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2011 年 1 月
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Organizing Committee of the Symposium on Ring Theory and Representation Theory

The Symposium on Ring Theory and Representation Theory has been held annually in Japan and the Proceedings have been published by the organizing committee. The first Symposium was organized in 1968 by H. Tominaga, H. Tachikawa, M. Harada and S. Endo. After their retirement, a new committee was organized in 1997 for managing the Symposium. The present members of the committee are H. Asashiba (Shizuoka Univ.), S. Ikehata (Okayama Univ.), S. Kawata (Osaka City Univ.), M. Sato (Yamanashi Univ.) and K. Yamagata (Tokyo Univ. of Agriculture and Technology).

The Proceedings of each Symposium is edited by program organizer. Anyone who wants these Proceedings should ask to the program organizer of each Symposium or one of the committee members.

The Symposium in 2011 will be held at Saitama University for Sep. 25 (Sun.)–27 (Tue.) and the program will be arranged by O. Iyama (Nagoya Univ.).

Concerning several information on ring theory and representation theory of group and algebras containing schedules of meetings and symposiums as well as ring mailing list service for registered members, you should refer to the following homepage, which is arranged by M. Sato (Yamanashi Univ.):

http://fuji.cec.yamanashi.ac.jp/˜ring/ (in Japanese)
(Mirror site: www.cec.yamanashi.ac.jp/˜ring/)

http://fuji.cec.yamanashi.ac.jp/˜ring/japan/ (in English)

Masahisa Sato
Yamanashi, Japan
December, 2010

The memory of Professor Goro Azumaya

Picture taken in 1995

Goro Azumaya (February 26, 1920 – July 8, 2010)
His advisor was Shokichi Iyanaga.
He was Professor at Indiana University after Hokkaido University
and introduced the notion of Azumaya algebra in 1951.
The Krull-Schmidt-Azumaya Theorem
is one of the most important theorem in Mathematics.
He got the second "Chunichi-Culture Award"
with Professor Tadasi Nakayama in 1949.

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Preface

The 43rd Symposium on Ring Theory and Representation Theory was held at Naruto University of Education on September 10th – 12th, 2010. The symposium and this proceedings are financially supported by Kunio Yamagata (Tokyo University of Agriculture and Technology) JSPS Grant-in-Aid for 2010 Scientific Research (B), No. 21340003 and Jun-ichi Miyachi (Tokyo Gakugei University) JSPS Grant-in-Aid for 2010 Scientific Research (C), No. 22540042.

This volume consists of the articles presented at the symposium. We would like to thank all speakers and coauthors for their contributions.

We would also like to express our thanks to all the members of the organizing committee (Professors Hideto Asashiba, Shûichi Ikehata, Shigeto Koshitani, Masahisa Sato and Kunio Yamagata) for their helpful suggestions concerning the symposium. Finally we would like to express our gratitude to Professor Yasuyuki Hirano and students of Naruto University of Education who contributed in the organization of the symposium.

Hiroaki Komatsu
Soja
January, 2011

第 43 回環論および表現論シンポジウム

プログラム

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Prime factor rings of Ore extensions over a commutative Dedekind domain

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Graded Morita equivalences for AS-regular algebras

14:30 – 15:15　大関 一秀（明治大学）
The first Hilbert coefficients of parameters

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Modules left orthogonal to modules of finite projective dimension

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Graded Morita equivalences for AS-regular algebras

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The first Hilbert coefficients of parameters

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Rings with indecomposable right modules uniform

REFLECTION FOR SELFINJECTIVE ALGEBRAS

HIROKI ABE

Abstract. We introduce the notion of reflections for selfinjective algebras and determine the transformations of Brauer trees associated with reflections. In particular, we provide a way to transform every Brauer tree into a Brauer line.

Reflection functors introduced in [4] are induced by transformations of the quiver making a certain sink vertex changed into a source vertex. Let Λ be a finite dimensional algebra over a field K. In [3], it was shown that reflection functors are of the form $\mathrm{Hom}_\Lambda(T, -)$ with T a certain type of tilting modules. Let $P_1, \cdots, P_n$ be a complete set of nonisomorphic indecomposable projective modules in mod-Λ, the category of finitely generated right Λ-modules. Set $I = \{1, \cdots, n\}$. Assume that there exists a simple projective module $S \in$ mod-Λ which is not injective. Take $t \in I$ with $P_t \cong S$ and set

$$T = T_1 \oplus \tau^{-1}S \quad \text{with} \quad T_1 = \bigoplus_{i \in I \setminus \{t\}} P_i,$$

where τ denotes the Auslander-Reiten translation. Then T is a tilting module, called an APR-tilting module, and $\mathrm{Hom}_\Lambda(T, -)$ is a reflection functor.

In [5], APR-tilting modules were generalized as follows. Assume that there exists a simple module $S \in$ mod-Λ with $\mathrm{Ext}^1_\Lambda(S, S) = 0$ and $\mathrm{Hom}_\Lambda(D\Lambda, S) = 0$, where $D = \mathrm{Hom}_K(-, K)$. Let P_t be the projective cover of S and let T be the same as above. Then T is a tilting module, called a BB-tilting module. We are interested in a minimal projective presentation of T, which is a two-term tilting complex. Take a minimal injective presentation $0 \to S \to E^0 \xrightarrow{f} E^1$ and define a complex $E^\bullet$ as the mapping cone of $f : E^0 \to E^1$. Then $\mathrm{Hom}^\bullet_\Lambda(D\Lambda, E^\bullet)$ is a minimal projective presentation of $\tau^{-1}S$ and hence

$$T^\bullet = T_1 \oplus \mathrm{Hom}^\bullet_\Lambda(D\Lambda, E^\bullet)$$

is a minimal projective presentation of T. In this note, we demonstrate that this type of tilting complexes play an important role in the theory of derived equivalences for selfinjective algebras.

Let K be a commutative artinian local ring and Λ an Artin K-algebra, i.e., Λ is a ring endowed with a ring homomorphism $K \to \Lambda$ whose image is contained in the center of Λ and Λ is finitely generated as a K-module. We always assume that Λ is connected, basic and not simple. We denote by mod-Λ the category of finitely generated right Λ-modules and by $\mathcal{P}_\Lambda$ the full subcategory of mod-Λ consisting of projective modules. For a module $M \in$ mod-Λ, we denote by $P(M)$ (resp., $E(M)$) the projective cover (resp., injective

The detailed version of this note has been submitted for publication elsewhere.

envelope) of M. We denote by $\mathcal{K}(\text{mod-}\Lambda)$ the homotopy category of cochain complexes over mod-Λ and by $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_\Lambda)$ the full triangulated subcategory of $\mathcal{K}(\text{mod-}\Lambda)$ consisting of bounded complexes over $\mathcal{P}_\Lambda$. We consider modules as complexes concentrated in degree zero.

Throughout the rest of this note, we assume that Λ is selfinjective. Let $S \in \text{mod-}\Lambda$ be a simple module with $\mathrm{Ext}^1_\Lambda(S, S) = 0$ and $E(S) \cong P(S)$. Note that $E(S) \cong P(S)$ if and only if $\mathrm{Hom}_\Lambda(D\Lambda, S) \cong S$, where D denotes the Matlis dual over K. Take a minimal injective presentation $0 \to S \to E^0 \xrightarrow{f} E^1$ and define a complex $E^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_\Lambda)$ as the mapping cone of $f : E^0 \to E^1$. Note that E^1 is the 0th term of $E^\bullet$ and E^0 is the (-1)th term of $E^\bullet$. Let $P_1, \cdots, P_n$ be a complete set of nonisomorphic indecomposable modules in $\mathcal{P}_\Lambda$ and set $I = \{1, \cdots, n\}$. We assume that $n > 1$. Take $t \in I$ with $P_t \cong P(S)$ and set

$$T^\bullet = T_1 \oplus E^\bullet \quad \text{with} \quad T_1 = \bigoplus_{i \in I \setminus \{t\}} P_i.$$

The following holds.

Theorem 1. *The complex $T^\bullet$ is a tilting complex for Λ and $\mathrm{End}_{\mathcal{K}(\text{mod-}\Lambda)}(T^\bullet)$ is a selfinjective Artin K-algebra whose Nakayama permutation coincides with that of Λ.*

Definition 2. The derived equivalence induced by the tilting complex $T^\bullet$ is said to be *the reflection for Λ at t*. Sometimes, we also say that $\mathrm{End}_{\mathcal{K}(\text{mod-}\Lambda)}(T^\bullet)$ is the reflection of Λ at t.

We will apply Theorem 1 to Brauer tree algebras and determine the transformations of Brauer tree algebras induced by reflections. We assume that K is an algebraically closed field. Recall that a Brauer tree (B, v, m) consists of a finite tree B, called the underlying tree, together with a distinguished vertex v, called the exceptional vertex and a positive integer m, called the multiplicity. In case $m = 1$, (B, v, m) is identified with the underlying tree B and is called a Brauer tree without exceptional vertex. The pair of the number of edges of B and the multiplicity m is said to be the numerical invariants of (B, v, m). Each Brauer tree determines a symmetric K-algebra Λ up to Morita equivalence (see [2] for details), called a Brauer tree algebra, which is given as the path algebra defined by some quiver with relations $(\Lambda_0, \Lambda_1, \rho)$, where Λ_0 is the set of vertices, Λ_1 is the set of arrows between vertices and ρ is the set of relations (see [6] for details). We have the following.

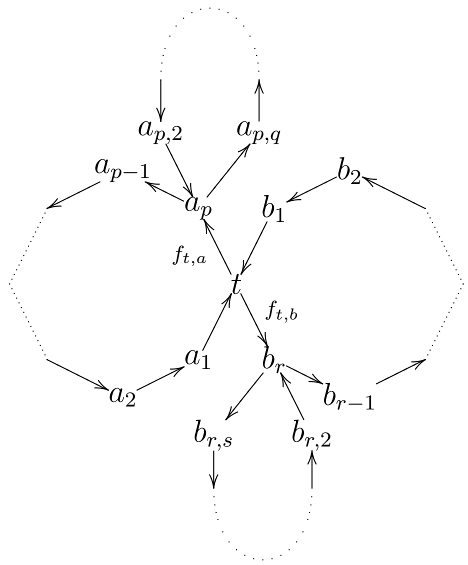
Remark 3. Let Λ be a Brauer tree algebra.

(1) Every ring Γ derived equivalent to Λ is a Brauer tree algebra having the same numerical invariants as Λ ([7, Theorem 4.2]).

(2) For any simple module $S \in \text{mod-}\Lambda$ we have $E(S) \cong P(S)$.

Throughout the rest of this note, we deal only with Brauer trees without exceptional vertex. Let Λ be a Brauer tree algebra, $(\Lambda_0, \Lambda_1, \rho)$ the quiver with relations of Λ and

$t \in \Lambda_0$. We consider the following cycles in $(\Lambda_0, \Lambda_1, \rho)$:



with $p, q, r, s \geq 0$, where $a_{p,1} = a_p$ and $b_{r,1} = b_r$ in case $p, r \geq 1$. We denote by S_t the simple module corresponding to t and by P_t the projective cover of S_t.

Lemma 4. *The following hold.*

(1) *We have a minimal injective presentation*

$$0 \to S_t \to P_t \xrightarrow{f} P_{a_p} \oplus P_{b_r} \text{ with } f = \begin{pmatrix} f_{t,a} \\ f_{t,b} \end{pmatrix}.$$

(2) *For any $t \in \Lambda_0$, we have* $\mathrm{Ext}^1_\Lambda(S_t, S_t) = 0$.

Take a minimal injective presentation $0 \to S_t \to E^0_t \xrightarrow{f} E^1_t$ and define a complex $E^\bullet_t$ as the mapping cone of $f : E^0_t \to E^1_t$. Set

$$T^\bullet_t = T_1 \oplus E^\bullet_t \quad \text{with} \quad T_1 = \bigoplus_{i \in \Lambda_0 \setminus \{t\}} P_i.$$

Then $T^\bullet_t$ is a tilting complex and $\mathrm{End}_{\mathcal{K}(\text{mod-}\Lambda)}(T^\bullet_t)$ is the reflection of Λ at t. Set $\Gamma = \mathrm{End}_{\mathcal{K}(\text{mod-}\Lambda)}(T^\bullet_t)$ and let $(\Gamma_0, \Gamma_1, \sigma)$ be the quiver with relations of Γ. Note that $\Gamma_0 = (\Lambda_0 \setminus \{t\}) \cup \{t'\}$, where t' is the vertex corresponding to $E^\bullet_t$. Since Γ is a Brauer tree algebra, the relations σ is determined automatically by Γ_0 and Γ_1. To determine Γ_1, we need the next lemma.

