ON A GENERALIZATION OF STABLE TORSION THEORY

YASUHIKO TAKEHANA

ABSTRACT. Throughout this paper R is a ring with a unit element, every right R-module is unital and Mod-R is the category of right R-modules. A subfunctor of the identity functor of Mod-R is called a preradical. A torsion theory $(\mathcal{T}, \mathcal{F})$ is called stable if $\mathcal{T}$ is closed under taking injective hulls. We denote $E(M)$ the injective hull of a module M. For a preradical σ, we denote $E_\sigma(M)$ the σ-injective hull of a module M, where $E_\sigma(M)$ is defined by $E_\sigma(M)/M := \sigma(E(M)/M)$. For a preradical σ we call a torsion theory $(\mathcal{T}, \mathcal{F})$ is σ-stable if $\mathcal{T}$ is closed under taking σ-injective hulls. In this note, we characterize σ-stable torsion theories and give some related facts.

0. FUNDAMENTAL FACTS OF TORSION THEORY

For a preradical t it hold that $t(N) \subseteq t(M)$ and $t(M/N) \supseteq (t(M) + N)/N$ for any $M \in$ Mod-R and its submodule N. A preradical t is called idempotent (radical) if $t(t(M)) = t(M)$ $(t(M/t(M)) = 0)$ for any module M, respectively. For a preradical σ, $\mathcal{T}_\sigma := \{M \in \text{Mod-}R \mid \sigma(M) = M\}$ is the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma := \{M \in \text{Mod-}R \mid \sigma(M) = 0\}$ is the class of σ-torsionfree right R-modules. For a subclass $\mathcal{C}$ of Mod-R, it is said that $\mathcal{C}$ is closed under taking extensions if: if $N, M/N \in \mathcal{C}$ then $M \in \mathcal{C}$ for any $M \in$ Mod-R and its submodule N. A preradical t is called left exact if $t(N) = N \cap t(M)$ for any submodule N of a module M. It is also well known that a preradical t is idempotent and $\mathcal{T}_t$ is closed under taking submodules if and only if t is left exact. A right R-module M is called σ-injective if the functor $\text{Hom}_R(-, M)$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{T}_\sigma$. For a preradical σ a submodule N of a module M is called σ-dense in M if M/N is σ-torsion, and N is called σ-essential in M if N is σ-dense and essential in M. It holds that a module M is σ-injective if and only if M has no proper σ-essential extension.

Let σ be an idempotent radical. If X is minimal in $\{X \mid X \text{ is } \sigma\text{-injective and } X \supseteq M\}$, X is called to be a minimal σ-injective extension of M. If Y is maximal in $\{Y \mid Y \supseteq M \text{ and } M \text{ is } \sigma\text{-essential in } Y\}$, Y is called to be a maximal σ-essential extension of M. If $X \supseteq M$ and X is σ-injective and M is σ-essential in X, X is called to be a σ-injective σ-essential extension of M. For any module M a σ-injective σ-essential extension of M exists and is unique to within isomorphism. The σ-injective σ-essential extension of M coincides with the minimal σ-injective extension of M and the maximal σ-essential extension of M and is called to be the σ-injective hull of M. We put $\sigma(E(M)/M) = E_\sigma(M)/M$. For an idempotent radical σ, the σ-injective hull of M is isomorphic to $E_\sigma(M)$. But even if a preradical σ is not an idempotent radical, we call $E_\sigma(M)$ the σ-injective hull of a module M.

The detailed version of this paper will be submitted for publication elsewhere.

Let $\mathcal{C}$ be a subclass of Mod-R. A torsion theory for $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathcal{C}$ such that

(i) $\mathrm{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$

(ii) If $\mathrm{Hom}_R(M, F) = 0$ for all $F \in \mathcal{F}$, then $M \in \mathcal{T}$

(iii) If $\mathrm{Hom}_R(T, N) = 0$ for all $T \in \mathcal{T}$, then $N \in \mathcal{F}$.

