T-STRUCTURES AND LOCAL COHOMOLOGY FUNCTORS

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Abstract. The section functor $\varGamma_W$ with support in a specialization closed subset W of $\mathrm{Spec}(R)$ is one of the most important radical functors and basic tools not only for the theory of commutative algebra but also for algebraic geometry. The aim of this article is to characterize the section functor $\varGamma_W$ (resp. the right derived functor $\mathbf{R}\varGamma_W$ of $\varGamma_W$) as elements of the set of all functors on the category of all R-modules (resp. the derived category consisting of all left bounded complexes of R-modules).

1. Introduction

This is a joint work with Yuji Yoshino.

Let R be a commutative noetherian ring. We denote the category of all R-modules by R-Mod and also denote the derived category consisting of all left bounded complexes of R-modules by $\mathcal{D}^+(R\text{-Mod})$.

A radical functor, or more generally a preradical functor, has its own long history in the theory of categories and functors. See [2] or [3] for the case of module category. One of the most useful and important facts is that there is a bijective correspondence between the set of all left exact radical functors on R-Mod and the set of all hereditary torsion theories for R-Mod (See [5, Chapter VI, Proposition 3.1]).

In this paper, one of our purpose is to observe some necessary and sufficient conditions for a functor on R-Mod to be left exact radical functor. Furthermore, we give the notion of abstract local cohomology functors, that is, we say a triangle functor δ on $\mathcal{D}^+(R\text{-Mod})$ is an abstract local cohomology functor if it defines a stable t-structure on $D^+(R\text{-Mod})$ which divides indecomposable injective R-modules. (See Definition 6 for the precise meaning.) We note here that the notion of t-structure was introduced and studied first in the paper [1], but what we need in this paper is the notion of stable t-structure introduced by Miyachi in [4]. We shall also prove that an abstract local cohomology functor is of the form $\mathbf{R}\varGamma_W$ with W being a specialization closed subset of $\mathrm{Spec}(R)$ and show that the set of specialization closed subsets of $\mathrm{Spec}(R)$ bijectively corresponds to $\mathbb{A}(R)$ which is the set of all isomorphism classes of abstract local cohomology functors on $\mathcal{D}^+(R\text{-Mod})$.

2. The definition of abstract local cohomology functors

Let us recall some definitions for functors from the category theory.

Definition 1. Let γ be a functor on R-Mod.

(1) A functor γ is called a preradical functor if γ is a subfunctor of identity functor $\mathbf{1}$.

The detailed version of this paper has been submitted for publication elsewhere.

(2) A preradical functor γ is called a radical functor if $\gamma(M/\gamma(M)) = 0$ for every R-module M.

(3) A functor γ is said to preserve injectivity if $\gamma(I)$ is an injective R-module whenever I is an injective R-module.

Example 2. Let W be a subset of $\mathrm{Spec}(R)$. Recall that W is said to be specialization closed if $\mathfrak{p} \in W$ and $\mathfrak{p} \subseteq \mathfrak{q} \in \mathrm{Spec}(R)$ imply $\mathfrak{q} \in W$.

When W is a specialization closed subset, we can define the section functor Γ_W with support in W as

$$\Gamma_W(M) = \{x \in M \mid \mathrm{Supp}(Rx) \subseteq W\}$$

for all $M \in R$-Mod. Then it is easy to see that Γ_W is a left exact radical functor that preserves injectivity.

The notion of stable t-structure was introduced by J. Miyachi.

Definition 3. A pair $(\mathcal{U}, \mathcal{V})$ of full subcategories of a triangulated category $\mathcal{T}$ is called a stable t-structure on $\mathcal{T}$ if it satisfies the following conditions:

(1) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$.

(2) $\mathcal{U} = \mathcal{U}[1]$ and $\mathcal{V} = \mathcal{V}[1]$.

(3) For any $X \in \mathcal{T}$, there is a triangle $U \to X \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

For a triangle functor δ on triangulated category $\mathcal{T}$, we define two full subcategories of $\mathcal{T}$

$$\begin{aligned} \mathrm{Im}(\delta) &= \{X \in \mathcal{T} \mid X \cong \delta(Y) \ \text{ for some } \ Y \in \mathcal{T}\}, \\ \mathrm{Ker}(\delta) &= \{X \in \mathcal{T} \mid \delta(X) \cong 0\}. \end{aligned}$$

The following theorem proved by J. Miyachi is a key to our argument. We shall refer to this theorem as Miyachi's Theorem.

