta(H) \nsubseteq H$. Now H is also projective, for some $u \in H$ and some indecomposable idempotent $e \in R$, $H = uR$, $ue = u$. We write $\eta(u) = x + y$ for some $x \in L$, $y \in M$ such that $xe = x$, $ye = y$. As $x \notin S$, xR is projective. By using the fact that $\frac{E}{S} = \frac{L}{S} \oplus \frac{M}{S}$, we get an automorphism $\lambda \in End(E)$ such that $\lambda(\eta(u)) = x$. Now $A = usR$ for some $s \in R$. Let $\mu = \lambda\eta$. Then $\eta(us) = xs + ys \in A$. This gives $ys \in S$, $xs \in A$. We also have homomorphism $\rho : uR \to yR$, $\rho(u) = y$.

Case 1. $ys = 0$. Then $us = xs$, $\eta(us) = xs = \mu(us)$. It follows that μ is an extension of σ.

Case 2. $ys \neq 0$. Then A is homogeneous. Now $B \cong \frac{fR}{fJ^2}$ for some indecomposable idempotent $f \in R$. As B is not projective, $fJ^2 \neq 0$. Therefore $\frac{fJ}{fJ^3} \cong A$, $fJ^3 = 0$. Set $K = \rho^{-1}(B)$. Then $K \cong fR$, K is not homogeneous, KJ is quasi-injective. As $\frac{K}{KJ^2}$ is isomorphic to B, it is also quasi-injective. Further K is projective as $A \subset K$ and A is projective. Therefore by (i), there exists a $\tau \in End(K)$ extending σ. Let $\varphi \in End(E)$ be an extension of τ. Suppose $\varphi(H) \nsubseteq H$. Set $F = H \cap \varphi(H)$. Then $K \subseteq F$. By (4.2), we get $F < H' \leqslant H$, such that $\frac{H'}{F} \cong \frac{C}{S} \cong \frac{K}{A}$, thus $\frac{H}{A}$ have two isomorphic composition factors. However as every submodule of H containing A is projective, no two composition factors of $\frac{H}{A}$ can be isomorphic, which is a contradiction. Hence $\varphi(H) = H$, which proves the result. □

Lemma 4.9. *Let R be an artinian ring, $N_R = N_1 \oplus N_2 \oplus ..... \oplus N_t$ be such that for some simple module S, $soc(N_i) = S$ and let T be a simple submodule of N generated by an element $(x_1, x_2, ..., x_t)$ with $x_i \neq 0$ for every i. If for some $i \neq j$, there exists a homomorphism $\lambda : N_i \to N_j$ such that $\lambda(x_i) = x_j$, then T is contained in a summand of N.*

Proof. By re-indexing, we take $i = 1$, $j = 2$. Let $C_1 = \{(x, \lambda x) : x \in N_1\}$. Then $N = N_2 \oplus (C_1 \oplus N_3 \oplus N_4 \oplus ... \oplus N_t)$ and $T \subseteq C_1 \oplus N_3 \oplus N_4 \oplus ... \oplus N_t$, a summand of N. □

Lemma 4.10. *Let R be an artinian ring, M_R an indecomposable module of finite composition length and $M = K_1 + K_2 + K_3 + ... + K_n$ for some uniform modules $K_i \nsubseteq MJ$, such that $n > 1$, $N = K_2 + K_3 + ... + K_n = K_2 \oplus K_3 \oplus .... \oplus K_n$, $K_1 \cap N = soc(K_1)$. Then the following hold.*

(i) *$T = xR = soc(K_1)$ is not contained in a summand of N.*

(ii) *For any $1 \leq i \leq n$, $N_i = \sum_{j \neq i} K_j = \oplus \sum_{j \neq i} K_j$, $K_i \cap N_i = soc(K_i)$.*

Proof. (i) Suppose $N = A \oplus B$ for some non-zero submodules A, B and $T \subseteq A$. Then $M = (K_1 + A) \oplus B$, which is a contradiction.