Lemma 5. *The following hold.*

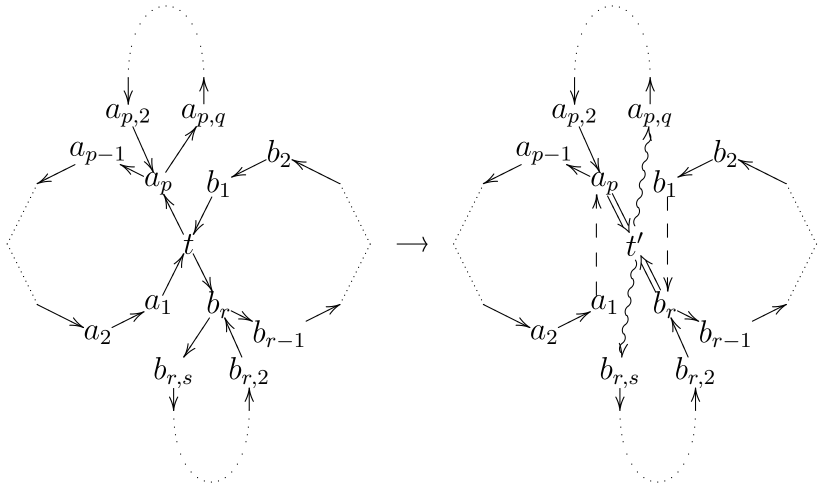
(1) *There exist* $\zeta_{a_p} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(P_{a_p}, E^\bullet_t)$ *with* $\zeta_{a_p} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$*and* $\zeta_{b_r} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(P_{b_r}, E^\bullet_t)$ *with* $\zeta_{b_r} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$.

(2) *There exist* $\eta_{a_{p,q}} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(E^\bullet_t, P_{a_{p,q}})$ *with* $\eta_{a_{p,q}} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$*and* $\eta_{b_{r,s}} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(E^\bullet_t, P_{b_{r,s}})$ *with* $\eta_{b_{r,s}} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$.

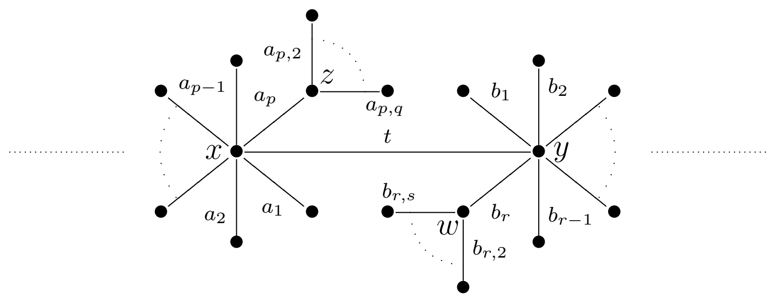
(3) *There exist* $\theta_{a_p} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(P_{a_1}, P_{a_p})$ *with* $\theta_{a_p} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$ *and* $\theta_{b_r} \in \mathrm{Hom}_{\mathcal{K}(\text{mod-}\Lambda)}(P_{b_1}, P_{b_r})$ *with* $\theta_{b_r} \in \mathrm{rad}(\Gamma) \setminus \mathrm{rad}^2(\Gamma)$.

According to Lemma 5, we have the following new arrows in Γ_1. We denote by $\Longrightarrow$ the arrows defined by ζ_*, by $\rightsquigarrow$ the arrows defined by η_* and by $- - \rightarrow$ the arrows defined by θ_*. In the next theorem, the left hand side diagram denotes cycles in $(\Lambda_0, \Lambda_1, \rho)$ and the right hand side diagram denotes cycles in $(\Gamma_0, \Gamma_1, \sigma)$.

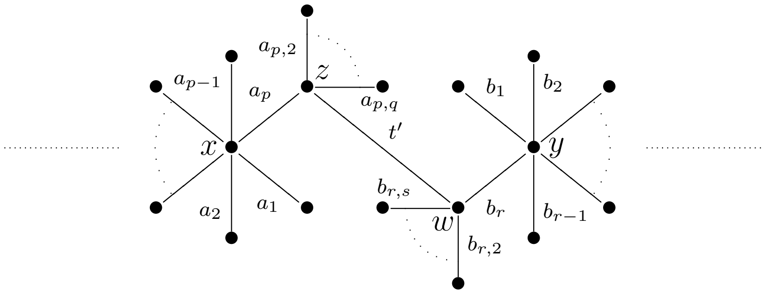
Theorem 6. *The reflection for Λ at t gives rise to the following transformation:*



Let Λ be determined by a Brauer tree B whose edges are identified with the vertices of $(\Lambda_0, \Lambda_1, \rho)$. We will describe a way to transform B into a Brauer tree B' determining Γ. Consider the tree



with $p, q, r, s \geq 0$, where $a_p = a_{p,1}, b_r = b_{r,1}$ in case $p, r \geq 1$. Turn the edge t anti-clockwise around the vertex x and select the edge a_p which t first meets. Then select the vertex z of the edge a_p different from x. Similarly, turn the edge t anti-clockwise around the vertex y and select the edge b_r which t first meets. Then select the vertex w of the edge b_r different from y. Add a new edge t' connecting the vertices z and w, and remove the edge t. As a consequence, we get the following Brauer tree B':



Corollary 7. *The Brauer tree B' determines Γ.*

Corollary 8 (cf. [1, Theorem 3.7])**.** *There exists a sequence of Brauer tree algebras $\Lambda = \Delta_0, \Delta_1, \cdots, \Delta_l$ such that Δ_{i+1} is the reflection of Δ_i at a suitable vertex for $0 \le i < l$ and Δ_l is a Brauer line algebra, i.e., the path algebra defined by the quiver*

$$1 \underset{\beta_1}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_2}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_{n-2}}{\overset{\alpha_{n-2}}{\rightleftarrows}} n-1 \underset{\beta_{n-1}}{\overset{\alpha_{n-1}}{\rightleftarrows}} n$$

with relations

$$\alpha_{i+1}\alpha_i = \beta_i\beta_{i+1} = 0, \quad \alpha_i\beta_i = \beta_{i+1}\alpha_{i+1}$$

for $1 \le i < n-1$, where n is the number of vertices of $(\Lambda_0, \Lambda_1, \rho)$.

References

[1] H. Abe and M. Hoshino, On derived equivalences for selfinjective algebras, Comm. Algebra **34** (2006), 4441–4452.

[2] J. L. Alperin, *Local Representation Theory*, Cambridge Studies in Advanced Mathematics **11**, Cambridge University Press, Cambridge, 1986.

[3] M. Auslander, M. I. Platzeck and I. Reiten, Coxeter functors without diagrams, Trans. Amer. Math. Soc. **250** (1979), 1–46.

[4] I. N. Bernstein, I. M. Gelfand and V. A. Ponomarev, Coxeter functors and Gabriel's theorem, Uspechi Mat. Nauk. **28** (1973), 19–38 = Russian Math. Surveys **28** (1973), 17–32.

[5] S. Brenner and M. C. R. Butler, Generalizations of the Bernstein-Gelfand-Ponomarev reflection functors, in: *Representation theory II*, 103–169, Lecture Notes in Math. **832**, Springer, 1980.

[6] P. Gabriel and C. Riedtmann, Group representations without groups, Comment. Math. Helv. **54** (1979), 240–287.

[7] J. Rickard, Derived categories and stable equivalence, J. Pure Appl. Algebra **61** (1989), 303–317.

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MODULES LEFT ORTHOGONAL TO MODULES OF FINITE PROJECTIVE DIMENSION

TOKUJI ARAYA, KEI-ICHIRO IIMA AND RYO TAKAHASHI

Abstract. In this proceeding, we characterize several properties of commutative noetherian local rings in terms of the left perpendicular category of the category of finitely generated modules of finite projective dimension. As an application we prove that a local ring is regular if (and only if) there exists a strong test module for projectivity having finite projective dimension.

Key Words: perpendicular category, projective dimension, semidualizing module, totally reflexive module, strong test module for projectivity.

2010 *Mathematics Subject Classification*: Primary 13C60; Secondary 13D05, 13H10.

1. Introduction

Throughout this proceeding, let R be a commutative noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. All modules considered in this proceeding are assumed to be finitely generated.

An R-module C is said to be *semidualizing* if the natural homomorphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^i_R(C, C) = 0$ for all $i > 0$. A semidualizing module admits a duality property, which has been defined by Foxby [5] and Golod [6]. A free module of rank one and a dualizing module are semidualizing modules. Various homological dimensions with respect to a fixed semidualizing R-module C are invented and investigated (cf. [2, 6, 9]). Among them, the *C-projective dimension* of a nonzero R-module M, denoted by $C\text{-}\mathrm{pd}_R M$, is defined as the infimum of integers n such that there exists an exact sequence of the form

$$0 \to C^{b_n} \to C^{b_{n-1}} \to \cdots \to C^{b_1} \to C^{b_0} \to M \to 0,$$

where each b_i is a positive integer.

An R-module M is called *totally C-reflexive*, where C is a semidualizing R-module, if the natural homomorphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism and $\mathrm{Ext}^i_R(M, C) = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C) = 0$ for all $i > 0$. The *complete intersection dimension* of M, which has been introduced in [4], is defined as the infimum of $\mathrm{pd}_S(M \otimes_R R') - \mathrm{pd}_S R'$ where $R \to R' \leftarrow S$ runs over all quasi-deformations. Here, a diagram $R \xrightarrow{f} R' \xleftarrow{g} S$ of homomorphisms of local rings is said to be a quasi-deformation if f is faithfully flat and g is a surjection whose kernel is generated by an S-regular sequence.

We denote by $\mathrm{mod}\, R$ the category of (finitely generated) R-modules. Let $\mathcal{G}_C(R)$, $\mathcal{I}(R)$, and $\mathrm{add}\, C$ denote the full subcategories of $\mathrm{mod}\, R$ consisting of all totally C-reflexive

The detailed version of this proceeding will be submitted for publication elsewhere.

R-modules, consisting of all R-modules of complete intersection dimension zero, and consisting of all direct summands of finite direct sums of copies of C, respectively. Let $\mathcal{X}_C(R)$ be the *left perpendicular category* of the category of R-modules of finite C-projective dimension, that is, the subcategory of $\operatorname{mod} R$ consisting of all R-modules X satisfying $\operatorname{Ext}_R^1(X, M) = 0$ for each R-module M of finite C-projective dimension. We write $\mathcal{G}(R) = \mathcal{G}_R(R)$ and $\mathcal{X}(R) = \mathcal{X}_R(R)$. There are inclusion relations of subcategories of $\operatorname{mod} R$:

$$\mathcal{X}(R) \supset \mathcal{G}(R) \supset \mathcal{I}(R) \supset \operatorname{add} R,$$
$$\mathcal{X}_C(R) \supset \mathcal{G}_C(R) \supset \operatorname{add} C, \ \operatorname{add} R.$$

The main purpose of this proceeding is to find out what property is characterized by the equalities of $\mathcal{X}(R)$ (respectively, $\mathcal{X}_C(R)$) and each of $\mathcal{G}(R)$, $\mathcal{I}(R)$, $\operatorname{add} R$ (respectively, each of $\mathcal{G}_C(R)$, $\operatorname{add} C$, $\operatorname{add} R$). The main result of this proceeding is the following theorem.

Theorem 1. *Let R be a commutative noetherian local ring.*

(1) The following are equivalent for a semidualizing R-module C.

(a) C is dualizing.

(b) $\mathcal{X}_C(R) = \mathcal{G}_C(R)$ holds.

If this is the case, then R is Cohen-Macaulay.

(2) The following are equivalent.

(a) R is Gorenstein.

(b) $\mathcal{X}(R) = \mathcal{G}(R)$ holds.

(3) The following are equivalent.

(a) R is a complete intersection.