Theorem 4. [4, Proposition 2.6] *Let $\mathcal{T}$ be a triangulated category and $\mathcal{U}$ be a full triangulated subcategory of $\mathcal{T}$. Then the following conditions are equivalent for $\mathcal{U}$.*

(1) *There is a full subcategory $\mathcal{V}$ of $\mathcal{T}$ such that $(\mathcal{U}, \mathcal{V})$ is a stable t-structure on $\mathcal{T}$.*

(2) *The natural embedding functor $i : \mathcal{U} \to \mathcal{T}$ has a right adjoint $\rho : \mathcal{T} \to \mathcal{U}$.*

If it is the case, setting $\delta = i \circ \rho : \mathcal{T} \to \mathcal{T}$, we have the equalities

$$\mathcal{U} = \mathrm{Im}(\delta) \quad \textit{and} \quad \mathcal{V} = \mathcal{U}^{\perp} = \mathrm{Ker}(\delta).$$

Remark 5. Let $(\mathcal{U}, \mathcal{V})$ be a stable t-structure on $\mathcal{T}$, ρ be a right adjoint functor of $i : \mathcal{U} \to \mathcal{T}$ and set $\delta = i \circ \rho$ as in the theorem. The functor ρ, hence δ as well, is unique up to isomorphisms by the uniqueness of right adjoint functors.

Now we can define an abstract local cohomology functor.

Definition 6. We denote $\mathcal{T} = \mathcal{D}^+(R\text{-Mod})$ in this definition. Let $\delta : \mathcal{T} \to \mathcal{T}$ be a triangle functor. We call that δ is an abstract local cohomology functor if the following conditions are satisfied:

(1) The natural embedding functor $i : \mathrm{Im}(\delta) \to \mathcal{T}$ has a right adjoint $\rho : \mathcal{T} \to \mathrm{Im}(\delta)$ and $\delta \cong i \circ \rho$. (Hence, by Miyachi's Theorem, $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ is a stable t-structure on $\mathcal{T}$.)

(2) The t-structure $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ divides indecomposable injective R-modules, by which we mean that each indecomposable injective R-module belongs to either $\mathrm{Im}(\delta)$ or $\mathrm{Ker}(\delta)$.

Example 7. We denote by $E_R(R/\mathfrak{p})$ the injective hull of an R-module $R/\mathfrak{p}$ for a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$.

Let W be a specialization closed subset of $\mathrm{Spec}(R)$. Since the section functor Γ_W is a left exact radical functor on R-Mod, we can define the right derived functor $\mathbf{R}\Gamma_W$ on $\mathcal{D}^+(R\text{-Mod})$. We claim that $\mathbf{R}\Gamma_W$ is an abstract local cohomology functor on $\mathcal{D}^+(R\text{-Mod})$.

In fact, it is known that $\mathcal{D}^+(R\text{-Mod})$ is triangle-equivalent to the triangulated category $\mathcal{K}^+(\mathrm{Inj}(R))$, which is the homotopy category consisting of all left-bounded injective complexes over R. Through this equivalence, for any injective complex $I \in \mathcal{K}^+(\mathrm{Inj}(R))$, $\mathbf{R}\Gamma_W(I) = \Gamma_W(I)$ is the subcomplex of I consisting of injective modules supported in W. Hence every object of $\mathrm{Im}(\mathbf{R}\Gamma_W)$ (resp. $\mathrm{Ker}(\mathbf{R}\Gamma_W)$) is an injective complex whose components are direct sums of $E_R(R/\mathfrak{p})$ with $\mathfrak{p} \in W$ (resp. $\mathfrak{p} \in \mathrm{Spec}(R)\backslash W$). In particular, if $\mathfrak{p} \in W$ (resp. $\mathfrak{p} \in \mathrm{Spec}(R)\backslash W$), then $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\mathbf{R}\Gamma_W)$ (resp. $E_R(R/\mathfrak{p}) \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$). Since $\mathrm{Hom}_R(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{q})) = 0$ for $\mathfrak{p} \in W$ and $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$, we can see that

