ON GRADED MORITA EQUIVALENCES FOR AS-REGULAR ALGEBRAS

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Abstract. One of the most active projects in noncommutative algebraic geometry is to classify AS-regular algebras. The motivation of this article is to find a nice criterion of graded Morita equivalence for AS-regular algebras. In this article, we associate to a geometric AS-regular algebra A a new algebra $\overline{A}$, and it is proved that $\overline{A}$ is isomorphic to $\overline{A'}$ as graded algebras if A is graded Morita equivalent to A'. In particular, if A, A' are generic geometric 3-dimensional AS-regular algebras, then $\overline{A}$ is isomorphic to $\overline{A'}$ as graded algebras if and only if A is graded Morita equivalent to A'.

Key Words: graded Morita equivalence, AS-regular algebra, generalized Nakayama automorphism.

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1. Introduction

This is based on a joint work with Izuru Mori.

In noncommutative algebraic geometry, classification of AS-regular algebras has been one of the major projects since its beginning. In fact, AS-regular algebras (of finite GK-dimension) up to dimension 3 were classified (cf. [1], [2], [9], [10]). Since classifying 4-dimensional AS-regular algebras up to isomorphism of graded algebras is difficult, it is natural to try to classify them up to something weaker than graded isomorphism such as graded Morita equivalence. In general, it is difficult to check whether two graded algebras are graded Morita equivalent. The main result of this article (Theorem 8) gives a new criterion of graded Morita equivalences for geometric AS-regular algebras.

2. Preliminaries

Throughout this paper, we fix an algebraically closed field k. Let A be a graded k-algebra. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules and right A-module homomorphisms preserving degree. We say that two graded algebras A and A' are graded Morita equivalent if there exists an equivalence of categories between $\operatorname{GrMod} A$ and $\operatorname{GrMod} A'$. For $M \in \operatorname{GrMod} A$ and $n \in \mathbb{Z}$, the shift of M, denoted by $M(n)$, is the graded right A-module such that $M(n)_i = M_{i+n}$. For $M, N \in \operatorname{GrMod} A$, we define the

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graded k-vector spaces

$$\underline{\mathrm{Hom}}_A(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{GrMod}\,A}(M, N(n)), \text{and}$$
$$\underline{\mathrm{Ext}}^i_A(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^i_{\mathrm{GrMod}\,A}(M, N(n)).$$

We say A is connected if $A_i = 0$ for all $i < 0$, and $A_0 = k$.

Definition 1. Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be a connected graded algebra such that $\dim_k A_i < \infty$ for all i. We define the Gelfand-Kirillov dimension (GK-dimension) of A by

$$\mathrm{GKdim}\, A = \limsup_{n\to\infty} \frac{\log(\sum_{i=0}^n \dim_k A_i)}{\log n}$$

If A is a commutative algebra, then $\mathrm{GKdim}\, A = \mathrm{Kdim}\, A$, the Krull dimension of A.

An AS-regular algebra defined below is one of the first classes of algebras studied in noncommutative algebraic geometry.

Definition 2. Let A be a connected graded k-algebra. Then A is called a d-dimensional AS-regular (resp. AS-Gorenstein) algebra of Gorenstein parametar ℓ if it satisfies the following conditions:

- $\mathrm{gldim}\, A = d < \infty$ (resp. $\mathrm{id}(A) = d < \infty$), and
- (Gorenstein condition)

$$\underline{\mathrm{Ext}}^i_A(k, A) \cong \begin{cases} 0 & \text{if } i \neq d, \\ k(\ell) & \text{if } i = d. \end{cases}$$

We do not assume that $\mathrm{GKdim}\, A < \infty$ in the definition.

Every 1-dimensional AS-regular algebra of Gorenstein parameter ℓ is isomorphic to a polynomial algebra $k[x]$ with $\deg x = \ell$.

The classification of 2-dimensional AS-regular algebras were completed by Zhang [13].