We put $t(M) = \sum_{\mathcal{T} \ni N \subset M} N \; (= \bigcap_{M/N \in \mathcal{F}} N)$, then $\mathcal{T} = \mathcal{T}_t$ and $\mathcal{F} = \mathcal{F}_t$ hold and t is an idempotent radical. Conversely if t is an idempotent radical, then $(\mathcal{T}_t, \mathcal{F}_t)$ is a torsion theory.

1. A stable torsion theory relative to torsion theories

P. Gabriel studied a hereditary stable torsion theory in [3] (Or see p. 152 in [12]). We generalize hereditary stable torsion theory. First we generalize left exact preradicals. For preradicals σ and t, we call t a σ-left exact preradical if $t(N) = N \cap t(M)$ holds for any σ-dense submodule N of a module M.

Lemma 1. *If σ is a radical, then $E_\sigma(M)$ is σ-injective for any module M.*

Lemma 2. *For a preradical σ, the following hold.*

(1) *If σ is idempotent, then $\mathcal{F}_\sigma$ is closed under taking extensions. Conversely if σ is a radical and $\mathcal{F}_\sigma$ is closed under taking extensions, then σ is idempotent.*

(2) *If σ is a radical, then $\mathcal{T}_\sigma$ is closed under taking extensions. Conversely if σ is idempotent and $\mathcal{T}_\sigma$ is closed under taking extensions, then σ is a radical.*

In [14] we generalized hereditary torsion theories. For the sake of reader's convenience, we state the following propositions.

Proposition 3. *For a left exact preradical σ and an idempotent preradical t, t is σ-left exact if and only if $\mathcal{T}_t$ is closed under taking σ-dense submodules.*

Proof. $(\rightarrow)$: Let N be a σ-dense submodule of a module $M \in \mathcal{T}_t$. Then $t(N) = N \cap t(M) = N \cap M = N$, as desired.

$(\leftarrow)$: Let N be a σ-dense submodule of a module M. Since $t(M)/(N \cap t(M)) \simeq (N + t(M))/N \subseteq M/N \in \mathcal{T}_\sigma$ and $t(M) \in \mathcal{T}_t$, $N \cap t(M) \in \mathcal{T}_t$. Then it holds that $N \cap t(M) = t(N \cap t(M)) \subseteq t(N)$. Since it is clear that $N \cap t(M) \supseteq t(N)$, $N \cap t(M) = t(N)$ holds. □

Proposition 4. *For an idempotent radical σ and a radical t, t is σ-left exact if and only if $\mathcal{F}_t$ is closed under taking σ-injective hulls.*

Proof. $(\rightarrow)$: Let M be in $\mathcal{F}_t$. Then $0 = t(M) = M \cap t(E_\sigma(M))$, and so $t(E_\sigma(M)) = 0$, as desired.

$(\leftarrow)$: Let N be a σ-dense submodule of a module $M \in \mathcal{T}_t$. Consider the following diagram.

$$\begin{array}{ccccccc} 0 \to & N & \xrightarrow{g} & M & \to M/N \to 0 \\ & \downarrow_j & & \downarrow_f & \\ 0 \to & N/t(N) & \xrightarrow{i} & E_\sigma(N/t(N)), & \end{array}$$

where g and i are the inclusion maps, j is the canonical epimorphism and f is a homomorphism determined by the σ-injectivity of $E_\sigma(N/t(N))$. Since t is a radical, $E_\sigma(N/t(N)) \in \mathcal{F}_t$ by the assumption. Since $f(t(M)) \subseteq t(E_\sigma(N/t(N))) = 0$, it holds that $t(M) \subseteq \ker f$. Let $f|_N$ be a restriction map of f to N. Then it follows that $t(N) = \ker j = \ker f|_N = N \cap \ker f \supseteq N \cap t(M) \supseteq t(N)$, and so $t(N) = N \cap t(M)$, as desired. $\square$

Lemma 5. *Let σ be an idempotent radical. If M is a σ-essential extension of a module N, then $E_\sigma(M) = E_\sigma(N)$ holds. Conversely if σ is a left exact radical, $N \subseteq M$ and $E_\sigma(M) = E_\sigma(N)$, then M is a σ-essential extension of N.*

Lemma 6. *Let σ be a left exact radical and L a submodule of a module M. Then the following are equivalent.*