$$\mathrm{Hom}_{\mathcal{K}^+(\mathrm{Inj}(R))}(I, J) = \mathrm{Hom}_{\mathcal{K}^+(\mathrm{Inj}(R))}(I, \Gamma_W(J))$$

for any $I \in \mathrm{Im}(\mathbf{R}\Gamma_W)$ and $J \in \mathcal{K}^+(\mathrm{Inj}(R))$. Hence it follows from the above equivalence that $\mathbf{R}\Gamma_W$ is a right adjoint of the natural embedding $i : \mathrm{Im}(\mathbf{R}\Gamma_W) \to \mathcal{D}^+(R\text{-Mod})$.

3. Main result

Let W be a specialization closed subset of $\mathrm{Spec}(R)$ and Γ_W be a section functor with support in W. We have pointed out in Example 7 that the right derived functor $\mathbf{R}\Gamma_W$ is an abstract local cohomology functor. In this section we shall prove that every abstract local cohomology functor is of this form. The main result of this paper is the following.

Theorem 8. (1) *The following conditions are equivalent for a left exact preradical functor γ on R-Mod.*

(a) *γ is a radical functor.*

(b) *γ preserves injectivity.*

(c) *γ is a section functor with support in a specialization closed subset of* $\mathrm{Spec}(R)$.

(d) *$\mathbf{R}\gamma$ is an abstract local cohomology functor.*

(2) *Given an abstract local cohomology functor δ on $\mathcal{D}^+(R\text{-Mod})$, there exists a specialization closed subset $W \subseteq \mathrm{Spec}(R)$ such that δ is isomorphic to the right derived functor $\mathbf{R}\Gamma_W$ of the section functor Γ_W.*

The equivalences among the conditions (a), (b) and (c) of the statment (1) in Theorem 8 already appear in several literatures, but they are not explicitly written. A new and significant feature of the statement (1) is that they are equivalent as well to the condition (d) and we have already seen that it holds the implication (c) $\Rightarrow$ (d) in Example 7.

Therefore, we shall prove that it holds the implication (d) $\Rightarrow$ (a) of the statement (1) and the statement (2) in Theorem 8. (For details, see the paper [6].)

To prove the statement (2), we introduce several lemmas.

Lemma 9. *Let $X \in \mathcal{D}^+(R\text{-Mod})$ and let W be a specialization closed subset of* $\mathrm{Spec}(R)$*.*

(1) $X \cong 0 \iff \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{p}, X)_\mathfrak{p} = 0$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)$*.*

(2) $X \in \mathrm{Im}(\mathbf{R}\varGamma_W) \iff \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{q}, X)_\mathfrak{q} = 0$ *for all* $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$*.*

(3) $X \in \mathrm{Ker}(\mathbf{R}\varGamma_W) \iff \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{p}, X)_\mathfrak{p} = 0$ *for all* $\mathfrak{p} \in W$*.*

Corollary 10. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let $X \not\cong 0 \in \mathcal{D}^+(R\text{-Mod})$. If $X \in \mathrm{Im}(\mathbf{R}\varGamma_\mathfrak{m})$, then $\mathbf{R}\mathrm{Hom}_R(E_R(k), X) \not\cong 0$.*

It follows from above results that we can show the following lemma.

Lemma 11. *Let $X \in \mathcal{D}^+(R\text{-Mod})$ and let W be a specialization closed subset of* $\mathrm{Spec}(R)$*.*

(1) *If $X \in \mathrm{Ker}(\mathbf{R}\varGamma_W)$ and $\mathbf{R}\mathrm{Hom}_R(X, E_R(R/\mathfrak{q})) = 0$ for all $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$, then $X \cong 0$.*

(2) *If $X \in \mathrm{Im}(\mathbf{R}\varGamma_W)$ and $\mathbf{R}\mathrm{Hom}_R(E_R(R/\mathfrak{p}), X) = 0$ for all $\mathfrak{p} \in W$, then $X \cong 0$.*

Now we can prove that it holds the implication (d) $\Rightarrow$ (a) of the statement (1) and the statement (2) in Theorem 8.