(b) $\mathcal{X}(R) = \mathcal{I}(R)$ holds.

(4) The following are equivalent.

(a) R is regular.

(b) $\mathcal{X}(R) = \operatorname{add} R$ holds.

(c) $\mathcal{X}_C(R) = \operatorname{add} C$ holds for some semidualizing R-module C.

(d) $\mathcal{X}_C(R) = \operatorname{add} R$ holds for some semidualizing R-module C.

On the other hand, the notion of a strong test module for projectivity has been introduced and studied by Ramras [8]. An R-module M is called a *strong test module for projectivity* if every R-module N with $\operatorname{Ext}_R^1(N, M) = 0$ is projective, or equivalently, free. The residue field k is a typical example of a strong test module for projectivity. Ramras shows that the maximal ideal $\mathfrak{m}$ is a strong test module for projectivity. He also proves that every strong test module for projectivity has depth at most one. Using the rigidity theorem for Tor modules, Jothilingam [7] proves that when R is a regular local ring, every R-module of depth at most one is a strong test module for projectivity. Our Theorem 1 yields that the converse of this Jothilingam's result also holds true.

Corollary 2. *The following seven conditions are equivalent.*

(1) R is regular.

(2) Every R-module of depth at most one is a strong test module for projectivity.

(3) Every R-module of depth zero is a strong test module for projectivity.

(4) Every R-module of depth zero and of finite projective dimension is a strong test module for projectivity.

(5) There exists a strong test R-module for projectivity of depth zero and of finite projective dimension.

(6) There exists a strong test R-module for projectivity of finite projective dimension.

(7) There exist a semidualizing R-module C and a strong test R-module for projectivity of finite C-projective dimension.

Now let us give a proof of the corollary.

Proof. (1) $\Rightarrow$ (2): This implication follows from [7, Theorem 1].

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) and (5) $\Rightarrow$ (6): These implications are obvious.

(4) $\Rightarrow$ (5): Take a maximal R-regular sequence $x_1, x_2, \dots, x_t$. Then $R/(x_1, x_2, \dots, x_t)$ is an R-module of depth zero and of finite projective dimension.

(6) $\Rightarrow$ (7): Letting $C = R$ shows this implication.

(7) $\Rightarrow$ (1): Let M be a strong test R-module for projectivity with $C\text{-pd}_R\, M < \infty$. Let N be a module in $\mathcal{X}_C(R)$. Then we have $\mathrm{Ext}^1_R(N, M) = 0$. Since M is a strong test module for projectivity, N is a free R-module. Thus $\mathcal{X}_C(R)$ is contained in $\operatorname{add} R$. Therefore $\mathcal{X}_C(R) = \operatorname{add} R$, and R is regular by Theorem 1(4). $\square$

References

[1] T. Araya; K. Iima; R. Takahashi, *On the left perpendicular category of the modules of finite projective dimension*, arXiv: 1008.3680v1 [math.AC] 22 Aug 2010.

[2] T. Araya; R. Takahashi; Y. Yoshino, *Homological invariants associated to semi-dualizing bimodules*, J. Math. Kyoto Univ. **45** (2005), no. 2, 287–306.

[3] M. Auslander; M. Bridger, *Stable module theory*, Memoirs of the American Mathematical Society, No. 94, American Mathematical Society, Providence, R.I., 1969.

[4] L. L. Avramov; V. N. Gasharov; I. V. Peeva, *Complete intersection dimension.* (English summary) Inst. Hautes Études Sci. Publ. Math. No. 86 (1997), 67–114 (1998).

[5] H.-B. Foxby, *Gorenstein modules and related modules*, Math. Scand. **31** (1972), 267–284 (1973).

[6] E. S. Golod, *G-dimension and generalized perfect ideals, Algebraic geometry and its applications*, Trudy Mat. Inst. Steklov. **165** (1984), 62–66.

[7] P. Jothilingam, *Test modules for projectivity*, Proc. Amer. Math. Soc. **94** (1985), no. 4, 593–596.

[8] M. Ramras, *On the vanishing of* Ext, Proc. Amer. Math. Soc. **27** (1971), 457–462.

[9] R. Takahashi; D. White, *Homological aspects of semidualizing modules*, Math. Scand. **106** (2010), no. 1, 5–22.

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WEAKLY SECTIONAL PATHS AND THE SHAPES OF AUSLANDER-REITEN QUIVERS

TAKAHIKO FURUYA

Abstract. We introduce weakly sectional paths in the Auslander-Reiten quiver of an artin algebra, which are generalizations of sectional paths as well as pre-sectional paths of Liu [10]. We show that there are n-irreducible maps lying on a weakly sectional path of length n such that their composite does not fall into the $(n+1)$-th power of the radical of the module category. As a corollary we see that there is no weakly sectional oriented cycle in the Auslander-Reiten quiver.

1. Introduction

Throughout this report let K be a commutative Artinian ring and A an artin algebra over K ([1]). Denote by $\mathrm{mod}\, A$ the category of all finitely generated right A-modules and by Γ_A the Auslander-Reiten quiver of A. We also denote by τ the Auslander-Reiten translation $D\mathrm{Tr}$ in $\mathrm{mod}\, A$ and by $\mathfrak{R}$ the Jacobson radical of $\mathrm{mod}\, A$.

Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a path in Γ_A. Then an integer i with $1 \leq i \leq n-1$ is a hook of Ω, if $X_{i+1} \simeq \tau X_{i-1}$ holds. Moreover Ω is called a sectional path if Ω has no hook.

Recall from [1] that a map f in $\mathrm{mod}\, A$ is called an irreducible map, if it satisfies the following conditions:

(1) f is neither a section nor a retraction.
(2) If $f = hg$ for some maps g and h in $\mathrm{mod}\, A$, then either g is a section or h is a retraction.

Let $f : X \to Y$ be a map in $\mathrm{mod}\, A$ with X and Y indecomposable. Then it is well-known, as a connection between irreducible maps and $\mathfrak{R}$, that f is an irreducible map if and only if f belongs to $\mathfrak{R}(X,Y)\backslash\mathfrak{R}^2(X,Y)$. Based on this fact we study here the composite of irreducible maps lying on a certain path in Γ_A, which is called a weakly sectional path.

The following question is essential in the investigation of the composite of irreducible maps.

Question. For each $i = 1, \dots, n$, let $f_i : X_i \to X_{i-1}$ be an irreducible map in $\mathrm{mod}\, A$ with X_i and X_{i-1} indecomposable. When do

$$f_1 f_2 \cdots f_n \neq 0 \qquad \text{and} \qquad f_1 f_2 \cdots f_n \in \mathfrak{R}^{n+1}(X_n, X_0)$$

hold?

This question has been studied for two irreducible maps in Γ_A (i.e. for the case $n = 2$)

The detailed version of this paper will be submitted for publication elsewhere.

[4], for irreducible maps lying on an almost sectional path in Γ_A [5], and for irreducible maps lying on a path in a standard component in Γ_A [7].

On the other hand, K. Igusa and G. Todorov [9] proved the following fact, which is a pioneering result in the study of the composite of irreducible maps:

Theorem 1 ([9]). *Let $X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a sectional path in Γ_A. Then for all irreducible maps $f_i : X_i \to X_{i-1}$ $(i = 1, \ldots, n)$ we have $f_1 \cdots f_n \notin \mathfrak{R}^{n+1}(X_n, X_0)$.*

Also, in [10], S. Liu introduced the following path which is a generalization of a sectional path and proved an analogue for Theorem 1.

Definition 2 ([10]). Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a path in Γ_A. Then Ω is called a pre-sectional path, if, for each hook i of Ω, $\tau X_{i-1} \oplus X_{i+1}$ $(\simeq X_{i+1} \oplus X_{i+1})$ is a summand of the domain of the sink map for X_i.

Theorem 3 ([10]). *Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a pre-sectional path in Γ_A. Then there are irreducible maps $f_i : X_i \to X_{i-1}$ $(i = 1, \ldots, n)$ such that $f_1 \cdots f_n \notin \mathfrak{R}^{n+1}(X_n, X_0)$.*

The aim of this report is to introduce new paths called weakly sectional paths in Γ_A (Definition 4). These paths are clearly generalizations of sectional paths as well as pre-sectional paths. We generalize Theorem 3 and show, as a corollary, that there is no weakly sectional oriented cycle in Γ_A (Threorem 7 and Corollary 8).

2. Weakly sectional paths

In this section we define weakly sectional paths in Γ_A and give some examples of them.

First recall that the pair (d_{XY}, d'_{XY}) of integers is the valuation of an arrow $X \to Y$ in Γ_A if X appears d_{XY}-times in the domain of the sink map for Y, and Y appears d'_{XY}-times in the codomain of the source map for X. Let I be one of the sets $\{0, 1, \ldots, n\}$ $(n \geq 1)$, $\mathbb{N} \cup \{0\}$, or $\{0, -1, -2, \ldots\}$. Moreover, if Ω is a path $\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ in Γ_A where the set of the indices i of X_i is I, then we set

$$J_\Omega := \{j \in I \mid j \text{ is a hook in } \Omega \text{ with } d_{X_{j+1}X_j} = 1\}.$$

Definition 4 ([8]). Let $\Omega = \cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ be a path in Γ_A, where the set of the indices i of X_i is I. Then Ω is said to be a *weakly sectional path* in Γ_A, if there is a set of (non-zero) indecomposable modules $\{M_i\}_{i \in J_\Omega}$, called a *support* of Ω, such that

(1) $X_j \oplus M_j \oplus \tau X_{j-2}$ is a summand of the domain of the sink map for X_{j-1} for all $j \in J_\Omega$ such that $j - 2 \notin J_\Omega$. (Here, if $I = \{0, 1, \ldots, n\}$ or $I = \mathbb{N} \cup \{0\}$ and if $1 \in J_\Omega$, then define τX_{-1} to be an indecomposable module in $\operatorname{mod} A$.)
(2) $X_j \oplus M_j \oplus \tau X_{j-2} \oplus \tau M_{j-2}$ is a summand of the domain of the sink map for X_{j-1} for all $j \in J_\Omega$ such that $j - 2 \in J_\Omega$.
(3) $X_j \oplus \tau X_{j-2} \oplus \tau M_{j-2}$ is a summand of the domain of the sink map for X_{j-1} for all $j \in I \backslash J_\Omega$ such that $j - 2 \in J_\Omega$.

It is easy to see that any subpath of a weakly sectional path is also a weakly sectional path.

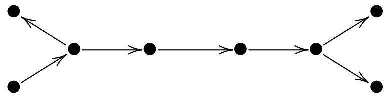
Remark 5. Let Ω be a path $\cdots \to X_{i+1} \to X_i \to X_{i-1} \to \cdots$ in Γ_A, where the set of indices i of X_i is I.

(1) Ω is a pre-sectional path if and only if $J_\Omega = \emptyset$.
(2) Suppose that Ω is a weakly sectional path with a support $\{M_i\}_{i \in J_\Omega}$. If $j \in J_\Omega$, then $j+1 \notin J_\Omega$.

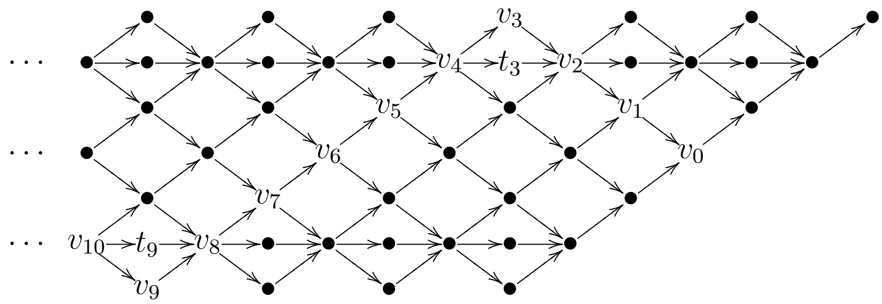
We now provide typical examples of weakly sectional paths.

Example 6. Suppose that K is an algebraically closed field.