(ii) Now $x = x_1 + x_2 + ... + x_n$, $x_i \in K_i$. By (i) $x_i \neq 0$, and $soc(K_i) = x_iR$ for any $1 \leq i \leq n$. Clearly $x_i \in K_i \cap N_i$. Suppose for some $i > 1$, $0 \neq z_i \in K_i \cap N_i$, then $z_i = \sum_{j \neq i} z_j$ for some $z_j \in K_j$. Then $z_1 = u_2 + ... + u_n$, where $u_i = z_i$, $u_j = -z_j$ for $j \neq i$, therefore $0 \neq z_1 \in K_1 \cap N_1 = soc(K_1)$. This gives that $z_i \in soc(K_i)$. Hence $K_i \cap N_i = soc(K_i)$. It also gives $N_i = \oplus \sum_{j \neq i} K_j$. □

Lemma 4.11. *Let R be a ring satisfying (***), S_R a simple module such that $E = E(S)$ is not uniserial but $\frac{soc^2(E)}{S}$ is homogeneous, and L a uniserial submodule of E with $d(L) \geq 2$. If $A = soc^2(E) \cap L$, and (D, D') is drpa of L, then (D, D') is also drpa of A.*

Proof. Let (D_1, D_1') be the *drpa* of A. By definition $D = End(S) = D_1$. Let $\sigma \in End(L)$. Then $\sigma \mid S = (\sigma \mid A) \mid S \in D_1'$, therefore $D' \subseteq D_1'$. Let $\eta \in End(A)$. As every submodule of L containing A is projective, no two composition factors of $\frac{L}{S}$ are isomorphic. Let $\overline{\eta} \in End(\frac{A}{S})$ be induced by η. As $\frac{L}{S}$ is quasi-injective and L is quasi-projective, there exists a $\lambda \in End(L)$ that induces $\overline{\eta}$. Suppose λ does not extend η, then $\lambda_1 = \lambda \mid A$ is such that $(\lambda_1 - \eta)(A) = S$, which gives that $\frac{A}{AJ} \cong S$ and S its only predecessor. By (4.3), $J^2 = 0$, therefore $d(L) = 2$, which is a contradiction. Thus λ is an extension of η, which gives $D_1' \subseteq D'$. Hence (D, D') is *drpa* of A. □

We now prove the main theorem. We prove that conditions (**) and (***) are equivalent.

Theorem 4.12. *Let R be an artinian ring. Then every finitely generated indecomposable right R-module is uniform if and only if it satisfies the following.*

(α) *R is right serial.*

(β) *Let e, f, g be any three indecomposable idempotents of R with eJ, fJ, gJ non-zero. Then the following hold.*

(i) *The drpa (D, D') of $A = \frac{eR}{eJ^2}$ is such that $[D : D']_r \leq 2$, $[D : D']_l \leq 2$.*

(ii) *If e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2}$, then $eJ^2 = 0$ or $fJ^2 = 0$.*

(iii) *If e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2} \cong \frac{gJ}{eJ^2}$, then g is isomorphic to e or f.*

(iv) *If $A = \frac{eR}{eJ^2}$ is not quasi-injective, then $eJ^2 = 0$, $\frac{eJ}{eJ^2} \ncong \frac{fJ}{fJ^2}$ whenever e is not isomorphic to f.*

Proof. If every finitely generated indecomposable right R-module is uniform, as seen before, R satisfies the given conditions, i.e R satisfies (***).

Conversely, let R satisfy (***). Suppose the contrary. We get an indecomposable module M_R of smallest composition length, which is not uniform. Then $soc(M) \subseteq MJ$. Firstly, we prove that $M = G + N$ for some uniserial submodule $G \nsubseteq MJ$, $N < M$ such that $soc(G) = G \cap N$. Let S be a simple submodule of M. As $\frac{M}{S}$ is a direct sum of uniform modules, we get two submodules K, N of M such that $M = K + N$, $S = K \cap N$ and $\frac{K}{S}$ is a non-zero uniform module. If $\frac{K}{S}$ is uniserial, then K is uniserial and we finish. Suppose $\frac{K}{S}$ is not uniserial.