Proof. In this proof we denote $\mathcal{T} = \mathcal{D}^+(R\text{-Mod})$.

(1) (d) $\Rightarrow$ (a) Assume that $\mathbf{R}\gamma$ is an abstract local cohomology functor. We have to show that $\gamma(M/\gamma(M)) = 0$ for any R-module M. It is enough to show that $\gamma(E/\gamma(E)) = 0$ for any injective R-module E. In fact, for any R-module M, taking the injective hull $E(M)$ of M, we have $\gamma(M/\gamma(M)) \subseteq \gamma(E(M)/\gamma(E(M)))$.

We note that the natural inclusion $\gamma \subset \mathbf{1}$ of functors on R-Mod induces a natural morphism $\phi : \mathbf{R}\gamma \to \mathbf{1}$ of functors on $\mathcal{T}$. Since $(\mathrm{Im}(\mathbf{R}\gamma), \mathrm{Ker}(\mathbf{R}\gamma))$ is a stable t-structure on $\mathcal{T}$, it follows from Miyachi's Theorem and the proof of it that every injective R-module E is embedded in a triangle

$$\mathbf{R}\gamma(E) \xrightarrow{\phi(E)} E \longrightarrow V \longrightarrow \mathbf{R}\gamma(E)[1]$$

with $\mathbf{R}\gamma(E) \in \mathrm{Im}(\mathbf{R}\gamma)$ and $V \in \mathrm{Ker}(\mathbf{R}\gamma)$. Since E is an injective R-module and since $\mathbf{R}\gamma$ is the right derived functor of a left exact functor, $\mathbf{R}\gamma(E) = \gamma(E)$ is a submodule of E via the morphism $\phi(E)$. Therefore we have $V \cong E/\gamma(E)$ in $\mathcal{T}$. In particular, $H^0(\mathbf{R}\gamma(E/\gamma(E))) \cong H^0(\mathbf{R}\gamma(V)) = 0$. Since γ is left exact functor, it is concluded that $\gamma(E/\gamma(E)) = 0$ as desired.

(2) Suppose that $\delta : \mathcal{T} \to \mathcal{T}$ is an abstract local cohomology functor. We divides the proof into several steps.

(1st step) : Consider the subset $W = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta)\}$ of $\mathrm{Spec}(R)$. Then W is a specialization closed subset. To see this, we have only to show that $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta)$ implies $E_R(R/\mathfrak{q}) \in \mathrm{Im}(\delta)$ for prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$. Assume contrarily that there are prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ so that $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta)$ but $E_R(R/\mathfrak{q}) \notin \mathrm{Im}(\delta)$. Since the t-structure $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ divides indecomposable injective modules, we must have $E_R(R/\mathfrak{q}) \in \mathrm{Ker}(\delta)$. Then, from the definition of t-structures, we have

$\mathrm{Hom}_{\mathcal{T}}(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{q})) = 0$, which says that there are no nontrivial R-module homomorphisms from $E_R(R/\mathfrak{p})$ to $E_R(R/\mathfrak{q})$. However, a natural nontrivial map $R/\mathfrak{p} \to R/\mathfrak{q} \hookrightarrow E_R(R/\mathfrak{q})$ extends to a non-zero map $E_R(R/\mathfrak{p}) \to E_R(R/\mathfrak{q})$. This is a contradiction, hence it is proved that W is specialization closed. ■

Our final goal is, of course, to show the isomorphism $\delta \cong \mathbf{R}\Gamma_W$. Notice that, since the both functors δ and $\mathbf{R}\Gamma_W$ are abstract local cohomology functors, we have two stable t-structures $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ and $(\mathrm{Im}(\mathbf{R}\Gamma_W), \mathrm{Ker}(\mathbf{R}\Gamma_W))$ on $\mathcal{T}$.

