PRIME FACTOR RINGS OF ORE EXTENSIONS OVER A COMMUTATIVE DEDEKIND DOMAIN

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Abstract. Let $R = D[x;\sigma]$ be a skew polynomial ring over a commutative Dedekind domain D and let P be a minimal prime ideal of R, where σ is an automorphism of D. There are two different types of P, namely, either $P = \mathfrak{p}[x;\sigma]$ or $P = P' \cap R$, where $\mathfrak{p}$ is a σ-prime ideal of D, P' is a prime ideal of $K[x;\sigma]$ and K is the quotient field of D. In the first case R/P is a hereditary prime ring and in the second case, it is shown that R/P is a hereditary prime ring if and only if $P \nsubseteq M^2$ for any maximal ideal M of R. We give some examples of minimal prime ideals such that the factor rings are not hereditary or hereditary or Dedekind, respectively. In the case $R = D[x;\sigma,\delta]$, an Ore extension, where δ is a left σ-derivation of D, we roughly speak of any prime ideal P of R which is not complete, by using Goodearl's classification.

1. Background

A ring is called left (resp. right) hereditary if every left (resp. right) ideal is projective. A Dedekind domain is a commutative domain which is hereditary.

When D is a Dedekind domain, Hillman [1] gave a criterion for D-torsion-free prime factor rings of $D[x]$ to be Dedekind. Indeed, let $f(x)$ generate a prime ideal $P = f(x)D[x]$ of $D[x]$ which is not maximal. Then it was shown that $D[x]/P$ is a Dedekind domain if and only if P is not contained in the square of any maximal ideal of $D[x]$.

Armendariz asked the following Question: Can Hillman's result be generalized from Dedekind domains to hereditary prime P.I. rings?

Park and Roggenkamp [2] studied this question and gave a partial answer under a strong condition. Later Lee, Marubayashi and Park [3] gave a precise answer to Armendariz's question. Let Λ be a hereditary prime P.I. ring, and suppose that a non-zero central polynomial $f(x)$ generates a prime ideal $P = f(x)\Lambda[x]$. They proved that $\Lambda[x]/P$ is hereditary if and only if P is not contained in the square of any maximal ideal of $\Lambda[x]$ by adopting localization, some properties of *v-HC* orders [4, 5] and Kaplansky's method [6].

2. Main results

Let D be a commutative Dedekind domain with σ, an automorphism of D. We denote by $R = D[x;\sigma]$ the skew polynomial ring over D in an indeterminate x. We denote by $\mathrm{Spec}(R) = \{P \mid P \text{ is a prime ideal of } R\}$ and $\mathrm{Spec}_0(R) = \{P \in \mathrm{Spec}(R) \mid P \cap D = 0\}$. We always assume that D is not a field to avoid the trivial case.

If P is not a minimal prime ideal of R, we can see that R/P is a simple Artinian ring by [7, (6.5.4), (7.5.3) and (6.3.11)]. So from now on, to study the factor rings R/P by prime ideals we can consider the minimal prime ideals only.

The detailed version of this paper has been submitted for publication elsewhere.

Firstly, we find out the set of minimal prime ideals of R.

Proposition 1. *$\{\mathfrak{p}[x;\sigma],\ P \mid \mathfrak{p}$ is a σ-prime ideal of D and $P \in \mathrm{Spec}_0(R)$ with $P \neq (0)\}$ is the set of all minimal prime ideals of R.*

Then when minimal prime ideal P is the former type, we have

Proposition 2. *Let $P = \mathfrak{p}[x;\sigma]$, where $\mathfrak{p}$ is a σ-prime ideal of D. Then R/P is a hereditary prime ring. In particular, R/P is a Dedekind prime ring if and only if $\mathfrak{p} \in \mathrm{Spec}(D)$.*

When the minimal prime ideal P is the latter type, i.e. $P \in \mathrm{Spec}_0(R)$, we discuss the factor ring R/P in terms of the order of σ. If σ is of infinite order, that is, $\sigma^n \neq 1$ for any $n > 0$, by [8], it is clear that:

Proposition 3. (1) *$P = xR$ is the only element in $\mathrm{Spec}_0(R)$.*

(2) *$R/P = D[x;\sigma]/xR \simeq D$ is a Dedekind Domain.*

If σ is of finite order, we may assume $\sigma^n = 1$ for some $n > 0$. By using localization, Kaplansky's method [6], Reiner's result [9, (3.24)], some lemmas and other known results, we obtain the following proposition which is similar to Hillman's.

Proposition 4. *Let $P \in \mathrm{Spec}_0(R)$ with $P \neq xR$ and $P \neq 0$. Then R/P is a hereditary prime ring if and only if $P \nsubseteq M^2$ for any maximal ideal M of R.*

Remark 5. (1) The center of R is $C = D_\sigma[x^n]$, where $D_\sigma = \{d \in D \mid \sigma(d) = d\}$, and C is a Dedekind Domain.

(2) Let $P \in \mathrm{Spec}_0(R)$ with $P \neq xR$. Then $\mathbb{Z}(R/P) = C/(P \cap C)$, where $\mathbb{Z}(R/P)$ is the center of the factor ring R/P.