(1) *$L = E_\sigma(L) \cap M$.*
(2) *L is σ-essentially closed in M, that is, if L is σ-essential in X such that $L \subseteq X \subseteq M$, then $L = X$.*

Lemma 7. *Let σ be an idempotent radical and M a module. Then M is σ-injective if and only if $E_\sigma(M) = M$.*

A preradical t is called stable if $\mathcal{T}_t$ is closed under taking injective hulls. Next we generalize stable torsion theory. We call a preradical t σ-stable if $\mathcal{T}_t$ is closed under taking σ-injective hulls for a preradical σ. We put $\mathcal{X}_t(M) := \{X : M/X \in \mathcal{T}_t\}$ and $N \cap \mathcal{X}_t(M) := \{N \cap X : X \in \mathcal{X}_t(M)\}$. The following theorem generalize Proposition 7.1 in [12] and (i) and (ii) of Theorem 2.8 in [2].

Theorem 8. *Let t be an idempotent preradical and σ an idempotent radical. Then the following conditions (1), (2) and (3) are equivalent.*

Assume that t is an idempotent radical and $\mathcal{T}_t$ is closed under taking σ-dense submodules and σ is a left exact radical, then all conditions (1)˜(10) except (6) are equivalent. Moreover if t is left exact, then all conditions are equivalent.

(1) *t is σ-stable, that is, $\mathcal{T}_t$ is closed under taking σ-injective hulls.*
(2) *The class of σ-injective modules are closed under taking $t(-)$, that is, $t(E)$ is σ-injective for any σ-injective module E.*
(3) *$E_\sigma(t(M)) \subseteq t(E_\sigma(M))$ holds for any module M.*
(4) *$\mathcal{T}_t$ is closed under taking σ-essential extensions.*
(5) *If M/N is σ-torsion, then $N \cap \mathcal{X}_t(M) = \mathcal{X}_t(N)$ holds.*
(6) *Every module $M \notin \mathcal{T}_t$ with $M/t(M) \in \mathcal{T}_\sigma$ contains a nonzero submodule $N \in \mathcal{F}_t$.*
(7) *For any module M, $t(M) = E_\sigma(t(M)) \cap M$ holds.*
(8) *For any module M, $t(M)$ is σ-essentially closed in M.*
(9) *For any σ-injective module E with $E/t(E) \in \mathcal{T}_\sigma$, $t(E)$ is a direct summand of E.*
(10) *$E_\sigma(t(M)) = t(E_\sigma(M))$ holds for any module M.*

Proof. (1)→(3): Let t be an idempotent preradical and $M \in$ Mod-R. Then $t(M) \in \mathcal{T}_t$, and by assumption $E_\sigma(t(M)) \in \mathcal{T}_t$. Since $E_\sigma(t(M)) \subseteq E_\sigma(M)$, it follows that $E_\sigma(t(M)) = t(E_\sigma(t(M))) \subseteq t(E_\sigma(M))$, as desired.

(3)→(2): Let σ be an idempotent radical and X be a σ-injective module, and then we have $E_\sigma(X) = X$ by Lemma 1. Then it follows that $E_\sigma(t(X)) \subseteq t(E_\sigma(X)) = t(X)$ by the assumption. Since $E_\sigma(t(X)) \supseteq t(X)$ holds clearly, it follows that $E_\sigma(t(X)) = t(X)$, and so $t(X)$ is σ-injective by Lemma 1, as desired.

(2)$\to$(1): Let σ be a radical and $M \in \mathcal{T}_t$. By the assumption, $t(E_\sigma(M))$ is σ-injective. Since $t(E_\sigma(M)) \supseteq t(M) = M$, $E_\sigma(M)/t(E_\sigma(M))$ is an epimorphic image of $E_\sigma(M)/M$, and so $E_\sigma(M)/t(E_\sigma(M)) \in \mathcal{T}_\sigma$. Thus the exact sequence $(0 \to t(E_\sigma(M)) \to E_\sigma(M) \to E_\sigma(M)/t(E_\sigma(M)) \to 0)$ splits. Then there exists a submodule K of $E_\sigma(M)$ such that $E_\sigma(M) = t(E_\sigma(M)) \oplus K$, and so $0 = K \cap t(E_\sigma(M)) \supseteq K \cap M$. Since M is essential in $E_\sigma(M)$, it follows that $K = 0$, and so $E_\sigma(M) = t(E_\sigma(M))$, as desired.