(2nd step) : Note that if $\mathfrak{p} \in W$, then $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta) \cap \mathrm{Im}(\mathbf{R}\Gamma_W)$. On the other hand, if $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$, then $E_R(R/\mathfrak{q}) \in \mathrm{Ker}(\delta) \cap \mathrm{Ker}(\mathbf{R}\Gamma_W)$. ■

(3rd step) : To prove the theorem, it is enough to show that $\mathrm{Im}(\delta) = \mathrm{Im}(\mathbf{R}\Gamma_W)$ by Miyachi's Theorem. (See also Remark 5.) ■

(4th step) : Now we prove the inclusion $\mathrm{Im}(\delta) \subseteq \mathrm{Im}(\mathbf{R}\Gamma_W)$.

To do this, assume $X \in \mathrm{Im}(\delta)$. Then there is a triangle in $\mathcal{T}$: $\mathbf{R}\Gamma_W(X) \to X \to V \to \mathbf{R}\Gamma_W(X)[1]$, where $V \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$. Let $\mathfrak{q}$ be an arbitrary element of $\mathrm{Spec}(R)\backslash W$. Since $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ and $(\mathrm{Im}(\mathbf{R}\Gamma_W), \mathrm{Ker}(\mathbf{R}\Gamma_W))$ are stable t-structures and since $E_R(R/\mathfrak{q})$ belongs to $\mathrm{Ker}(\delta) \cap \mathrm{Ker}(\mathbf{R}\Gamma_W)$, it follows that

$$\mathrm{Hom}_{\mathcal{T}}(X, E_R(R/\mathfrak{q})[n]) = \mathrm{Hom}_{\mathcal{T}}(\mathbf{R}\Gamma_W(X), E_R(R/\mathfrak{q})[n]) = 0$$

for any integer n. Then by the above triangle we have

$$\mathrm{Hom}_{\mathcal{T}}(V, E_R(R/\mathfrak{q})[n]) = 0$$

for any integer n. This is equivalent to that $\mathbf{R}\mathrm{Hom}_R(V, E_R(R/\mathfrak{q})) \cong 0$. In fact, the n-th cohomology module of $\mathbf{R}\mathrm{Hom}_R(V, E_R(R/\mathfrak{q}))$ is just $\mathrm{Hom}_{\mathcal{T}}(V, E_R(R/\mathfrak{q})[n]) = 0$. Since $V \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$, Lemma 11(1) forces $V \cong 0$, therefore $X \cong \mathbf{R}\Gamma_W(X)$. Hence we have $X \in \mathrm{Im}(\mathbf{R}\Gamma_W)$ as desired. ■

(5th step) : For the final step of the proof, we show the inclusion $\mathrm{Im}(\delta) \supseteq \mathrm{Im}(\mathbf{R}\Gamma_W)$.

Let $X \in \mathrm{Im}(\mathbf{R}\Gamma_W)$. Then there are triangles $\delta(X) \to X \to Y \to \delta(X)[1]$ with $Y \in \mathrm{Ker}(\delta)$, and $\mathbf{R}\Gamma_W(Y) \to Y \to V \to \mathbf{R}\Gamma_W(Y)[1]$ with $V \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$. Let $\mathfrak{p}$ be an arbitrary prime ideal belonging to W. Similarly to the 4th step, since $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta) \cap \mathrm{Im}(\mathbf{R}\Gamma_W)$, we see that $\mathrm{Hom}_{\mathcal{T}}(E_R(R/\mathfrak{p})[n], Y) = \mathrm{Hom}_{\mathcal{T}}(E_R(R/\mathfrak{p})[n], V) = 0$ for any integer n, hence we have $\mathrm{Hom}_{\mathcal{T}}(E_R(R/\mathfrak{p})[n], \mathbf{R}\Gamma_W(Y)) = 0$ for any n. This shows $\mathbf{R}\mathrm{Hom}_R(E_R(R/\mathfrak{p}), \mathbf{R}\Gamma_W(Y)) = 0$, then by Lemma 11(2) we have $\mathbf{R}\Gamma_W(Y) = 0$. Thus $Y \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$. Then, since $(\mathrm{Im}(\mathbf{R}\Gamma_W), \mathrm{Ker}(\mathbf{R}\Gamma_W))$ is a stable t-structure, the morphism $X \to Y$ in the triangle $\delta(X) \to X \to Y \to \delta(X)[1]$ is zero. It then follows that $\delta(X) \cong X \oplus Y[-1]$. Since there is no nontrivial morphisms $\delta(X) \to Y[-1]$ in $\mathcal{T}$, it is concluded that $\delta(X) \cong X$, hence $X \in \mathrm{Im}(\delta)$ as desired, and the proof is completed. □

4. Lattice structure of the set of abstract local cohomology functors

We consider the following sets.