Summarizing all the results we have obtained, we have the following theorem:

Theorem 6. *Let $R = D[x;\sigma]$ be a skew polynomial ring over a commutative Dedekind domain, where σ is an automorphism of D and let P be a prime ideal of R. Then*

1. *P is a minimal prime ideal of R if and only if either $P = \mathfrak{p}[x;\sigma]$, where $\mathfrak{p}$ is a non-zero σ-prime ideal of D or $P \in \mathrm{Spec}_0(R)$ with $P \neq (0)$.*
2. *If $P = \mathfrak{p}[x;\sigma]$, where $\mathfrak{p}$ is a non-zero σ-prime ideal of D, then R/P is a hereditary prime ring.*
3. *If $P \in \mathrm{Spec}_0(R)$ with $P = xR$, then R/P is a Dedekind domain.*
4. *If $P \in \mathrm{Spec}_0(R)$ with $P \neq (0)$ and $P \neq xR$, then R/P is a hereditary prime ring if and only if $P \nsubseteq M^2$ for any maximal ideal M of R.*

3. Examples

We give three kinds of rings which is not hereditary, hereditary but not Dedekind and Dedekind respectively by using Proposition 4.

Let $D = \mathbb{Z} + \mathbb{Z}i$ be the Gauss integers, where $i^2 = -1$, and let σ be the automorphism of D with $\sigma(a + bi) = a - bi$ where a, $b \in \mathbb{Z}$, the ring of integers. Let p be a prime number. Then the following properties are well known in the elementary number theory:

(1) If $p = 2$, then $2D = (1 + i)^2 D$ and $(1 + i)D$ is a prime ideal.

(2) If $p = 4n+1$, then $pD = \pi\sigma(\pi)D$ for some prime element π with $\pi D + \sigma(\pi)D = D$.

(3) If $p = 4n+3$, then pD is a prime ideal of R.

We let $R = D[x;\sigma]$ be the skew polynomial ring, $P = (x^2+p)R$. It is obvious that $P = (x^2+p)R \in \mathrm{Spec}_0(R)$.

(1) If $p = 2$, then R/P is not a hereditary prime ring.
($P \subseteq M^2$, where $M = (1+i)D + xR$.)

(2) If $p = 4n+1$, then R/P is a hereditary prime ring but not a Dedekind prime ring.
($\exists\ M = \pi D + xR \supset P$, $M^2 + P = M$.)

(3) If $p = 4n+3$, then R/P is not a hereditary prime ring.
($P \subseteq M^2$, where $M = (1+x)R + 2R$.)
However, let $S = \{2^i |\, i = 0, 1, 2, \cdots\}$, a central multiplicative set in R. Then R_S/P_S is a Dedekind prime ring.

4. Questions

Let Λ be a hereditary prime P.I. ring with σ, an automorphism of Λ.

(Q1) Does $R = \Lambda[x;\sigma]$ have the similar properties as in the first step about the factor ring R/P for a prime ideal P of R?

We can get similar results except Proposition 4. We have an example that the center is different from the Dedekind domain case. In general, we have the following example:

Let Q be a simple Artinian ring with its center $\mathbb{Z}(Q) = K$. Suppose $[Q : K] < \infty$. $\sigma \in \mathrm{Aut}_K(Q)$ (i.e. $\sigma \in \mathrm{Aut}(Q)$ s.t. $\sigma(k) = k$ for all $k \in K$), then σ is an inner automorphism [9], that is, there exists $q \in U(Q)$ such that $\sigma(a) = q^{-1}aq$ for all $a \in Q$. Suppose $\sigma^n = 1$, $n > 1$. $Q[x;\sigma] \supseteq \mathbb{Z}(Q[x;\sigma]) \supsetneqq K_\sigma[x^n]$ since there exists $c(x) = qx + b \in \mathbb{Z}(Q[x;\sigma]) - K_\sigma[x^n]$, where $b \in K_\sigma$.

Let D be a commutative Dedekind domain with σ, an automorphism of D and δ, a left σ-derivation of D ($\delta \neq 0$). Let $R = D[x;\sigma,\delta]$ be the Ore extension over D with

$$xa = \sigma(a)x + \delta(a) \ \text{ for all } \ a \in D.$$

We study the structure of R/P for any prime ideal P of R in terms of Goodearl's classification [10, (3.1)] on prime ideals as the following.

$R = D[x;\sigma,\delta]$ where D is a commutative Noetherian ring, σ is an automorphism of D and δ is a left σ-derivation of D ($\delta \neq 0$). Let P be a prime ideal of R, $\mathfrak{p} = P \cap D$. Then one of the following three cases must hold:

(1) $\mathfrak{p}$ is a (σ,δ)-prime ideal of D. In this case,
  (a) $\mathfrak{p}$ is a σ-prime ideal of D, or
  (b) $\mathfrak{p}$ is a δ-prime ideal of D and R/P has a unique associated prime ideal, which contains $(1-\sigma)D$.

(2) $\mathfrak{p}$ is a prime ideal of D and $\sigma(\mathfrak{p}) \neq \mathfrak{p}$.

We assume that D is a commutative Dedekind domain.

When P is in the case (2), then P is not a minimal prime ideal. Hence R/P is a simple Artinian ring.

When P is in the case (1) (a). If $\mathfrak{p} \neq 0$, then $P = \mathfrak{p}[x; \sigma, \delta]$. Hence R/P is hereditary and R/P is Dedekind if and only if $\mathfrak{p} \in \mathrm{Spec}(D)$. If $\mathfrak{p} = 0$, then $P \in \mathrm{Spec}_0(R)$. We predict:

(Q2) R/P is hereditary if and only if $P \nsubseteq M^2$ for any maximal ideal M of R.

When P is in the case (1) (b). $\mathfrak{p} = \mathfrak{p}_o{}^e$ for some prime ideal $\mathfrak{p}_o$ of D and $P = \mathfrak{p}[x; \sigma, \delta]$.

(Q3) Whether gld (R/P) is finite or not?

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