Case 1. $M = K$. As M is not uniform and $\frac{M}{S}$ is uniform, $soc(M) = S \oplus S'$ for some simple submodule S'. As $\frac{M}{S}$ is not uniserial, its critical uniserial submodule is $soc(\frac{M}{S})$. By (4.7), $\frac{M}{S+S'}$ is a direct sum of two uniserial modules. Therefore there exist non simple uniserial submodules A, B of M such that $M = A+B$, $(A+S)\cap(B+S) = S+S'$. Then one of A, B say A does not contain S. But $S + S' = S + A \cap (B + S)$. Thus $A \cap (B + S)$ $= soc(A)$ and $M = A + (B + S)$.

Case 2. $M \neq K$. Then K is a direct sum of uniform modules. So there exists a uniform summand L of K. Now $K = L \oplus W$ for some $W \leqslant K$. Then $M = L + (W + N)$ with $L \cap (W + N) = soc(L)$. If L is uniserial, we finish. Otherwise, by (4.7) $L = A + B$ for some non-simple uniserial submodules A, B such that $soc(L) = A \cap B$. Now neither of A, B is contained in MJ. Then $M = A + (B + W + N)$, $A \cap (B + W + N) = soc(A)$.

We get a uniserial submodule A of M of minimum composition length such that $A \nsubseteq MJ$, and for some $N < M$, $M = A + N$, $soc(A) = A \cap N$. Set $S = soc(A) = xR$. Now $N = K_1 \oplus K_2 \oplus .... \oplus K_t$ for some uniform submodules K_i. Suppose $t \geq 2$.

Suppose some K_i say K_1 is not uniserial. Then $K_1 = A_1 + B_1$ for some uniserial submodules A_1, B_1 such that $d(A_1) \geq 2$, $d(B_1) \geq 2$, $A_1 \cap B_1 = S_1 = soc(K_1)$. Now M $= A_1 + N_1$, where $N_1 = A + B_1 + K_2 + ... + K_t$. As $A_1 \cap N_1 = soc(A_1)$, the choice of A implies $d(A) \leq d(A_1)$. Similarly, $d(A) \leq d(B_1)$. Let $k = min\{d(A_1), d(B_1)\}$. Then A embeds in $soc^k(E) = soc^k(K_1)$, where $E = E(K_1)$. Hence K_1 is A-injective. Now M $= K_2 + H_2$, where $H_2 = A \oplus K_1 \oplus K_3 \oplus .... \oplus K_t$. and $soc(K_2) = K_2 \cap H_2$. Let $yR =$ $soc(K_2)$. Then $y = a + y_1 + y_3 + ... + y_t$ for some $a \in A$, $y_i \in K_i$, $i \neq 2$. We get a monomorphism $\sigma : A \to K_1$ for which $\sigma(a) = y_1$. Then by (4.9), yR is contained in a summand of H_2, which contradicts (4.10). Hence every K_i is uniserial and $d(A) \leq d(K_i)$. Set $A = K_0$. Arrange K_i's in a such way that $d(K_i) \leq d(K_{i+1})$ for $i > 0$. Fix an $x_0 \neq 0$

in $soc(K_0)$. Then $x_0 = x_1 + x_2 + ... + x_t$ for some uniquely determined non-zero $x_i \in K_i$. Now $M \cong \frac{K_0 \times K_1 \times .... \times K_t}{L}$, where $L = (x_0, -x_1, -x_2, ..., -x_t)R$ is a simple submodule not contained in any summand of $K_0 \times K_1 \times .... \times K_t$. Let $E = E(K_0)$, $S = soc(E)$. Then every K_i embeds in E. If E is uniserial, then K_1 is K_0-injective, and we get an embedding $\sigma : K_0 \rightarrow K_1$ such that $\sigma(x_0) = -x_1$, which gives that L is contained in a summand of $K_0 \times K_1 \times .... \times K_t$, therefore M is decomposable, which is a contradiction. Hence E is not uniserial.