(a) Let Δ be the quiver

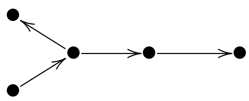


of the Euclidean type $\widetilde{\mathbb{D}}_7$. Then the pre-injective component $\mathcal{Q}$ of the path algebra $K\Delta$ is of the form:

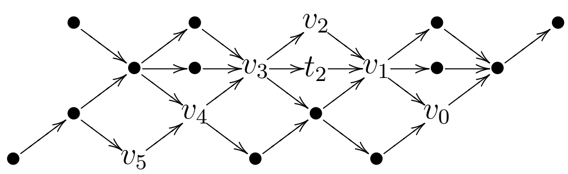


The infinite path $\cdots \to v_{n+1} \to v_n \to v_{n-1} \to \cdots \to v_1 \to v_0$ in $\mathcal{Q}$ is not a pre-sectional path but is a weakly sectional path with a support $\{t_j \mid j = 6i+3 \text{ for } i \geq 0\}$. (Note that, in this case, for each vertex v there is a weakly sectional path ending with v.)

(b) Let Δ be the quiver



of the Dynkin type $\mathbb{D}_5$. Then it is well-known that $K\Delta$ is of representation-finite, and the Auslander-Reiten quiver of $K\Delta$ is of the form:



The path $v_5 \to v_4 \to \cdots \to v_1 \to v_0$ is not a pre-sectional path but is a weakly sectional path with a support $\{t_2\}$.

3. Main result

Now, using same technique in the proof of Theorem 3 given in [10], we have the following result. (see [8] for the detail of proof).

Theorem 7 ([8]). *Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a weakly sectional path in Γ_A. Then there are irreducible maps $f_i : X_i \to X_{i-1}$ $(i = 1, \ldots, n)$ such that $f_1 \cdots f_n \in \mathfrak{R}^{n+1}(X_n, X_0)$.*

Using Harada-Sai lemma (see for example [1]) we immediately have the following.

Corollary 8. *There is no weakly sectional oriented cycle in Γ_A.*

Remark 9. It is shown by Bautista and Smalø [3] (see also [2]) that there is no sectional oriented cycle in Γ_A, and by Liu [10] that there is no pre-sectional oriented cycle in Γ_A.

In particular, if A is a finite-dimensional algebra over an algebraically closed field K, we have the following.

Corollary 10. *Let A be finite-dimensional algebra over an algebraically closed field K, and let $\mathcal{C}$ be a component in Γ_A such that the valuation of every arrow in $\mathcal{C}$ is trivial. Let $\Omega = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$ be a weakly sectional path in $\mathcal{C}$. Then for any irreducible maps $f_i : X_i \to X_{i-1}$ $(i = 1, \ldots, n)$ in $\mathcal{C}$ we have $f_1 \cdots f_n \notin \mathfrak{R}^{n+1}(X_n, X_0)$.*

Example 11. Consider again the paths of Example 6. Then since the path $\cdots \to v_{n+1} \to v_n \to v_{n-1} \to \cdots \to v_1 \to v_0$ of Example 6 (a) is weakly sectional, it follows by Corollary 10 that for all irreducible maps $f_i : v_i \to v_{i-1}$ $(i = 1, 2, \ldots)$ we have $f_m \cdots f_\ell \notin \mathfrak{R}^{m+1}(v_m, v_0)$ for any integers $\ell > m > 0$. Similarly since the path $v_5 \to v_4 \to \cdots \to v_1 \to v_0$ of Example 6 (b) is a weakly sectional path, it follows that, for all irreducible maps $f_i : v_i \to v_{i-1}$ $(i = 1, \ldots, 5)$, the composite $f_1 \cdots f_5$ is not in $\mathfrak{R}^6(v_5, v_0)$.

References

[1] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge studies in advanced mathematics **36**, Cambridge University Press, 1995.

[2] K. Bongartz, *On a result of Bautista and Smalø on cycles*, Comm. Algebra **11**, (1983) 2123–2124.

[3] R. Bautista and S. Smalø, *Nonexistent cycles*, Comm. Algebra **11** (1983), 1755–1767.

[4] C. Chaio, F. Coelho and S. Trepode, *On the composite of two irreducible morphisms in radical cube*, J. Algebra **312** (2007), 650–667.

[5] C. Chaio, F. Coelho and S. Trepode, *On the composite of irreducible morphisms in almost sectional paths*, J. Pure Appl. Algebra **212** (2008), 244–261.

[6] C. Chaio, M. Platzeck and S. Trepode, *On the degree of irreducible morphisms*, J. Algebra **281**(2004) 200–224.

[7] C. Chaio and S. Trepode, *The composite of irreducible morphisms in standard components*, J. Algebra **323** (2010), 1000–1011.

[8] T. Furuya, *Weakly sectional paths and the degrees of irreducible maps*, preprint.

[9] K. Igusa and G. Todorov, *A characterization of finite Auslander-Reiten quivers*, J. Algebra **89** (1984) 148–177.

[10] S. Liu, *Degrees of irreducible maps and the shapes of Auslander-Reiten quivers*, J. London Math. Soc. **45** (1992), 32–54.

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AUSLANDER-GORENSTEIN RESOLUTION

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Abstract. We introduce the notion of Auslander-Gorenstein resolution and show that a noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring.

1. Introduction

1.1. **Notation.** Let A be a ring. We denote by Mod-A the category of right A-modules and by mod-A the full subcategory of Mod-A consisting of finitely presented modules. We denote by $\mathcal{P}_A$ the full subcategory of mod-A consisting of projective modules. We denote by A^{op} the opposite ring of A and consider left A-modules as right A^{op}-modules. In particular, we denote by $\mathrm{Hom}_A(-,-)$ (resp., $\mathrm{Hom}_{A^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-A (resp., Mod-A^{op}). Sometimes, we use the notation M_A (resp., ${}_AM$) to stress that the module M considered is a right (resp., left) A-module. We denote by $\mathrm{Hom}^\bullet(-,-)$ the associated single complex of the double hom complex. As usual, we consider modules as complexes concentrated in degree zero. For an object X of an additive category $\mathcal{A}$ we denote by $\mathrm{add}(X)$ the full subcategory of $\mathcal{A}$ consisting of direct summands of finite direct sums of copies of X. For a commutative ring R, we denote by $\mathrm{Spec}(R)$ the set of prime ideals of R. For each $\mathfrak{p} \in \mathrm{Spec}(R)$ we denote by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p}$ and for each $M \in$ Mod-R we denote by $\mathrm{Supp}_R(M)$ the set of $\mathfrak{p} \in \mathrm{Spec}(R)$ with $M_\mathfrak{p} \neq 0$.

1.2. **Introduction.** In this note, a noetherian ring A is a ring which is left and right noetherian, and a noetherian R-algebra A is a ring endowed with a ring homomorphism $R \to A$, with R a commutative noetherian ring, whose image is contained in the center of A and A is finitely generated as an R-module. Note that a noetherian algebra is a noetherian ring.

Let R be a commutative Gorenstein local ring and A a noetherian R-algebra with $\mathrm{Ext}^i_R(A,R) = 0$ for $i \neq 0$. Set $\Omega = \mathrm{Hom}_R(A,R)$. Then proj dim ${}_A\Omega < \infty$ and proj dim $\Omega_A < \infty$ if and only if Ω_A is a tilting module in the sense of [12] (see Remark 4). In Section 2, we will show that inj dim ${}_AA \le \dim R + 1$ if and only if inj dim $A_A \le \dim R + 1$ (Theorem 5). In case inj dim $A_A = \dim R$, such an algebra A is called a Gorenstein algebra and extensively studied in [10]. In particular, a Gorenstein algebra is an Auslander-Gorenstein ring (see Definition 9). On the other hand, even if A is an Auslander-Gorenstein ring, it may happen that inj dim $A_A \neq \dim R$. For instance, if $A = \mathrm{T}_m(R)$, the ring of $m \times m$ upper triangular matrices over R, for $m \ge 2$, then A is an Auslander-Gorenstein ring with inj dim $A_A = \dim R + 1$ (see Example 16). Also, consider the case where R is a complete Gorenstein local ring of dimension one

The detailed version of this paper will be submitted for publication elsewhere.

and Λ is a noetherian R-algebra with $\mathrm{Ext}^i_R(\Lambda, R) = 0$ for $i \neq 0$. Denote by $\mathcal{L}_\Lambda$ the full subcategory of mod-Λ consisting of modules X with $\mathrm{Ext}^i_R(X, R) = 0$ for $i \neq 0$ and assume that $\mathcal{L}_\Lambda = \mathrm{add}(M)$ with $M \in$ mod-Λ non-projective. Then we know from [3] that $A = \mathrm{End}_\Lambda(M)$ is an Auslander-Gorenstein ring of global dimension two (see Example 15). These examples can be formulated as follows. If Ω admits a projective resolution $0 \to P^{-1} \to P^0 \to \Omega \to 0$ in mod-A^{op} with $P^0 \in \mathrm{add}(\Omega)$, then A is an Auslander-Gorenstein ring with inj dim $A_A \leq \dim R + 1$ (see Example 14), the converse of which holds true if R is complete (see Proposition 7).

Consider the case where Ω_A is a tilting module of arbitrary finite projective dimension. Take a projective resolution $P^\bullet \to \Omega$ in mod-A^{op}. Then, setting $Q^\bullet = \mathrm{Hom}^\bullet_R(P^\bullet, R)$, we have a right resolution $A \to Q^\bullet$ in mod-A such that every $Q^i \in$ mod-R is a reflexive module with $\mathrm{Ext}^j_R(\mathrm{Hom}_R(Q^i, R), R) = 0$ for $j \neq 0$, $\oplus_{i \geq 0} \mathrm{Hom}_R(Q^i, R) \in$ mod-A^{op} is a projective generator and proj dim $Q^i < \infty$ in mod-A for all $i \geq 0$ (Remark 6). We will show that A is an Auslander-Gorenstein ring if proj dim $Q^i \leq i$ in mod-A for all $i \geq 0$ and that the converse holds true if R is complete and $P^\bullet \to \Omega$ is a minimal projective resolution (Proposition 7). In Section 3, formulating these facts, we will introduce the notion of Auslander-Gorenstein resolution. Let R, A be noetherian rings. In this note, a right resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ in Mod-A is said to be an Auslander-Gorenstein resolution of A over R if the following conditions are satisfied: (1) every Q^i is an R-A-bimodule; (2) every $Q^i \in$ Mod-R^{op} is a finitely generated reflexive module with $\mathrm{Ext}^j_R(\mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R), R) = 0$ for $j \neq 0$; (3) $\oplus_{i \geq 0} \mathrm{Hom}_{R^{\mathrm{op}}}(Q^i, R) \in$ Mod-A^{op} is faithfully flat; and (4) flat dim $Q^i \leq i$ in Mod-A for all $i \geq 0$. We will show that A is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over R and if R is an Auslander-Gorenstein ring (Theorem 13). In Section 4, we will provide several examples of Auslander-Gorenstein resolution.

We refer to [6], [7], [2], [9] and so on for information on Auslander-Gorenstein rings. Also, we refer to [8] for standard homological algebra, to [11] for standard commutative ring theory.

2. Auslander-Gorenstein algebras

Throughout this section, R is a commutative noetherian ring with a minimal injective resolution $R \to I^\bullet$ and A is a noetherian R-algebra such that $R_\mathfrak{p}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq 0$. Set $\Omega = \mathrm{Hom}_R(A, R)$.

In this section, assuming R being a complete Gorenstein local ring, we will provide a necessary and sufficient condition for A to be an Auslander-Gorenstein ring (see Definition 9 below). We refer to [5] for commutative Gorenstein rings.

Definition 1 ([4])**.** A family of idempotents $\{e_\lambda\}_{\lambda \in \Lambda}$ is said to be orthogonal if $e_\lambda e_\mu = 0$ unless $\lambda = \mu$. An idempotent $e \in A$ is said to be primitive if eA_A is indecomposable and to be local if $eAe \cong \mathrm{End}_A(eA)$ is local. A ring A is said to be semiperfect if $1 = e_1 + \cdots + e_n$ in A with the e_i orthogonal local idempotents.

Remark 2. Assume that R is a complete local ring. Then every noetherian R-algebra A is semiperfect.