(1)$\to$(4): Assume that σ is an idempotent radical and $\mathcal{T}_t$ is closed under taking σ-dense submodules. Let $M \in \mathcal{T}_t$ be σ-essential in a module X. By the assumption it follows that $E_\sigma(M) \in \mathcal{T}_t$. By Lemma 5 $E_\sigma(M) = E_\sigma(X)$. Thus $E_\sigma(X) \in \mathcal{T}_t$. Since X is a σ-dense submodule of $E_\sigma(X)$, it follows that $X \in \mathcal{T}_t$, as desired.

(4)$\to$(1): It is clear.

(3)$\to$(7): Let t be a σ-left exact preradical. By the assumption it follows that $t(M) \subseteq M \cap E_\sigma(t(M)) \subseteq M \cap t(E_\sigma(M)) = t(M)$. Thus $t(M) = M \cap E_\sigma(t(M))$.

(7)$\to$(9): Let σ be an idempotent radical, E be σ-injective and $E/t(E) \in \mathcal{T}_\sigma$. Then it follows that $t(E) = E_\sigma(t(E)) \cap E$ and $E_\sigma(t(E)) \subseteq E_\sigma(E) = E$, and so $t(E) = E_\sigma(t(E))$. Hence $t(E)$ is σ-injective. Thus the sequence $0 \to t(E) \to E \to E/t(E) \to 0$ splits, as desired.

(9)$\to$(1): Let σ be an idempotent radical and t be an idempotent preradical and $M \in \mathcal{T}_t$, then it follows that $M = t(M) \subseteq t(E_\sigma(M))$. Thus $E_\sigma(M)/t(E_\sigma(M))$ is a factor module of $E_\sigma(M)/M \in \mathcal{T}_\sigma$. By the assumption there exists a submodule K of $E_\sigma(M)$ such that $E_\sigma(M) = K \oplus t(E_\sigma(M))$. Thus it follows that $0 = K \cap t(E_\sigma(M)) \supseteq K \cap M$, and so $K = 0$. Hence $E_\sigma(M) = t(E_\sigma(M)) \in \mathcal{T}_t$.

(10)$\to$(2): It is clear.

(3)$\to$(10): Here we assume that σ is a left exact radical and t is a σ-left exact preradical.

First we claim that $t(M)$ is σ-essential in $t(E_\sigma(M))$. Suppose that $L \cap t(M) = 0$ for a submodule L of $t(E_\sigma(M))$. Then it follows that $0 = L \cap t(M) = L \cap M \cap t(E_\sigma(M)) = L \cap M$. Since M is essential in $E_\sigma(M)$, $L = 0$, and so $t(M)$ is essential in $t(E_\sigma(M))$. It is clear that $t(M)$ is a σ-dense submodule of $t(E_\sigma(M))$ since $t(E_\sigma(M))/t(M) = t(E_\sigma(M))/(M \cap t(E_\sigma(M))) \simeq (M + t(E_\sigma(M)))/M \subseteq E_\sigma(M)/M \in \mathcal{T}_\sigma$.

Thus $t(M)$ is σ-essential in $t(E_\sigma(M))$, and so by Lemma 5 $E_\sigma(t(M)) = E_\sigma(t(E_\sigma(M))) \supseteq t(E_\sigma(M))$. By the assumption $E_\sigma(t(M)) \subseteq t(E_\sigma(M))$, and so $E_\sigma(t(M)) = t(E_\sigma(M))$, as desired.

(4)$\to$(5): Assume that $\mathcal{T}_t$ is closed under taking σ-dense submodules. Let N be a σ-dense submodule of a module M.