Case 1. $\frac{soc^2(E)}{S}$ is homogeneous. Then given any two uniserial submodules V, W of E with $d(V) \leq d(W)$, there exists an automorphism of E that maps V into W. Thus if $K = K_t$, we take every $K_i \subseteq K$ Let (D, D') be the *drpa* of $B = A \cap soc^2(E)$, therefore $[D : D']_l = 2$. It can be seen that (D, D') is also *drpa* of K. Now $M \cong \frac{K_0 \times K_1 \times .... \times K_t}{L} \subseteq \frac{K^{(t+1)}}{L}$, where for some non-zero ω_i, $1 \leq i \leq t$ in D, $-x_i = \omega_i x_0$. But I, ω_1, ω_2, ..., ω_t are left linearly dependent over D'. Therefore for some $1 \leq i \leq t$, $\omega_i = \mu_0 I + \mu_1\omega_1 + ... + \mu_{i-1}\omega_{i-1}$, where $\omega_0 = I$ and each μ_j is the restriction to S of some $\rho_j \in End(K)$. Let $\rho_{tj} : K_j \rightarrow K_t$ $(= K) = \rho_j \mid K_j$. Then for $\mu_{tj} = (\rho_{tj} \mid S) = \mu_j$, $\omega_i = \mu_{i0} I + \mu_{i1}\omega_1 + ... + \mu_{ii-1}\omega_{i-1}$. By (2.2), T is contained in a summand of $K_0 \times K_1 \times .... \times K_t$, which is a contradiction.

Case 2. $\frac{soc^2(E)}{S}$ is not homogeneous. Then $E = F + H$, for some uniserial submodules F, H such that $d(F) \geq 2$, $d(H) \geq 2$ and $S = F \cap H$. Let G, H be the intersection of F, H respectively with $soc^2(E)$. Then both G, H are quasi-injective, one of them say G is projective, and any uniserial submodule L of E of composition length at least 2 contains G or H. Once again, we suppose that all $K_i \subseteq E$. Suppose the number of K_i that contain H is more than one, say $H \subseteq K_1 \cap K_2$. Consider $W = \frac{K_1 \times K_2}{T'}$, where $T' = (x_1, x_2)R$. We know that there is no homomorphism $\sigma : K_1 \rightarrow K_2$ for which $\sigma(x_1) = x_2$, therefore W is indecomposable. However as H is quasi-injective, there exists a homomorphism $\eta : H \rightarrow H$ for which $\eta(x_1) = x_2$. By (1.1), W is not uniform, but $d(W) < d(M)$, which gives a contradiction to the choice of M. Thus there is only one K_i containing H. Similarly there is only one K_i containing G. Thus $t = 1$.

In any case $t = 1$, $M \cong \frac{K_0 \times K_1}{L}$, where $L = (x_0, -x_1)R$, and K_0 is uniserial. As argued earlier, K_1 is also uniserial. We regard K_0, $K_1 \subseteq E$, then for some $\omega \in End(S))$, $\omega x_0 = x_1$. Let $A = K_0 \cap soc^2(E)$, $B = K_1 \cap soc^2(E)$. As M is not uniform, by (1.1) ω extends to an isomorphism $\sigma : A \rightarrow B$.

Case 1. $\frac{soc^2(E)}{S}$ is homogeneous. Then for any extension $\lambda \in End(E)$ of σ, $\lambda(K_0) \subseteq K_1$, which proves that M is decomposable, which is a contradiction.

Case 2. $\frac{soc^2(E)}{S}$ is not homogeneous. Then $\sigma(A) = B$ gives $A = B$. Suppose A is not projective. Then $K_0 \subseteq K_1$. As there is unique maximal uniserial submodule P of E containing A, for any extension $\lambda \in End(E)$ of $\sigma, \lambda(K_0) \subseteq K_1$. Thus M is decomposable, which is a contradiction. This shows that A is projective, $soc^2(E) = A + C$ for some uniquely determined uniserial submodule C with $d(C) = 2$, $A \cap C = S$. Then there exists unique maximal uniserial submodule Q of E containing C. Let P be a maximal uniserial submodule of E containing K_1. By (4.9)(ii), there exists an $\eta \in End(E)$ which extend σ and $\eta(K_0) = K_0$. Now $K_0 = xR$ for some $x \in K_0$, such that for some indecomposable idempotent $e \in R$, $xe = x$. Then $\eta(x) = a + b$, for some $a \in P$, $b \in Q$ with $ae = a$, $be = b$. As $\frac{E}{S} = \frac{P}{S} \oplus \frac{Q}{S}$, and K_0, aR are projective, we get isomorphism $\rho : K_0 \rightarrow aR$ for which $\rho(x) = a$. As $d(K_0) \leq d(K_1)$, it follows that $a \in K_1$. Now $A = xsR$. Then $\eta(xs) \in A$, as $\eta(xs) = as + bs$, $as \in A$, $bs \in S$. We also have homomorphism $\lambda : K_0 \to Q$, $\lambda(x) = b$. It follows that if $xsr \in S$, then $bsr = 0$, therefore $\rho(xsr) = asr = \sigma(xrs)$. Hence $\rho : K_0 \to K_1$ extends σ, which is a contradiction. This proves the result. $\square$