Definition 3 ([12]). A module $T \in \text{Mod-}A$ is said to be a tilting module if for some integer $m \geq 0$ the following conditions are satisfied:

(1) T admits a projective resolution $0 \to P^{-m} \to \cdots \to P^{-1} \to P^0 \to T \to 0$ in Mod-A with $P^{-i} \in \mathcal{P}_A$ for all $i \geq 0$.
(2) $\mathrm{Ext}^i_A(T,T) = 0$ for $i \neq 0$.
(3) A admits a right resolution $0 \to A \to T^0 \to T^1 \to \cdots \to T^m \to 0$ in Mod-A with $T^i \in \mathrm{add}(T)$ for all $i \geq 0$.

Remark 4. The following hold:

(1) A has Gorenstein dimension zero as an R-module, i.e., $A \xrightarrow{\sim} \mathrm{Hom}_R(\Omega, R)$ and $\mathrm{Ext}^i_R(\Omega, R) = 0$ for $i \neq 0$.
(2) $A \xrightarrow{\sim} \mathrm{End}_A(\Omega)$ and $A \xrightarrow{\sim} \mathrm{End}_{A^{\mathrm{op}}}(\Omega)^{\mathrm{op}}$ canonically.
(3) $\mathrm{Ext}^i_A(\Omega,\Omega) = \mathrm{Ext}^i_{A^{\mathrm{op}}}(\Omega,\Omega) = 0$ for $i \neq 0$.
(4) The following are equivalent:
   (i) proj dim $\Omega_A < \infty$ and proj dim ${}_A\Omega < \infty$;
   (ii) Ω_A is a tilting module with proj dim ${}_A\Omega =$ proj dim Ω_A;
   (iii) inj dim ${}_AA =$ inj dim $A_A < \infty$.

Theorem 5. *Assume that R is a Gorenstein local ring. Then the following are equivalent:*

(1) inj dim $A_A \leq$ dim $R + 1$.
(2) inj dim ${}_AA \leq$ dim $R + 1$.

Throughout the rest of this section, we assume that R is a Gorenstein local ring and that proj dim ${}_A\Omega =$ proj dim $\Omega_A = m < \infty$. Take a projective resolution $P^\bullet \to \Omega$ in mod-A^{op} and set $Q^\bullet = \mathrm{Hom}_R(P^\bullet, R)$. Then we have a right resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ in mod-A with $Q^i = \mathrm{Hom}_R(P^{-i}, R) \in \mathrm{add}(\Omega)$ for all $i \geq 0$. Recall that a module $M \in \text{Mod-}A$ is said to be reflexive if the canonical homomorphism

$$M \to \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(M, A), A), x \mapsto (f \mapsto f(x))$$

is an isomorphism.

Remark 6. The following hold:

(1) Every $Q^i \in \text{mod-}R$ is a reflexive module with $\mathrm{Ext}^j_R(\mathrm{Hom}_R(Q^i, R), R) = 0$ for $j \neq 0$.
(2) $\oplus_{i\geq 0} \mathrm{Hom}_R(Q^i, R) \in \text{mod-}A^{\mathrm{op}}$ is a projective generator.
(3) proj dim $Q^i < \infty$ in mod-A for all $i \geq 0$.

In the following, we assume further that R is complete and that $P^\bullet \to \Omega$ is a minimal projective resolution. Let $A \to E^\bullet$ be a minimal injective resolution in Mod-A.

In the next proposition, the implication (1) $\Rightarrow$ (2) holds true without the completeness of R.

Proposition 7. *The following are equivalent:*

(1) proj dim $Q^i \leq i$ *in* mod-A *for all* $i \geq 0$.
(2) flat dim $E^n \leq n$ *in* Mod-A *for all* $n \geq 0$.

3. Auslander-Gorenstein resolution

In this section, formulating Remark 6 and Proposition 7, we will introduce the notion of Auslander-Gorenstein resolution and show that a noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring.

We start by recalling the Auslander condition. In the following, Λ stands for an arbitrary noetherian ring.

Proposition 8 (Auslander). *For any $n \geq 0$ the following are equivalent:*

(1) *In a minimal injective resolution $\Lambda \to I^\bullet$ in $\text{Mod-}\Lambda$, $\text{flat dim } I^i \leq i$ for all $0 \leq i \leq n$.*
(2) *In a minimal injective resolution $\Lambda \to J^\bullet$ in $\text{Mod-}\Lambda^{\text{op}}$, $\text{flat dim } J^i \leq i$ for all $0 \leq i \leq n$.*
(3) *For any $1 \leq i \leq n+1$, any $M \in \text{mod-}\Lambda$ and any submodule X of $\text{Ext}^i_\Lambda(M, \Lambda) \in \text{mod-}\Lambda^{\text{op}}$ we have $\text{Ext}^j_{\Lambda^{\text{op}}}(X, \Lambda) = 0$ for all $0 \leq j < i$.*
(4) *For any $1 \leq i \leq n+1$, any $X \in \text{mod-}\Lambda^{\text{op}}$ and any submodule M of $\text{Ext}^i_{\Lambda^{\text{op}}}(X, \Lambda) \in \text{mod-}\Lambda$ we have $\text{Ext}^j_\Lambda(M, \Lambda) = 0$ for all $0 \leq j < i$.*

Definition 9 ([6]). We say that Λ satisfies the Auslander condition if it satisfies the equivalent conditions in Proposition 8 for all $n \geq 0$, and that Λ is an Auslander-Gorenstein ring if $\text{inj dim }{}_\Lambda\Lambda = \text{inj dim }\Lambda_\Lambda < \infty$ and if it satisfies the Auslander condition.

Definition 10. We denote by $\mathcal{G}_\Lambda$ the full subcategory of $\text{mod-}\Lambda$ consisting of reflexive modules $M \in \text{mod-}\Lambda$ with $\text{Ext}^i_{\Lambda^{\text{op}}}(\text{Hom}_\Lambda(M, \Lambda), \Lambda) = 0$ for $i \neq 0$.

Throughout the rest of this section, R and A are noetherian rings. We do not require the existence of a ring homomorphism $R \to A$. Also, even if we have a ring homomorphism $R \to A$ with R commutative, the image of which may fail to be contained in the center of A (cf. [1]).

Definition 11. A right resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ in $\text{Mod-}A$ is said to be a Gorenstein resolution of A over R if the following conditions are satisfied:

(1) Every Q^i is an R-A-bimodule.
(2) $Q^i \in \mathcal{G}_{R^{\text{op}}}$ in $\text{Mod-}R^{\text{op}}$ for all $i \geq 0$.
(3) $\oplus_{i\geq 0} \text{Hom}_{R^{\text{op}}}(Q^i, R) \in \text{Mod-}A^{\text{op}}$ is faithfully flat.
(4) $\text{flat dim } Q^i < \infty$ in $\text{Mod-}A$ for all $i \geq 0$.

Definition 12. A Gorenstein resolution $0 \to A \to Q^0 \to \cdots \to Q^m \to 0$ of A over R is said to be an Auslander-Gorenstein resolution if the following stronger condition is satisfied:

(4)′ $\text{flat dim } Q^i \leq i$ in $\text{Mod-}A$ for all $i \geq 0$.

Theorem 13. *Assume that A admits a Gorenstein resolution*

$$0 \to A \to Q^0 \to \cdots \to Q^m \to 0$$

over R and that $\text{inj dim }{}_RR = \text{inj dim } R_R = d < \infty$. Then the following hold:

(1) *For an injective resolution* $R \to I^\bullet$ *in* Mod-R *we have an injective resolution* $A \to E^\bullet$ *in* Mod-A *such that*

$$E^n = \bigoplus_{i+j=n} I^j \otimes_R Q^i$$

for all $n \geq 0$*. In particular,* inj dim ${}_A A =$ inj dim $A_A \leq m + d$ *and*

$$\text{flat dim } E^n \leq \sup\{\text{flat dim } I^j + \text{flat dim } Q^i \mid i + j = n\}$$

for all $n \geq 0$*.*

(2) *If* R *is an Auslander-Gorenstein ring, and if* $A \to Q^\bullet$ *is an Auslander-Gorenstein resolution, then* A *is an Auslander-Gorenstein ring.*

In case $m = 0$, a Gorenstein resolution of A over R is just an R-A-bimodule Q such that $Q \cong A$ in Mod-A, $Q \in \mathcal{G}_{R^{\mathrm{op}}}$ in Mod-R^{op} and $\mathrm{Hom}_{R^{\mathrm{op}}}(Q, R) \in$ Mod-A^{op} is faithfully flat. In particular, if A is a Frobenius extension of R in the sense of [1], then both A itself and $\mathrm{Hom}_R(A, R)$ are Gorenstein resolutions of A over R, where $A \cong \mathrm{Hom}_R(A, R)$ in Mod-A but $A \ncong \mathrm{Hom}_R(A, R)$ as R-A bimodules in general.

4. Examples

In this section, we will provide several examples of Auslander-Gorenstein resolution.

Example 14. Let R be a commutative noetherian ring and A a noetherian R-algebra such that $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ and $\mathrm{Ext}^i_R(A, R) = 0$ for $i \neq 0$. Set $\Omega = \mathrm{Hom}_R(A, R)$ and assume that Ω admits a projective resolution $0 \to P^{-1} \to P^0 \to \Omega \to 0$ in mod-A^{op} with $P^0 \in \mathrm{add}(\Omega)$. Then applying $\mathrm{Hom}_R(-, R)$ we have a right resolution $0 \to A \to Q^0 \to Q^1 \to 0$ in mod-A with $Q^0 \in \mathrm{add}(\Omega)$, where $Q^i = \mathrm{Hom}_R(P^{-i}, R)$ for $0 \leq i \leq 1$, which must be an Auslander-Gorenstein resolution of A over R.

Example 15 (cf. [3])**.** Let R be a complete Gorenstein local ring of dimension one and Λ a noetherian R-algebra with $\mathrm{Ext}^i_R(\Lambda, R) = 0$ for $i \neq 0$. Denote by $\mathcal{L}_\Lambda$ the full subcategory of mod-Λ consisting of modules X with $\mathrm{Ext}^i_R(X, R) = 0$ for $i \neq 0$. It should be noted that $\mathcal{L}_\Lambda$ is closed under submodules. Assume that $\mathcal{L}_\Lambda = \mathrm{add}(M)$ with $M \in$ mod-Λ non-projective and set $A = \mathrm{End}_\Lambda(M)$.

Set $F = \mathrm{Hom}_\Lambda(M, -) : \mathcal{L}_\Lambda \xrightarrow{\sim} \mathcal{P}_A$ and $D = \mathrm{Hom}_R(-, R)$. Take a minimal projective presentation $P^{-1} \to P^0 \to DM \to 0$ in mod-Λ^{op}. Applying $F \circ D$, we have an exact sequence in mod-A

$$0 \to A \to F(DP^{-1}) \xrightarrow{f} F(DP^0).$$

Setting $Q^0 = F(DP^{-1})$ and $Q^1 = \mathrm{Im}\, f$, we have an Auslander-Gorenstein resolution of A over R.

Example 16. Let $m \geq 2$ be an integer and $A = \mathrm{T}_m(R)$, the ring of $m \times m$ upper triangular matrices over a noetherian ring R. Namely, A is a free right R-module with a basis $\mathfrak{B} = \{e_{ij} \mid 1 \leq i \leq j \leq m\}$ and the multiplication in A is defined subject to the following axioms: (A1) $e_{ij}e_{kl} = 0$ unless $j = k$ and $e_{ij}e_{jk} = e_{ik}$ for all $i \leq j \leq k$; and (A2) $xv = vx$ for all $x \in R$ and $v \in \mathfrak{B}$. Set $e_i = e_{ii}$ for all i. Then A is a noetherian ring with $1 = e_1 + \cdots + e_m$, where the e_i are orthogonal idempotents. We consider R as a subring of A via the injective ring homomorphism $\varphi : R \to A, x \mapsto x1$. Denote by

$\mathfrak{B}^* = \{e_{ij}^* \mid 1 \le i \le j \le m\}$ the dual basis of $\mathfrak{B}$ for the left R-module $\mathrm{Hom}_R(A, R)$, i.e., we have $a = \Sigma_{v \in \mathfrak{B}} v v^*(a)$ for all $a \in A$. It is not difficult to check the following:

(1) $e_1 A \xrightarrow{\sim} \mathrm{Hom}_R(Ae_m, R), a \mapsto e_{1m}^* a$ as R-A-bimodules.