We now focus on 3-dimensional AS-regular algebras generated in degree 1 of finite GK-dimension. These algebras were completely classified by Artin, Tate and Van den Bergh [2] using geometric techniques. In this article, we will use their classification only in the quadratic case.

Let $T(V)$ be the tensor algebra on V over k where V is a finite dimensional vector space. We say that A is a quadratic algebra if A is a graded algebra of the form $T(V)/(R)$ where $R \subseteq V \otimes_k V$ is a subspace and (R) is the ideal of $T(V)$ generated by R. For a quadratic algebra $A = T(V)/(R)$, we define

$$\Gamma_2 := \{(p,q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid f(p,q) = 0 \text{ for all } f \in R\}.$$

Definition 3. [5] A quadratic algebra $A = T(V)/(R)$ is called geometric if there exists a geometric pair (E, σ) where $E \subseteq \mathbb{P}(V^*)$ is a closed k-subscheme and σ is a k-automorphism of E such that

(G1) $\Gamma_2 = \{(p, \sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$, and
(G2) $R = \{f \in V \otimes_k V \mid f(p, \sigma(p)) = 0 \text{ for all } p \in E\}$.

Let $A = T(V)/(R)$ be a quadratic algebra. If A satisfies the condition **(G1)**, then A determines a geometric pair (E, σ). If A satisfies the condition **(G2)**, then A is determined by a geometric pair (E, σ), so we will write $A = \mathcal{A}(E, \sigma)$.

If A is a 3-dimensional quadratic AS-regular algebra of finite GK-dimension, then $A = \mathcal{A}(E, \sigma)$ is geometric, and E is either $\mathbb{P}^2$ or a cubic curve in $\mathbb{P}^2$. Artin, Tate and Van den Bergh [2] gave a list of geometric pairs (E, σ) for "generic" 3-dimensional quadratic AS-regular algebras. In their generic classification, E is one of the following:

(1) a triangle.
(2) a union of a line and a conic meeting at two points.
(3) an elliptic curve.

Example 4. Let

$$A = k\langle x, y, z\rangle/(\alpha yz + \beta zy + \gamma x^2, \alpha zx + \beta xz + \gamma y^2, \alpha xy + \beta yx + \gamma z^2)$$

where $\alpha, \beta, \gamma \in k$. Unless $\alpha^3 = \beta^3 = \gamma^3$, or two of $\{\alpha, \beta, \gamma\}$ are zero, $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional quadratic AS-regular algebra of GK-dimension 3 such that

$$E = \mathcal{V}(\alpha\beta\gamma(x^3 + y^3 + z^3) - (\alpha^3 + \beta^3 + \gamma^3)xyz) \subset \mathbb{P}^2$$

is an elliptic curve, and $\sigma \in \operatorname{Aut}_k E$ is given by the translation automorphism by a fixed point $(\alpha, \beta, \gamma) \in E$. In this case, A is called a 3-dimensional Sklyanin algebra.

For the purpose of this article, we define the Types of some geometric pairs (E, σ) of 3-dimensional quadratic AS-regular algebras as follows:

- Type $\mathbb{P}^2$: E is $\mathbb{P}^2$, and $\sigma \in \operatorname{Aut}_k \mathbb{P}^2$.
- Type S_1: E is a triangle, and σ stabilizes each component.
- Type S_2: E is a triangle, and σ interchanges two components.
- Type S_3: E is a triangle, and σ circulates three components.
- Type S_1': E is a union of a line and a conic meeting at two points, and σ stabilizes each component and two intersection points.
- Type S_2': E is a union of a line and a conic meeting at two points, and σ stabilizes each component and interchanges two intersection points.

Remark 5. If E is a union of a line and a conic meeting at two points, and σ interchanges these two components, then $\mathcal{A}(E, \sigma)$ is not an AS-regular algebra [2, Proposition 4.11]. Thus the above types completely cover the generic singular cases and $E = \mathbb{P}^2$.