First we claim that $N \cap \mathcal{X}_t(M) \supseteq \mathcal{X}_t(N)$. Let $N_0 \in \mathcal{X}_t(N)$. Then $N/N_0 \in \mathcal{T}_t$. We put $\Gamma = \{M_i/N_0 \subseteq M/N_0 : (M_i/N_0) \cap (N/N_0) = 0\}$. Then by Zorn's argument, Γ has a maximal element M_0/N_0 which is a complement of N/N_0 in M/N_0, and then $M_0 \cap N = N_0$. Hence $(M_0/N_0) \oplus (N/N_0)$ is essential in M/N_0, and so $[(M_0/N_0) \oplus (N/N_0)]/[M_0/N_0]$ is essential in $[M/N_0]/[M_0/N_0]$. Therefore $(M_0+N)/M_0$ is essential in M/M_0. Since $M/N \in \mathcal{T}_\sigma$, it follows that $M/(M_0 + N_0) \in \mathcal{T}_\sigma$. Thus $\mathcal{T}_t \ni N/N_0 = N/(M_0 \cap N) \simeq (N + M_0)/M_0$. So $(N + M_0)/M_0$ is σ-essential in M/M_0. By the assumption it follows that $M/M_0 \in \mathcal{T}_t$. Since $M_0 \cap N = N_0$, it conclude that $N \cap \mathcal{X}_t(M) \supseteq \mathcal{X}_t(N)$.

Next we will show that $N \cap \mathcal{X}_t(M) \subseteq \mathcal{X}_t(N)$. Let $M_1 \in \mathcal{X}_t(M)$, and then $M/M_1 \in \mathcal{T}_t$. Simce $N/(N \cap M_1) \simeq (N + M_1)/M_1 \subseteq M/M_1 \in \mathcal{T}_t$ and $\mathcal{T}_\sigma \ni M/N \to M/(N + M_1) \to 0$, it follows that $N/(N \cap M_1) \in \mathcal{T}_t$ by the assumption, and so $N \cap M_1 \in \mathcal{X}_t(N)$.

(5)→(1): Let σ be an idempotent preradical and M be in $\mathcal{T}_t$. Since $E_\sigma(M)/M \in \mathcal{T}_\sigma$, $\mathcal{X}_t(E_\sigma(M)) \cap M = \mathcal{X}_t(M) \ni 0$ for $M \in \mathcal{T}_t$. Thus there exists a submodule X of $E_\sigma(M)$ such that $E_\sigma(M)/X \in \mathcal{T}_t$ and $X \cap M = 0$. Since M is essential in $E_\sigma(M)$, it follows that $X = 0$, and so $E_\sigma(M) \in \mathcal{T}_t$.

(1)→(6): Let $M \notin \mathcal{T}_t$ with $M/t(M) \in \mathcal{T}_\sigma$. Suppose that any nonzero submodule N of M is not t-torsionfree. Since $0 \neq t(N) \subseteq N \cap t(M)$, $N \cap t(M) \neq 0$ holds for any nonzero submodule N of M, and so $t(M)$ is essential in M. By the assumption it follows that $t(M)$ is σ-essential in M. By Lemma 5, $E_\sigma(t(M)) = E_\sigma(M)$ holds. Since t is an idempotent preradical, it follows that $t(M) \in \mathcal{T}_t$ and so $E_\sigma(t(M)) \in \mathcal{T}_t$ by the assumption. Thus $E_\sigma(M) \in \mathcal{T}_t$. Then $t(M) = M \cap t(E_\sigma(M)) = M \cap E_\sigma(M) = M$, and so $M \in \mathcal{T}_t$. This is a contradiction, and so $M \notin \mathcal{T}_t$ with $M/t(M) \in \mathcal{T}_\sigma$ contains a nonzero submodule $N \in \mathcal{F}_t$.