(2) For each $2 \le i \le m$, setting $f : e_1 A \to \mathrm{Hom}_{R^{\mathrm{op}}}(Ae_{i-1}, R), a \mapsto e_{1,i-1}^* a$ and $g : e_i A \to e_1 A, a \mapsto e_{1i} a$, we have an exact sequence of R-A-bimodules

$$0 \to e_i A \xrightarrow{g} e_1 A \xrightarrow{f} \mathrm{Hom}_R(Ae_{i-1}, R) \to 0.$$

(3) $\mathrm{Hom}_{R^{\mathrm{op}}}(\mathrm{Hom}_R(Ae_i, R), R) \cong Ae_i$ as A-R-bimodules for all $1 \le i \le m$.

Consequently, we have an exact sequence of R-A-bimodules

$$0 \to A \to \overset{m}{\oplus} e_1 A \to \bigoplus_{i=2}^{m} \mathrm{Hom}_R(Ae_{i-1}, R) \to 0,$$

which is an Auslander-Gorenstein resolution of A over R.

References

[1] H. Abe an M. Hoshino, Frobenius extensions and tilting complexes, Algebras and Representation Theory **11** (2008), no. 3, 215–232.

[2] K. Ajitabh, S. P. Smith and J. J. Zhang, Auslander-Gorenstein rings, Comm. Algebra **26** (1998), no. 7, 2159–2180.

[3] M. Auslander, Isolated singularities and existence of almost split sequences, *Representation theory, II (Ottawa, Ont., 1984)*, 194–242, Lecture Notes in Math., **1178**, Springer, Berlin, 1986.

[4] H. Bass, Finitistic dimension and a homological generalization of semi-primary rings, Trans. Amer. Math. Soc. **95** (1960), 466–488.

[5] H. Bass, On the ubiquity of Gorenstein rings, Math. Z. **82** (1963), 8–28.

[6] J. -E. Björk, The Auslander condition on noetherian rings, in: *Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988)*, 137–173, Lecture Notes in Math., **1404**, Springer, Berlin, 1989.

[7] J. -E. Björk and E. K. Ekström, Filtered Auslander-Gorenstein rings, Progress in Math. **92**, 425–447, Birkhäuser, Boston-Basel-Berlin, 1990.

[8] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton Univ. Press, Princeton, N. J., 1956.

[9] J. Clark, Auslander-Gorenstein rings for beginners, *International Symposium on Ring Theory (Kyongju, 1999)*, 95–115, Trends Math., Birkhäuser Boston, Boston, MA, 2001.

[10] S. Goto and K. Nishida, Towards a theory of Bass numbers with application to Gorenstein algebras, Colloquium Math. **91** (2002), 191–253.

[11] H. Matsumura, *Commutative Ring Theory* (M. Reid, Trans.), Cambridge Univ. Press, Cambridge, 1986 (original work in Japanese).

[12] Y. Miyashita, Tilting modules of finite projective dimension, Math. Z. **193** (1986), no. 1, 113–146.

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HIGH ORDER CENTERS AND LEFT DIFFERENTIAL OPERATORS

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Abstract. We give a new frame for derivations and high order left differential operators of algebras. This frame is based on high order centers of bimodules. We show the relation between separable algebras and high order centers.

Key Words: Center, Derivation, Bimodule, Separable algebra.

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1. Introduction

Hattori [1] and Sweedler [10] generalized the notion of high order differential operators of commutative algebras to the notion of high order *left* differential operators of noncommutative algebras. The author generalized them from algebras to noncommutative ring extensions and studied them in [2], [3], [4], and [5]. Recently in [6] and [7], he gave a new view point under which derivations and left differential operators are treated in the same frame. This new frame is based on high order centers of bimodules. He also studied the relation between separable algebras and high order centers. The purpose of this note is to introduce the results of [6] and [7].

Throughout this note, all rings have identity 1, all ring homomorphisms preserve 1, all modules are unitary, and k represents a commutative ring and $\mathbb{N}$ the set of nonnegative integers. For a k-algebra A, we denote by $\mathfrak{M}_k(A)$ the category of bimodules over a k-algebra A. An object of $\mathfrak{M}_k(A)$ is an A-bimodule M such that $\alpha u = u\alpha$ for all $\alpha \in k$ and $u \in M$.

2. High order centers (simple version)

In this section we introduce the notion of high order centers for algebras.

Definition 1. Let A be a k-algebra and let $M \in \mathfrak{M}_k(A)$. For $u \in M$ and $a \in A$, we set $[u, a] = ua - au$. For $U \subseteq M$, we set $[U, A] = \{[u, a] \mid u \in U,\ a \in A\}$. Furthermore, we set $[U, A]_0 = U$ and $[U, A]_{q+1} = [[U, A]_q,\ A]$ $(q \in \mathbb{N})$. If U is a singleton $\{u\}$, we use the notations $[u, A]$ and $[u, A]_q$ instead of $[U, A]$ and $[U, A]_q$, respectively. For $q \in \mathbb{N}$, we set

$$\mathcal{C}_A^q(M) = \{u \in M \mid [u, A]_q = 0\},$$

which is called the *qth order center of* M.

If $\varphi : M \to N$ is a morphism of $\mathfrak{M}_k(A)$, then it is easy to see that $\varphi\big(\mathcal{C}_A^q(M)\big) \subseteq \mathcal{C}_A^q(N)$. Hence $\mathcal{C}_A^q(-)$ gives a functor from $\mathfrak{M}_k(A)$ to the category of k-modules. We shall show that the functor $\mathcal{C}_A^q$ is representable.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 2. For a k-algebra A and $q \in \mathbb{N}$, we set

$$\mathcal{J}_A^q = (A \otimes_k A)/A[1 \otimes 1,\, A]_q A \quad \text{and} \quad j_A^q = 1 \otimes 1 + A[1 \otimes 1,\, A]_q A \in \mathcal{J}_A^q.$$

Theorem 3. *Let A be a k-algebra and $q \in \mathbb{N}$. Then we have a natural isomorphism*

$$\mathrm{Hom}_{\mathfrak{M}_k(A)}(\mathcal{J}_A^q,\, M) \ni \varphi \mapsto \varphi(j_A^q) \in \mathcal{C}_A^q(M)$$

for $M \in \mathfrak{M}_k(A)$.

For any k-algebra A, we have a sequence of surjective A-bimodule homomorphisms

$$A = \mathcal{J}_A^1 \twoheadleftarrow \mathcal{J}_A^2 \twoheadleftarrow \mathcal{J}_A^3 \twoheadleftarrow \cdots \twoheadleftarrow \mathcal{J}_A^q \twoheadleftarrow \cdots \twoheadleftarrow A \otimes_k A.$$

Therefore $\mathcal{J}_A^q$ is closely related to the separability of A. An k-algebra A is said to be *separable* if $A[1 \otimes 1,\, A]A$ is a direct summand of $A \otimes_k A$ as A-bimodule. According to [9], A is said to be *purely inseparable* if $A[1 \otimes 1,\, A]A$ is a small submodule of $A \otimes_k A$ as A-bimodule. The next theorem is known.

Theorem 4. *Let A be a k-algebra. Then the following hold.*

(1) ([10, Theorem 1.21] and [4, Theorem 2.4]) *If A is separable, then $\mathcal{J}_A^q = A$ for all $q > 0$.*
(2) ([10, Theorem 2.1]) *If $\mathcal{J}_A^q = A \otimes_k A$ for some $q \in \mathbb{N}$, then A is purely inseparable.*

Combining Theorems 3 and 4, we get the next

Corollary 5. *Let A be a k-algebra. Then the following hold.*

(1) *If A is separable, then $\mathcal{C}_A^q(M) = \mathcal{C}_A^1(M)$ for all $M \in \mathfrak{M}_k(A)$ and for all $q > 1$.*
(2) *If there exists $q \in \mathbb{N}$ such that $\mathcal{C}_A^q(M) = M$ for all $M \in \mathfrak{M}_k(A)$, then A is purely inseparable.*

Remark 6. In case that A is a finite dimensional algebra over a field k, Sweedler [10, Theorem 2.1] showed that A is purely inseparable if and only if $\mathcal{J}_A^q = A \otimes_k A$ for some $q \in \mathbb{N}$.

3. High order left differential operators (old version)

In this section, we introduce the results of Sweedler [10].

Definition 7 ([10])**.** Let A be a k-algebra, and let M and N be left A-modules. As usual, $\mathrm{Hom}_k(M, N)$ belongs to $\mathfrak{M}_k(A)$ by the multiplications

$$(afb)(u) = af(bu) \quad (f \in \mathrm{Hom}_k(M, N),\ a,\ b \in A,\ u \in M).$$

We set

$$\mathcal{D}_A^q(M, N) = \mathcal{C}_A^{q+1}(\mathrm{Hom}_k(M, N)) \quad \text{and}$$
$$\mathrm{LDer}_k^q(A, M) = \left\{ d \in \mathcal{D}_A^q(A, M) \;\middle|\; d(1) = 0 \right\}.$$

An element of $\mathcal{D}_A^q(M, N)$ is called a *qth order left differential operator*, and an element of $\mathrm{LDer}_k^q(A, M)$ is called a *qth order left derivation*.

Remark 8. For a left A-module M and $d \in \mathrm{Hom}_k(A, M)$, $d \in \mathrm{LDer}_k^1(A, M)$ if and only if $d(xy) = xd(y) + yd(x)$ for all $x,\, y \in A$. In commutative ring theory, d is regarded as a derivation, i.e., $d(xy) = xd(y) + d(x)y$ for all $x,\, y \in A$.

Example 9. Set $A = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \;\middle|\; a,\ b,\ c,\ d \in k \right\}$. Then the mapping

$$\begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \mapsto \begin{pmatrix} 0 & b & d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

belongs to $\mathrm{LDer}_k^1(A, A)$. We can see that $\mathcal{J}_A^5 = A \otimes_k A$. Hence we have $\mathcal{D}_A^4(M, N) = \mathrm{Hom}_k(M, N)$ for all left A-modules M and N, and A is a purely inseparable algebra.

Definition 10. In $\mathcal{J}_A^{q+1}$, we set $\mathcal{K}_A^q = A[j_A^{q+1},\ A]A$, which is called the *qth Kähler module*, and define $d_A^q \in \mathrm{LDer}_k^q(A, \mathcal{K}_A^q)$ by $d_A^q(x) = [j_A^{q+1},\ x]$.

The next theorem corresponds to Theorems 1.17, 1.18 and 5.12 of [10].

Theorem 11. *Let A be a k-algebra and $q \in \mathbb{N} \setminus \{0\}$. Then we have the following natural isomorphisms for left A-modules M and N.*

$$\mathrm{Hom}_A(\mathcal{J}_A^{q+1} \otimes_A M,\ N) \ni \varphi \mapsto \varphi(j_A^{q+1} \otimes -) \in \mathcal{D}_A^q(M, N)$$
$$\mathrm{Hom}_A(\mathcal{K}_A^q,\ M) \ni \varphi \mapsto \varphi d_A^q \in \mathrm{LDer}_k^q(A, M)$$

Sweedler used $\mathcal{C}_A^q$ only to define differential operators, and did not investigate the functor $\mathcal{C}_A^q$. And so he did not know that $\mathcal{J}_A^q$ also represents $\mathcal{C}_A^q$.

4. High order centers (general version)

We shall generalize the notion of high order centers defined in §2.

Definition 12. We denote by **Alg** the category of k-algebras and by $\mathbf{Alg}^n$ the product of n copies of **Alg**. For any $\mathbf{A} = (A_1, \ldots, A_n) \in \mathbf{Alg}^n$, we set $\widehat{\mathbf{A}} = A_1 \otimes_k \cdots \otimes_k A_n$. For any morphism $\alpha = (\alpha_1, \ldots, \alpha_n) : \mathbf{A} \to \mathbf{B}$ in $\mathbf{Alg}^n$, we set $\widehat{\alpha} = \alpha_1 \otimes \cdots \otimes \alpha_n : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$.