It follows from the above theorem that any balanced ring, as discussed in [2], and which is right serial satisfies (**)

Definition 4.13. [6]. Let M be a local module, $D = End(\frac{M}{J(M)})$ and D' the division subring of D consisting of those $\sigma \in D$ which can be lifted to some endomorphisms of M. Then the pair (D, D') is called the dual division ring pair associate (in short *ddpa*) of M.

By suitable dualization of the arguments involved in proving the above theorem, we can prove the following dual of the above theorem.

Theorem 4.14. *Let R be an artinian ring. Then every finitely generated indecomposable right R-module is local if and only if the following hold.*
(α) *Any uniform right R-module is uniserial.*
(β) *For any three uniserial right R-modules A, B, C with $d(A) = d(B) = d(C) = 2$, the following hold.*
(i) *The ddpa (D, D') is such that $[D, D']_r \leq 2$, $[D, D]_l \leq 2$;*
(ii) *if A, B are not isomorphic and $\frac{A}{AJ} \cong \frac{B}{BJ}$, then A is injective or B is injective.*
(iii) *if A, B are not isomorphic and $\frac{A}{AJ} \cong \frac{B}{BJ} \cong \frac{C}{CJ}$, then $C \cong A$ or $C \cong B$;*
(iv) *if A is not quasi-projective, then A is injective and $\frac{A}{AJ} \ncong \frac{B}{BJ}$, whenever $A \ncong B$.*

Examples of rings satisfying (**) or (*) can be easily constructed.

Example 4.15. Let D be a division ring having a subdivision ring D' such that $[D, D']_r = [D : D']_l = 2$. Let $R = \begin{bmatrix} D' & D \\ 0 & D \end{bmatrix}$. Then R is right serial but not left serial, and its radical J satisfies $J^2 = 0$. Only uniserial right R-module with composition length 2 is $A = e_{11}R$. Its *drpa* is (D, D'). It follows from (4.12) that R satisfies (**). To within isomorphism, R admits only one uniserial module $A = \frac{Re_{22}}{D'e_{12}}$, it is injective and its *ddpa* is (D, D'). By (4.14), every finitely generated indecomposable left module is local. Now consider the ring $R' = \begin{bmatrix} D' & D' & D \\ 0 & D' & D \\ 0 & 0 & D \end{bmatrix}$. Then R' is right serial and $J^2 \neq 0$. There are only two uniserial right R-modules of composition length 2, viz $A = \frac{e_{11}R}{e_{11}J^2}$, $B = e_{22}R$. Here A is injective. As seen for R the *drpa* of B is (D, D'). By (4.12), R' satisfies (**). R' is also such that every finitely generated indecomposable left module is local.

Example 4.16. Let D be a division ring, and $R = \begin{bmatrix} D & 0 & D \\ 0 & D & D \\ 0 & 0 & D \end{bmatrix}$. Then R is right serial, but not left serial. Here $J^2 = 0$. It admits only two uniserial right modules of composition length 2, viz $A = e_{11}R$, $B = e_{22}R$. Both A, B are quasi-injective, and $soc(A) \cong soc(B) \cong e_{33}R$. It follows from (4.12) that R satisfies (**). R admits two uniserial left modules of composition length 2, viz modules $M = \frac{Re_{33}}{De_{23}}$, $N = \frac{Re_{33}}{De_{13}}$, both of them are quasi-projective and injective. Once again, by (4.14). every finitely generated indecomposable left R-module is local.

Example 4.17. Let D be a division ring admitting a division subring D' such that $[D, D']_r = 2$, $[D, D']_l > 2$. Such division rings exist [4]. Then $R = \begin{bmatrix} D' & D \\ 0 & D \end{bmatrix}$ is right serial, but it does not satisfy (**).

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