Recall that the Hilbert series of A is defined by

$$H_A(t) = \sum_{i=-\infty}^{\infty} (\dim_k A_i)t^i \quad \in \mathbb{Z}[[t, t^{-1}]].$$

If A is a 3-dimensional quadratic AS-regular algebra of finite GK-dimension, then A is a noetherian Koszul domain and $H_A(t) = (1-t)^{-3}$. In particular, the Gorenstein parameter of A is equal to 3.

At the end of this section, we prepare the definition of the generalized Nakayama automorphism to state our theorem.

Let A be a d-dimensional AS-Gorenstein algebra, and $\mathfrak{m} := A_{\geq 1}$ the unique maximal homogeneous ideal of A. We define the graded A-A bimodule ω_A by

$$\omega_A := \mathrm{H}^d_{\mathfrak{m}}(A)^* = \underline{\mathrm{Hom}}_k(\lim_{n\to\infty} \underline{\mathrm{Ext}}^d_A(A/A_{\geq n}, A), k).$$

It is known that $\omega_A \cong {}_\nu A(-\ell)$ as graded A-A bimodules for some graded k-algebra automorphism $\nu \in \mathrm{Aut}_k A$, where ${}_\nu A$ is the graded A-A bimodule defined by ${}_\nu A = A$ as a graded k-vector space with a new action $a * x * b := \nu(a)xb$ (see [3, Theorem 1.2], [4]).

Definition 6. [6] Let A be a d-dimensional AS-Gorenstein algebra. We call $\nu \in \mathrm{Aut}_k A$ the generalized Nakayama automorphism of A if $\omega_A \cong {}_\nu A(-\ell)$ as graded A-A bimodules. If the generalized Nakayama automorphism $\nu \in \mathrm{Aut}_k A$ is id_A, then A is called symmetric.

A finite dimensional algebra A is called graded Frobenius if $A^* \cong A(-\ell)$ as right and left graded A-modules. Let A be a graded Frobenius algebra. Then $A^* \cong {}_\nu A(-\ell)$ as graded A-A bimodules where ν is the usual Nakayama automorphism. Since A is a noetherian AS-Gorenstein algebra of $\mathrm{id}(A) = 0$ and

$$\omega_A = \mathrm{H}^0_{\mathfrak{m}}(A)^* \cong A^* \cong {}_\nu A(-\ell),$$

the generalized Nakayama automorphism of A is the usual Nakayama automorphism ([6]).

Let $A = \mathcal{A}(E, \sigma)$ be a geometric AS-Gorenstein algebra of Gorenstein parameter ℓ. If ν is the generalized Nakayama automorphism of A, then it restricts to an automorphism $\nu \in \mathrm{Aut}_k V = \mathrm{Aut}_k A_1$. So its dual induces an automorphism $\nu^* \in \mathrm{Aut}_k \mathbb{P}(V^*)$, and which induces an automorphism $\nu^* \in \mathrm{Aut}_k E$ (see [6]). Therefore we define a new graded algebra $\overline{A}$ by

$$\overline{A} := \mathcal{A}(E, \nu^* \sigma^\ell).$$

3. Main results

The following theorem motivates this research.

Theorem 7. [5, Theorem 5.4] *Let $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$ be 3-dimensional Sklyanin algebras. If $\sigma^9, \sigma'^9 \neq \mathrm{id}$, then the following are equivalent:*

(1) $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A'$.
(2) $\mathcal{A}(E, \sigma^3) \cong \mathcal{A}(E', \sigma'^3)$ *as graded algebras.*

Now, we state our main theorem in this article.

Theorem 8. [7], [11]

(1) *Let A, A' be noetherian geometric AS-regular algebras. Then*

$$\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A' \implies \overline{A} \cong \overline{A'} \text{ as graded algebras.}$$

(2) *In particular, if $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$ are 3-dimensional quadratic AS-regular algebras of finite GK-dimension such that (E, σ) and (E', σ') are of the following Type: $\mathbb{P}^2$, S_1, S_2, S_3, S_1' or S_2', then*

$$\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A' \iff \overline{A} \cong \overline{A'} \text{ as graded algebras.}$$

The generalized Nakayama automorphism of a 3-dimensional Sklyanin algebra A is id_A (cf. [8, Example 10.1]). Thus Theorem 8 (2) also hold for 3-dimensional Sklyanin algebras $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$ with $\sigma^9, \sigma'^9 \neq \mathrm{id}$.