Let A be a k-algebra, and let $M,\ N \in \mathfrak{M}_k(A)$. Then $\mathrm{Hom}_k(M, N)$ has two A-bimodule structures.

$$\begin{cases} (afb)(u) = af(bu) \\ (a * f * b)(u) = f(ua)b \end{cases} \qquad (f \in \mathrm{Hom}_k(M, N),\ a,\ b \in A,\ u \in M)$$

We set $[f, a] = fa - af$ and $[f, a]^* = f * a - a * f$. As was mentioned in the previous section, $d \in \mathrm{Hom}_k(A, M)$ is a left derivation if and only if $[[d, A], A] = 0$ and $d(1) = 0$. Furthermore we can see that $d \in \mathrm{Hom}_k(A, M)$ is a derivation if and only if $[[d, A], A]^* = 0$ and $d(1) = 0$. This situation leads the next

Definition 13. Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism of $\mathbf{Alg}^n$ and $\mathbf{q} = (q_1, \ldots, q_n) \in \mathbb{N}^n$. Suppose that $\mathbf{B} = (B_1, \ldots, B_n)$. For $M \in \mathfrak{M}_k(\widehat{\mathbf{B}})$ and $u \in M$, we set

$$[u, \mathbf{B}]_{\mathbf{q}} = [\cdots [[u,\ B_1]_{q_1},\ B_2]_{q_2},\ \cdots,\ B_n]_{q_n}.$$

We note that $[[U,\ B_i],\ B_j] = [[U,\ B_j],\ B_i]$ for any $U \subseteq M$. We set

$$\mathcal{C}_\alpha^{\mathbf{q}}(M) = \left\{ u \in M \;\middle|\; [u, \mathbf{B}]_{\mathbf{q}} = [u, \widehat{\mathbf{A}}] = 0 \right\},$$

which is called the *center of M of type* $\mathbf{q}$.

If $\varphi : M \to N$ is a morphism of $\mathfrak{M}_k(\widehat{\mathbf{B}})$, then it is easy to see that $\varphi\big(\mathcal{C}^{\mathbf{q}}_{\alpha}(M)\big) \subseteq \mathcal{C}^{\mathbf{q}}_{\alpha}(N)$. Hence $\mathcal{C}^{\mathbf{q}}_{\alpha}(-)$ gives a functor from $\mathfrak{M}_k(\widehat{\mathbf{B}})$ to the category of k-modules. We shall show that the functor $\mathcal{C}^{\mathbf{q}}_{\alpha}$ is representable.

Definition 14. For a morphism $\alpha : \mathbf{A} \to \mathbf{B}$ of $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n$, we set

$$\mathcal{J}^{\mathbf{q}}_{\alpha} = (\widehat{\mathbf{B}} \otimes_{\widehat{\mathbf{A}}} \widehat{\mathbf{B}}) \,/\, \widehat{\mathbf{B}}\,[1 \otimes 1,\, \mathbf{B}]_{\mathbf{q}}\widehat{\mathbf{B}} \quad \text{and} \quad j^{\mathbf{q}}_{\alpha} = 1 \otimes 1 + \widehat{\mathbf{B}}\,[1 \otimes 1,\, \mathbf{B}]_{\mathbf{q}}\widehat{\mathbf{B}} \in \mathcal{J}^{\mathbf{q}}_{\alpha}.$$

Theorem 15. *Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism of $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n$. Then we have a natural isomorphism*

$$\mathrm{Hom}_{\mathfrak{M}_k(\widehat{\mathbf{B}})}(\mathcal{J}^{\mathbf{q}}_{\alpha},\, M) \ni \varphi \mapsto \varphi(j^{\mathbf{q}}_{\alpha}) \in \mathcal{C}^{\mathbf{q}}_{\alpha}(M)$$

for $M \in \mathfrak{M}_k(\widehat{\mathbf{B}})$.

5. High order left differential operators (general version)

Using new high order centers, we can define new high order differential operators.

Definition 16. Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n$. For $M, N \in \mathfrak{M}_k(\widehat{\mathbf{B}})$, we set

$$\begin{aligned} &\mathcal{D}^{\mathbf{q}}_{\alpha}(M, N) = \mathcal{C}^{\mathbf{q}}_{\alpha}(\mathrm{Hom}_k(M, N)) \quad \text{and} \\ &\mathrm{LDer}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}}, M) = \big\{ d \in \mathcal{D}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}}, M) \;\big|\; d(1) = 0 \big\}. \end{aligned}$$

An element of $\mathcal{D}^{\mathbf{q}}_{\alpha}(M, N)$ is called a *left differential operators of type* $\mathbf{q}$, and an element of $\mathrm{LDer}^{\mathbf{q}}_{\alpha}(\widehat{B}, M)$ is called a *left derivation of type* $\mathbf{q}$.

Example 17. Let $\mathbf{A} = (k,\, k)$, $\mathbf{B} = (R,\, R^{\circ})$, $\alpha = (\rho,\, \rho)$, $\mathbf{q} = (1,\, 1)$, where R° is the opposite algebra of R and $\rho : k \to R$ is the structure morphism of k-algebra R. Then, for any $M \in \mathfrak{M}_k(R)$, the set $\{d \in \mathcal{D}^{\mathbf{q}}_{\alpha}(R, M) \mid d(1) = 0\}$ coincides with the set of derivations of R to M.

Definition 18. In $\mathcal{J}^{\mathbf{q}}_{\alpha}$, we set $\mathcal{K}^{\mathbf{q}}_{\alpha} = \widehat{\mathbf{B}}[j^{\mathbf{q}}_{\alpha},\, \widehat{\mathbf{B}}]\widehat{\mathbf{B}}$, and define $d^{\mathbf{q}}_{\alpha} \in \mathrm{LDer}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}},\, \mathcal{K}^{\mathbf{q}}_{\alpha})$ by $d^{\mathbf{q}}_{\alpha}(x) = [j^{\mathbf{q}}_{\alpha},\, x]$.

Lemma 19. *Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n$. Then the following hold.*

(1) $\mathcal{D}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}}, M) = \mathrm{Hom}_{\widehat{\mathbf{B}}}(\widehat{\mathbf{B}}, M) \oplus \mathrm{LDer}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}}, M)$.

(2) $\mathcal{J}^{\mathbf{q}}_{\alpha} = \widehat{\mathbf{B}} j^{\mathbf{q}}_{\alpha} \oplus \mathcal{K}^{\mathbf{q}}_{\alpha} = j^{\mathbf{q}}_{\alpha}\widehat{\mathbf{B}} \oplus \mathcal{K}^{\mathbf{q}}_{\alpha}$ *and* $\big\{x \in \widehat{\mathbf{B}} \;\big|\; x j^{\mathbf{q}}_{\alpha} = 0\big\} = \big\{x \in \widehat{\mathbf{B}} \;\big|\; j^{\mathbf{q}}_{\alpha} x = 0\big\} = 0$.

Theorem 20. *Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism in $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$. Then we have following natural isomorphisms for left $\widehat{\mathbf{B}}$-modules M and N.*

$$\begin{aligned} &\mathrm{Hom}_{\widehat{\mathbf{B}}}(\mathcal{J}^{\mathbf{q}}_{\alpha} \otimes_{\widehat{\mathbf{B}}} M,\, N) \ni \varphi \mapsto \varphi(j^{\mathbf{q}}_{\alpha} \otimes -) \in \mathcal{D}^{\mathbf{q}}_{\alpha}(M, N) \\ &\mathrm{Hom}_{\widehat{\mathbf{B}}}(\mathcal{K}^{\mathbf{q}}_{\alpha},\, M) \ni \varphi \mapsto \varphi d^{\mathbf{q}}_{\alpha} \in \mathrm{LDer}^{\mathbf{q}}_{\alpha}(\widehat{\mathbf{B}},\, M) \end{aligned}$$

6. Fundamental properties of $\mathcal{J}_\alpha^{\mathbf{q}}$ and $\mathcal{K}_\alpha^{\mathbf{q}}$

Theorem 21. *Let $\mathbf{A} \xrightarrow{\alpha} \mathbf{B} \xrightarrow{\beta} \mathbf{C}$ be morphisms in $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$. Then $\mathcal{J}_\beta^{\mathbf{q}} \simeq \mathcal{J}_{\beta\alpha}^{\mathbf{q}}/\widehat{\mathbf{C}}[j_{\beta\alpha}^{\mathbf{q}}, \widehat{\mathbf{B}}]\widehat{\mathbf{C}}$ and $\mathcal{K}_\beta^{\mathbf{q}} \simeq \mathcal{K}_{\beta\alpha}^{\mathbf{q}}/\widehat{\mathbf{C}}d_{\beta\alpha}^{\mathbf{q}}\widehat{\beta}(\widehat{\mathbf{B}})\widehat{\mathbf{C}}$ as $\widehat{\mathbf{C}}$-bimodules.*

Corollary 22. *Let $\mathbf{A} \xrightarrow{\alpha} \mathbf{B} \xrightarrow{\beta} \mathbf{C}$ be morphisms in $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$. Then there exist exact sequences of $\widehat{\mathbf{C}}$-bimodules*

$$\widehat{\mathbf{C}} \otimes_{\widehat{\mathbf{B}}} \mathcal{K}_\alpha^{\mathbf{q}} \otimes_{\widehat{\mathbf{B}}} \widehat{\mathbf{C}} \to \mathcal{J}_{\beta\alpha}^{\mathbf{q}} \to \mathcal{J}_\beta^{\mathbf{q}} \to 0 \quad \textit{and} \quad \widehat{\mathbf{C}} \otimes_{\widehat{\mathbf{B}}} \mathcal{K}_\alpha^{\mathbf{q}} \otimes_{\widehat{\mathbf{B}}} \widehat{\mathbf{C}} \to \mathcal{K}_{\beta\alpha}^{\mathbf{q}} \to \mathcal{K}_\beta^{\mathbf{q}} \to 0.$$

Theorem 23. *Let $\mathbf{A} \xrightarrow{\alpha} \mathbf{B} \xrightarrow{\beta} \mathbf{C}$ be morphisms in $\mathbf{Alg}^n$ such that $\widehat{\beta} : \widehat{\mathbf{B}} \to \widehat{\mathbf{C}}$ is a surjective mapping, and let $\mathbf{q} \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$. Set $I = \operatorname{Ker} \widehat{\beta}$. Then the following hold.*

1. $\mathcal{J}_{\beta\alpha}^{\mathbf{q}} \simeq \mathcal{J}_\alpha^{\mathbf{q}}/(I\mathcal{J}_\alpha^{\mathbf{q}} + \mathcal{J}_\alpha^{\mathbf{q}}I) \simeq \widehat{\mathbf{C}} \otimes_{\widehat{\mathbf{B}}} \mathcal{J}_\alpha^{\mathbf{q}} \otimes_{\widehat{\mathbf{B}}} \widehat{\mathbf{C}}$
2. $\mathcal{K}_{\beta\alpha}^{\mathbf{q}} \simeq \mathcal{K}_\alpha^{\mathbf{q}}/\widehat{\mathbf{B}}\delta_\alpha^{\mathbf{q}}(I)\widehat{\mathbf{B}}$
3. *There exists an exact sequence of $\widehat{\mathbf{C}}$-bimodules: $I/I^2 \to \widehat{\mathbf{C}}\otimes_{\widehat{\mathbf{B}}}\mathcal{K}_\alpha^{\mathbf{q}}\otimes_{\widehat{\mathbf{B}}}\widehat{\mathbf{C}} \to \mathcal{K}_{\beta\alpha}^{\mathbf{q}} \to 0$.*

Theorem 24. *Let $\mathbf{A} \xrightarrow{\alpha} \mathbf{B} \xrightarrow{\beta} \mathbf{C}$ be morphisms in $\mathbf{Alg}^n$ such that $\widehat{\beta} : \widehat{\mathbf{B}} \to \widehat{\mathbf{C}}$ is a surjective mapping. Suppose that $\mathbf{B} = (B_1, \ldots, B_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n)$, and $\beta = (\beta_1, \ldots, \beta_n)$. Set $B_i' = \operatorname{Im} \alpha_i + \operatorname{Ker} \beta_i$ and denote by $\iota_i : B_i' \to B_i$ the inclusion mapping ($i = 1, \ldots, n$). Set $\iota = (\iota_1, \ldots, \iota_n) : (B_1', \ldots, B_n') \to \mathbf{B}$. Then $\mathcal{K}_{\beta\alpha}^{\mathbf{q}} \simeq \mathcal{K}_\iota^{\mathbf{q}}$.*

7. Separability

We shall generalize the separability of algebras to the separability of morphisms in $\mathbf{Alg}^n$.