(6)→(1): Let $M \in \mathcal{T}_t$, then $t(E_\sigma(M)) \supseteq t(M) = M$. Suppose that $E_\sigma(M) \notin \mathcal{T}_t$. Since $E_\sigma(M)/M \to E_\sigma(M)/t(E_\sigma(M)) \to 0$, it follows that $0 \neq E_\sigma(M)/t(E_\sigma(M)) \in \mathcal{T}_\sigma$. By the assumption there exists a nonzero submodule $N \in \mathcal{F}_t$ of $E_\sigma(M)$. Since M is essential in $E(M)$, it follows that $M \cap N \neq 0$, and so $\mathcal{F}_t \ni N \supseteq N \cap M \subseteq M \in \mathcal{T}_t$. As t is left exact, $N \cap M \in \mathcal{F}_t \cap \mathcal{T}_t = \{0\}$. This is a contradiction. Thus it follows that $E_\sigma(M) \in \mathcal{T}_t$, as desired. □

2. Some applications of σ-stable torsion theory

If R is right noetherian, t is stable if and only if every indecomposable injective module is t-torsion or t-torsionfree by Proposition 11.3 in [6]. We will generalize this. First we need the following torsion theoretic generalization of Matlis Papp's theorem in Theorem 1 in [10].

For a left exact radical σ, we denote $\mathcal{L}_\sigma := \{I \subseteq R; R/I \in \mathcal{T}_\sigma\}$

[10, Theorem 1] *Let σ be a left exact radical. Then $\mathcal{L}_\sigma$ satisfies ascending chain conditions if and only if every σ-injective σ-torsion R-module is a direct sum of σ-injective σ-torsion indecomposable submodules.*

The following theorem generalizes [6, Proposition 11.3].

Theorem 9. *Assume that t is an idempotent radical, σ is a left exact radical and $\mathcal{T}_t$ is closed under taking σ-dense submodules. Then the following hold.*

1. *If t is σ-stable, then $(*)$ every indecomposable σ-injective module E with $E/t(E) \in \mathcal{T}_\sigma$ is either t-torsion or t-torsionfree.*
2. *If the ring R satisfies the condition $(*)$ and $\mathcal{L}_\sigma$ satisfies ascending chain conditions, then $\mathcal{T}_t \cap \mathcal{T}_\sigma$ is closed under taking σ-injective hulls.*

Proof of (1): Let E be an indecomposable σ-injective module with $E/t(E) \in \mathcal{T}_\sigma$. By (9) in Theorem 8, $t(E)$ is a direct summand of E. As E is indecomposable, $t(E) = 0$ or $t(E) = E$, as desired.

Proof of (2): Let M be in $\mathcal{T}_t \cap \mathcal{T}_\sigma$. Since $\mathcal{T}_\sigma$ is closed under taking extensions, $E_\sigma(M)$ is σ-torsion. As $E_\sigma(M)$ is σ-injective and σ-torsion, it follows that $E_\sigma(M) = \sum_{i \in I} \oplus E_i$

by [10, Theorem 1], where I is an index set and E_i is a nonzero σ-injective σ-torsion indecomposable submodule of $E_\sigma(M)$. As $E_i \subseteq E_\sigma(M) \in \mathcal{T}_\sigma$, it follows that $E_i \in \mathcal{T}_\sigma$, and so $E_i/t(E_i) \in \mathcal{T}_\sigma$, it follows that E_i is t-torsion or t-torsionfree. Since M is essential in $E_\sigma(M)$, it follows that $M \cap E_i \neq 0$. Since $M \in \mathcal{T}_\sigma$, $M/(M \cap E_i) \in \mathcal{T}_\sigma$. As $M \in \mathcal{T}_t$ and $\mathcal{T}_t$ is closed under taking σ-dense submodules, $M \cap E_i \in \mathcal{T}_t$. Thus $t(E_i) \supseteq t(M \cap E_i) = M \cap E_i \neq 0$, and so $t(E_i) \neq 0$. Hence $t(E_i) = E_i$ holds for all i. Since every preradical preserves direct sums, it follows that $t(E_\sigma(M)) = t(\sum_{i \in I} \oplus E_i) = \sum_{i \in I} \oplus t(E_i) = \sum_{i \in I} \oplus E_i = E_\sigma(M)$, and so $E_\sigma(M) \in \mathcal{T}_t$. □

The following proposition generalizes [7, Proposition 1.2].