Definition 12. (1) We denote by $\mathbb{S}(R)$ the set of all left exact radical functors on R-Mod.
(2) We denote by $\mathbb{A}(R)$ the set of the isomorphism classes $[\delta]$ where δ ranges over all abstract local cohomology functors on $\mathcal{D}^+(R\text{-Mod})$.
(3) We denote by $\mathrm{sp}(R)$ the set of all specialization closed subsets of $\mathrm{Spec}(R)$.

All these sets are bijectively corresponding to one another. Actually we can define mappings among these sets. First of all, we are able to give a mapping

$$\mathbb{S}(R) \longrightarrow \mathrm{sp}(R) \quad : \quad \gamma \mapsto W_\gamma,$$

which has the inverse mapping

$$\mathrm{sp}(R) \longrightarrow \mathbb{S}(R) \quad : \quad W \mapsto \varGamma_W,$$

where $W_\gamma = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \gamma(R/\mathfrak{p}) = R/\mathfrak{p}\}$. We also have a mapping

$$\mathbb{S}(R) \longrightarrow \mathbb{A}(R) \quad : \quad \gamma \mapsto [\mathbf{R}\gamma],$$

which is surjective by Theorem 8. It is injective as well. In fact, since $\gamma(M) = H^0(\mathbf{R}\gamma(M))$ for $\gamma \in \mathbb{S}(R)$ and $M \in R$-Mod, γ is uniquely determined by $\mathbf{R}\gamma$.

Furthermore, we can see that these sets have complete lattice structure as follows. If $\{W_\lambda \mid \lambda \in \Lambda\}$ is a set of specialization closed subsets of $\mathrm{Spec}(R)$, then $\bigcap_\lambda W_\lambda$ and $\bigcup_\lambda W_\lambda$ are also specialization closed subset. By this reason $\mathrm{sp}(R)$ is a complete lattice.

By above correspondences, we can define $\bigcap$ and $\bigcup$ for any subsets of $\mathbb{S}(R)$. Actually, if $\{\gamma_\lambda \mid \lambda \in \Lambda\}$ is a set of elements in $\mathbb{S}(R)$, then $\gamma := \bigcap_\lambda \gamma_\lambda$ (resp. $\delta := \bigcup_\lambda \gamma_\lambda$) is well-defined as an element of $\mathbb{S}(R)$ so that $W_\gamma = \bigcap_\lambda W_{\gamma_\lambda}$ (resp. $W_\delta = \bigcup_\lambda W_{\gamma_\lambda}$). In this way we have shown that $\mathbb{S}(R)$ has a structure of complete lattice and the bijective mapping $\mathrm{sp}(R) \to \mathbb{S}(R)$ gives an isomorphism as lattices.

We can define a lattice structure as well on the set $\mathbb{A}(R)$ so that the bijection $\mathbb{A}(R) \cong \mathbb{S}(R)$ is an isomorphism as complete lattices. More precisely, we define the order on $\mathbb{A}(R)$ by

$$[\mathbf{R}\gamma_1] \subseteq [\mathbf{R}\gamma_2] \iff \gamma_1 \subseteq \gamma_2$$

for $\gamma_1, \gamma_2 \in \mathbb{S}(R)$. Notice that $\bigcap_\lambda [\mathbf{R}\gamma_\lambda] = [\mathbf{R}(\bigcap_\lambda \gamma_\lambda)]$, and $\bigcup_\lambda [\mathbf{R}\gamma_\lambda] = [\mathbf{R}(\bigcup_\lambda \gamma_\lambda)]$.

Summing all up we have the following result.

Theorem 13. *The mapping $\mathbb{S}(R) \to \mathbb{A}(R)$ which maps γ to $[\mathbf{R}\gamma]$ (resp. $\mathrm{sp}(R) \to \mathbb{A}(R)$ which sends W to $[\mathbf{R}\varGamma_W]$) gives an isomorphism of complete lattices.*

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