Theorem 9. [11] *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quadratic AS-regular algebra of finite GK-dimension such that (E, σ) is of the following Type: $\mathbb{P}^2$, S_1, S_2, S_3, S_1' or S_2', then $\overline{A}$ is a 3-dimensional symmetric AS-regular algebra.*

By Theorem 8 (2) and Theorem 9, graded Morita equivalences of 3-dimensional generic geometric AS-regular algebras are characterized by isomorphisms of 3-dimensional symmetric AS-regular algebras. In general, it is more difficult to check if two graded algebras are graded Morita equivalent than to check if they are isomorphic as graded algebras. In this sence, Theorem 8 is useful.

4. Example

In this last section, we give an example by applying Theorem 8 to 3-dimensional skew polynomial algebras.

Example 10. If

$$A = k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx)$$

where $\alpha, \beta, \gamma \in k$, $\alpha\beta\gamma \neq 0, 1$, then $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional quadratic AS-regular algebra of GK-dimension 3 and Gorenstein parameter 3 such that

$$E = l_1 \cup l_2 \cup l_3 \subset \mathbb{P}^2 \quad \text{where } l_1 = \mathcal{V}(x), l_2 = \mathcal{V}(y), l_3 = \mathcal{V}(z)$$

is a triangle, and $\sigma \in \operatorname{Aut}_k E$ is given by

$$\begin{aligned} \sigma|_{l_1}(0, b, c) &= (0, b, \alpha c) \\ \sigma|_{l_2}(a, 0, c) &= (\beta a, 0, c) \\ \sigma|_{l_3}(a, b, 0) &= (a, \gamma b, 0), \end{aligned}$$

so (E, σ) is of Type S_1. In this case, the automorphism $\nu^* \in \operatorname{Aut}_k E$ induced by the generalized Nakayama automorphism $\nu \in \operatorname{Aut}_k A$ is given by

$$\nu^*(a, b, c) = ((\beta/\gamma)a, (\gamma/\alpha)b, (\alpha/\beta)c)\,,$$

so $\nu^*\sigma^3 \in \operatorname{Aut}_k E$ is given by

$$\begin{aligned} \nu^*\sigma^3|_{l_1}(0, b, c) &= (0, b, \alpha\beta\gamma c) \\ \nu^*\sigma^3|_{l_2}(a, 0, c) &= (\alpha\beta\gamma a, 0, c) \\ \nu^*\sigma^3|_{l_3}(a, b, 0) &= (a, \alpha\beta\gamma b, 0). \end{aligned}$$

It follows that

$$\overline{A} = \mathcal{A}(E, \nu^*\sigma^3) = k\langle x, y, z\rangle/(yz - \alpha\beta\gamma zy,\ zx - \alpha\beta\gamma xz,\ xy - \alpha\beta\gamma yx).$$

Similarly, if

$$A' = k\langle x, y, z\rangle/(yz - \alpha' zy, zx - \beta' xz, xy - \gamma' yx)$$

where $\alpha', \beta', \gamma' \in k, \alpha'\beta'\gamma' \neq 0, 1$, then

$$\overline{A'} = k\langle x, y, z\rangle/(yz - \alpha'\beta'\gamma' zy,\ zx - \alpha'\beta'\gamma' xz,\ xy - \alpha'\beta'\gamma' yx).$$

By Theorem 8 (2),

$$\operatorname{GrMod} A \cong \operatorname{GrMod} A' \Longleftrightarrow \overline{A} \cong \overline{A'}.$$

Moreover,

$$\overline{A} \cong \overline{A'} \Longleftrightarrow \alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}$$

by [12, Lemma 2.1]. Hence we have

$$\operatorname{GrMod} A \cong \operatorname{GrMod} A' \Longleftrightarrow \alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}.$$

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