Proposition 10. *Let $(\mathcal{T}_t, \mathcal{F}_t)$ be a σ-hereditary σ-stable torsion theory, that is, t is an idempotent radical and $\mathcal{T}_t$ is closed under taking σ-injective hulls and σ-dense submodules, where σ is a left exact radical. Then there exists an isomorphism: $E_\sigma(M/t(M)) \simeq E_\sigma(M)/E_\sigma(t(M))$, if $M/t(M) \in \mathcal{T}_\sigma$.*

Proof. For a module M consider the following commutative diagram.

$$
\begin{array}{ccccc}
0 \to & M & \xrightarrow{j} & E_\sigma(M) \\
 & \downarrow g & & \downarrow f \\
0 \to & M/t(M) & \xrightarrow[i]{} & E_\sigma(M/t(M)),
\end{array}
$$

where i and j are inclusions and g is a canonical epimorphism and f is an induced morphism by σ-injectivity of $E_\sigma(M/t(M))$. By the above diagram, $t(M) = \ker g = \ker(f|_M) = \ker f \cap M$, and so $t(M) = \ker f \cap M$ follows. Since $M/t(M) \in \mathcal{F}_t$ and $\mathcal{F}_t$ is closed under taking σ-injective hulls and σ is a left exact preradical, it follows that $E_\sigma(M)/\ker f \subseteq E_\sigma(M/t(M)) \in \mathcal{F}_t$. Thus it follows that $t(E_\sigma(M)) \subseteq \ker f$. Since $\mathcal{T}_\sigma$ is closed under taking extensions and $M/t(M) \in \mathcal{T}_\sigma$ and $E_\sigma(M)/M \in \mathcal{T}_\sigma$, it follows that $E_\sigma(M)/t(M) \in \mathcal{T}_\sigma$. Since $E_\sigma(M)/t(E_\sigma(M))$ is an epimorphic image of $E_\sigma(M)/t(M)$, it follows that $E_\sigma(M)/t(E_\sigma(M)) \in \mathcal{T}_\sigma$. Since σ is left exact preradical and $\ker f/t(E_\sigma(M)) \subseteq E_\sigma(M)/t(E_\sigma(M)) \in \mathcal{T}_\sigma$, it follows that $\ker f/t(E_\sigma(M)) \in \mathcal{T}_\sigma$. By the assumption $t(E_\sigma(M))$ is σ-injective. Then the exact sequence $(0 \to t(E_\sigma(M)) \to \ker f \to \ker f/t(E_\sigma(M)) \to 0)$ splits. Then there exists a submodule S of $\ker f$ such that $\ker f = S \oplus t(E_\sigma(M))$. Then since $0 = S \cap t(E_\sigma(M)) \supseteq S \cap t(M)$, it follows that $0 = S \cap t(M) = S \cap \ker f \cap M$. As M is essential in $E_\sigma(M)$, it follows that $0 = S \cap \ker f = S$. Thus it follows that $t(E_\sigma(M)) = \ker f$. So $f(E_\sigma(M)) \simeq E_\sigma(M)/\ker f = E_\sigma(M)/t(E_\sigma(M)) \in \mathcal{T}_\sigma$. Thus the exact sequence $0 \to t(E_\sigma(M)) \to E_\sigma(M) \to f(E_\sigma(M)) \to 0$ splits as $t(E_\sigma(M))$ is σ-injective. Thus $f(E_\sigma(M))$ is a direct summand of σ-injective module $E_\sigma(M)$, and so $f(E_\sigma(M))$ is also σ-injective. Since $E_\sigma(M/t(M)) \supseteq f(E_\sigma(M)) \supseteq g(M) \supseteq M/t(M)$, it follows that $E_\sigma(M/t(M))/f(E_\sigma(M)) \in \mathcal{T}_\sigma$. Thus the exact sequence $0 \to f(E_\sigma(M)) \to E_\sigma(M/t(M)) \to E_\sigma(M/t(M))/f(E_\sigma(M)) \to 0$ splits. So there exists a submodule K of $E_\sigma(M/t(M))$ such that $E_\sigma(M/t(M)) = K \oplus f(E_\sigma(M))$. Since $f(E_\sigma(M)) \supseteq M/t(M)$, it follows that $K \cap (M/t(M)) = 0$. But $M/t(M)$ is essential in $E_\sigma(M/t(M))$, and so $K = 0$. Thus $E_\sigma(M/t(M)) = f(E_\sigma(M)) \simeq E_\sigma(M)/\ker f = E_\sigma(M)/t(E_\sigma(M))$, as desired. □

Hereafter we omit the proof of the following propositions.

We call A σ-M-injective if $\mathrm{Hom}_R(-, A)$ preserves the exactness for any exact sequence $0 \to N \to M \to M/N \to 0$, where $M/N \in \mathcal{T}_\sigma$. The following proposition is a generalization of Theorem 15 in [16].

Proposition 11. *A is σ-M-injective if and only if $f(M) \subseteq A$ for any $f \in \mathrm{Hom}_R(E_\sigma(M), E_\sigma(A))$.*

We obtain the following corollary as a torsion theoretic generalization of the Johnson Wong theorem by putting $M = A$ in Proposition 11. We call a module A σ-quasi-injective if A is σ-A-injective.

Corollary 12. *A is σ-quasi-injective if and only if $f(A) \subseteq A$ for any $f \in \mathrm{Hom}_R(E_\sigma(A), E_\sigma(A))$.*

The following lemma generalizes Proposition 2.3 in [17].

Lemma 13. *If A is σ-quasi-injective and $E_\sigma(A) = M \oplus N$, then $A = (M \cap A) \oplus (N \cap A)$.*

Now we can generalize [1, Theorem 2.3]

Theorem 14. *Assume that σ is a left exact radical and $\mathcal{T}_t$ is closed under taking σ-injective hulls, then every σ-quasi-injective R-module A with $A/t(A) \in \mathcal{T}_\sigma$ splits, that is, $A = t(A) \oplus N$ where $N \in \mathcal{F}_t$, and then if $t(A)$ is σ-torsion, then N is σ-quasi-injective.*

The following corollary generalizes Corollary 2.15 in [5].

Corollary 15. *Let M be a σ-quasi-injective module. Then any σ-essentially closed and σ-dense submodule of M is a direct summand of M, and any direct summand is σ-quasi-injective.*

References

[1] E. P. Armendariz, *Quasi-injective modules and stable torsion classes*, Pacific J. Math. **31** (1969), 227–280.

[2] L. Bican, P. Jambor, T. Kepka and P. Nemec, *Stable and costable preradicals*, Acta Universitatis Carolinae-Mathematicaet Physica, **16(2)** (1975), 63–69.

[3] P. Gabriel, *Des categories abeliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

[4] J. L. Gómez Pardo, *Spectral Gabriel topologies and relative singular functors*, Comm. in Algebra **13(1)** (1985), 21–57.

[5] K. R. Goodearl, *Ring Theory*, Marcel Dekker, New York, 1976.

[6] J. S. Golan, *Localization of Noncommutative Rings*, Marcel Dekker, New York, 1976.

[7] J. S. Golan, *Structure Sheaves over a Noncommutative Rings*, Marcel Dekker, 1980.

[8] J. S. Golan, *Torsion Theories*, Longman Scientific & Technical, 1986.

[9] J. N. Manocha, *Singular submodule relative to a kernel functor*, Comm. in Algebra, **10(5)** (1982), 474–491.

[10] K. Masaike and T. Horigome, *Direct sums of τ-injective modules*, Tsukuba J. Math. **4** (1980), 77–81.

[11] R. A. Rubin, *Semi-simplicity relative to kernel functors*, Can. J. Math. **26** (1974), 1405–1411.

[12] Bo Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, 1975.

[13] Y. Takehana, *On generalization of CQF-3′ modules and cohereditary torsion theories*, to appear in Math. J. Okayama Univ.

[14] Y. Takehana, *On generalization of QF-3′ modules and hereditary torsion theories*, to appear in Math. J. Okayama Univ.

[15] Z. Papp, *Semi-stability and topologies on R-sp*, Comm. in Algebra **4** (1976), 793–809.
[16] G. Azumaya, *M-projective and M-injectives*, unpublished.
[17] M. Harada, *Note on quasi-injective modules*, Osaka J. Math. **2** (1965), 351–356.

General Education
Hakodate National College of Technology
14-1 Tokura-cho Hakodate Hokkaido, 042-8501 JAPAN
E-mail address: takehana@hakodate-ct.ac.jp