Definition 25. Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism of $\mathbf{Alg}^n$. For $M \in \mathfrak{M}_k(\widehat{\mathbf{B}})$, we set

$$\mathcal{C}_\alpha(M) = \sum_{i=1}^{n} \big\{ u \in M \bigm| [u, B_i] = [u, \widehat{\mathbf{A}}] = 0 \big\}.$$

α is called $\mathbf{q}$-*quasi-separable* if $j_\alpha^{\mathbf{q}} \in \mathcal{C}_\alpha(\mathcal{J}_\alpha^{\mathbf{q}})$. α is called *left* $\mathbf{q}$-*differentially separable* if

$$\mathcal{D}_\alpha^{\mathbf{q}}(M, N) \subseteq \sum_{i=1}^{n} \operatorname{Hom}_{B_i}(M, N) \cap \operatorname{Hom}_{\widehat{\mathbf{A}}}(M, N) \quad \big(= \mathcal{C}_\alpha(\operatorname{Hom}_k(M, N)) \big)$$

for all left $\widehat{\mathbf{B}}$-modules M and N.

Lemma 26. *Let $\alpha : \mathbf{A} \to \mathbf{B}$ be a morphism of $\mathbf{Alg}^n$ and $\mathbf{q} \in \mathbb{N}^n$. Then the following hold.*

1. *α is $\mathbf{q}$-quasi-separable if and only if $\mathcal{C}_\alpha^{\mathbf{q}}(M) \subseteq \mathcal{C}_\alpha(M)$ for all $M \in \mathfrak{M}_k(\widehat{\mathbf{B}})$.*
2. *If α is $\mathbf{q}$-quasi-separable, then α is left $\mathbf{q}$-differentially-separable.*

Theorem 27. *Let $\mathbf{A} = (k, k)$, $\mathbf{B} = (R, R^\circ)$, and $\alpha = (\rho, \rho)$, where R° is the opposite algebra of R and $\rho : k \to R$ is the structure morphism of k-algebra R. Then the following are equivalent:*

1. *R is a separable algebra.*
2. *α is $(1, 1)$-quasi-separable.*

(3) α *is* $\mathbf{q}$*-quasi-separable for all* $\mathbf{q} \in \mathbb{N}^2 \setminus \{(0,0)\}$.
(4) α *is left* $(1,1)$*-differentially-separable.*
(5) α *is left* $\mathbf{q}$*-differentially-separable for all* $\mathbf{q} \in \mathbb{N}^2 \setminus \{(0,0)\}$.

Let $\rho : R \to S$ be a ring homomorphism. According to [8], ρ is said to be *separable* if $S[1 \otimes 1, S]S$ is a direct summand of $S \otimes_R S$ as S-bimodule. Usually, S is called a *separable extension* of R.

Theorem 28. *Let* $\alpha = (\alpha_1, \ldots, \alpha_n) : \mathbf{A} \to \mathbf{B}$ *be a morphism in* $\mathbf{Alg}^n$. *Suppose that all* α_i *are separable. Then the following hold:*
(1) α *is* $(1, \ldots, 1)$*-quasi-separable.*
(2) *If* $[1 \otimes 1, [B_i, A_i]] = 0$ *in* $B_i \otimes_{A_i} B_i$ $(i = 1, \ldots, n)$*, then* α *is* $\mathbf{q}$*-quasi-separable for all* $\mathbf{q} \in \mathbb{N}^n \setminus \{(0, \ldots, 0)\}$.

References

[1] A. Hattori, *On high order derivations from the view-point of two-sided modules*, Sci. Papers College Gen. Ed. Univ. Tokyo **20** (1970), 1–11.
[2] M. Hongan and H. Komatsu, *On the module of differentials of a noncommutative algebras and symmetric biderivations of a semiprime algebra*, Comm. Algebra **28** (2000), 669–692.
[3] H. Komatsu, *The module of differentials of a noncommutative ring extension*, International Symposium on Ring Theory, Birkhäuser, 2001, pp. 171–177.
[4] H. Komatsu, *Quasi-separable extensions of noncommutative rings*, Comm. Algebra **29** (2001), 1011–1019.
[5] H. Komatsu, *High order Kähler modules of noncommutative ring extensions*, Comm. Algebra **29** (2001), 5499–5524.
[6] H. Komatsu, *Differential operators of bimodules*, preprint.
[7] H. Komatsu, *High order centers and differential operators of modules*, preprint.
[8] Y. Miyashita, *Finite outer Galois theory of non-commutative rings*, J. Fac. Sci. Hokkaido Univ. Ser. I **19** (1966), 114–134.
[9] M. E. Sweedler, *Purely inseparable algebras*, J. Algebra **35** (1975), 342–355.
[10] M. E. Sweedler, *Right derivations and right differential operators*, Pacific J. Math. **86** (1980), 327–360.

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PRIME FACTOR RINGS OF ORE EXTENSIONS OVER A COMMUTATIVE DEDEKIND DOMAIN

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Abstract. Let $R = D[x;\sigma]$ be a skew polynomial ring over a commutative Dedekind domain D and let P be a minimal prime ideal of R, where σ is an automorphism of D. There are two different types of P, namely, either $P = \mathfrak{p}[x;\sigma]$ or $P = P' \cap R$, where $\mathfrak{p}$ is a σ-prime ideal of D, P' is a prime ideal of $K[x;\sigma]$ and K is the quotient field of D. In the first case R/P is a hereditary prime ring and in the second case, it is shown that R/P is a hereditary prime ring if and only if $P \nsubseteq M^2$ for any maximal ideal M of R. We give some examples of minimal prime ideals such that the factor rings are not hereditary or hereditary or Dedekind, respectively. In the case $R = D[x;\sigma,\delta]$, an Ore extension, where δ is a left σ-derivation of D, we roughly speak of any prime ideal P of R which is not complete, by using Goodearl's classification.

1. Background

A ring is called left (resp. right) hereditary if every left (resp. right) ideal is projective. A Dedekind domain is a commutative domain which is hereditary.

When D is a Dedekind domain, Hillman [1] gave a criterion for D-torsion-free prime factor rings of $D[x]$ to be Dedekind. Indeed, let $f(x)$ generate a prime ideal $P = f(x)D[x]$ of $D[x]$ which is not maximal. Then it was shown that $D[x]/P$ is a Dedekind domain if and only if P is not contained in the square of any maximal ideal of $D[x]$.

Armendariz asked the following Question: Can Hillman's result be generalized from Dedekind domains to hereditary prime P.I. rings?

Park and Roggenkamp [2] studied this question and gave a partial answer under a strong condition. Later Lee, Marubayashi and Park [3] gave a precise answer to Armendariz's question. Let Λ be a hereditary prime P.I. ring, and suppose that a non-zero central polynomial $f(x)$ generates a prime ideal $P = f(x)\Lambda[x]$. They proved that $\Lambda[x]/P$ is hereditary if and only if P is not contained in the square of any maximal ideal of $\Lambda[x]$ by adopting localization, some properties of *v-HC* orders [4, 5] and Kaplansky's method [6].

2. Main results

Let D be a commutative Dedekind domain with σ, an automorphism of D. We denote by $R = D[x;\sigma]$ the skew polynomial ring over D in an indeterminate x. We denote by $\mathrm{Spec}(R) = \{P \mid P \text{ is a prime ideal of } R\}$ and $\mathrm{Spec}_0(R) = \{P \in \mathrm{Spec}(R) \mid P \cap D = 0\}$. We always assume that D is not a field to avoid the trivial case.

If P is not a minimal prime ideal of R, we can see that R/P is a simple Artinian ring by [7, (6.5.4), (7.5.3) and (6.3.11)]. So from now on, to study the factor rings R/P by prime ideals we can consider the minimal prime ideals only.

The detailed version of this paper has been submitted for publication elsewhere.

Firstly, we find out the set of minimal prime ideals of R.

Proposition 1. $\{\mathfrak{p}[x;\sigma],\ P \mid \mathfrak{p}$ *is a* σ*-prime ideal of* D *and* $P \in \mathrm{Spec}_0(R)$ *with* $P \neq (0)\}$ *is the set of all minimal prime ideals of* R*.*

Then when minimal prime ideal P is the former type, we have

Proposition 2. *Let* $P = \mathfrak{p}[x;\sigma]$*, where* $\mathfrak{p}$ *is a* σ*-prime ideal of* D*. Then* R/P *is a hereditary prime ring. In particular,* R/P *is a Dedekind prime ring if and only if* $\mathfrak{p} \in \mathrm{Spec}(D)$*.*

When the minimal prime ideal P is the latter type, i.e. $P \in \mathrm{Spec}_0(R)$, we discuss the factor ring R/P in terms of the order of σ. If σ is of infinite order, that is, $\sigma^n \neq 1$ for any $n > 0$, by [8], it is clear that:

Proposition 3. (1) $P = xR$ *is the only element in* $\mathrm{Spec}_0(R)$*.*
(2) $R/P = D[x;\sigma]/xR \simeq D$ *is a Dedekind Domain.*

If σ is of finite order, we may assume $\sigma^n = 1$ for some $n > 0$. By using localization, Kaplansky's method [6], Reiner's result [9, (3.24)], some lemmas and other known results, we obtain the following proposition which is similar to Hillman's.

Proposition 4. *Let* $P \in \mathrm{Spec}_0(R)$ *with* $P \neq xR$ *and* $P \neq 0$*. Then* R/P *is a hereditary prime ring if and only if* $P \nsubseteq M^2$ *for any maximal ideal* M *of* R*.*

Remark 5. (1) The center of R is $C = D_\sigma[x^n]$, where $D_\sigma = \{d \in D \mid \sigma(d) = d\}$, and C is a Dedekind Domain.
(2) Let $P \in \mathrm{Spec}_0(R)$ with $P \neq xR$. Then $\mathbb{Z}(R/P) = C/(P \cap C)$, where $\mathbb{Z}(R/P)$ is the center of the factor ring R/P.

Summarizing all the results we have obtained, we have the following theorem:

Theorem 6. *Let* $R = D[x;\sigma]$ *be a skew polynomial ring over a commutative Dedekind domain, where* σ *is an automorphism of* D *and let* P *be a prime ideal of* R*. Then*

(1) P *is a minimal prime ideal of* R *if and only if either* $P = \mathfrak{p}[x;\sigma]$*, where* $\mathfrak{p}$ *is a non-zero* σ*-prime ideal of* D *or* $P \in \mathrm{Spec}_0(R)$ *with* $P \neq (0)$*.*
(2) *If* $P = \mathfrak{p}[x;\sigma]$*, where* $\mathfrak{p}$ *is a non-zero* σ*-prime ideal of* D*, then* R/P *is a hereditary prime ring.*
(3) *If* $P \in \mathrm{Spec}_0(R)$ *with* $P = xR$*, then* R/P *is a Dedekind domain.*
(4) *If* $P \in \mathrm{Spec}_0(R)$ *with* $P \neq (0)$ *and* $P \neq xR$*, then* R/P *is a hereditary prime ring if and only if* $P \nsubseteq M^2$ *for any maximal ideal* M *of* R*.*

3. Examples

We give three kinds of rings which is not hereditary, hereditary but not Dedekind and Dedekind respectively by using Proposition 4.

Let $D = \mathbb{Z} + \mathbb{Z}i$ be the Gauss integers, where $i^2 = -1$, and let σ be the automorphism of D with $\sigma(a+bi) = a - bi$ where a, $b \in \mathbb{Z}$, the ring of integers. Let p be a prime number. Then the following properties are well known in the elementary number theory:

(1) If $p = 2$, then $2D = (1+i)^2D$ and $(1+i)D$ is a prime ideal.

(2) If $p = 4n + 1$, then $pD = \pi\sigma(\pi)D$ for some prime element π with $\pi D + \sigma(\pi)D = D$.

(3) If $p = 4n + 3$, then pD is a prime ideal of R.

We let $R = D[x; \sigma]$ be the skew polynomial ring, $P = (x^2 + p)R$. It is obvious that $P = (x^2 + p)R \in \mathrm{Spec}_0(R)$.

(1) If $p = 2$, then R/P is not a hereditary prime ring.
($P \subseteq M^2$, where $M = (1 + i)D + xR$.)

(2) If $p = 4n+1$, then R/P is a hereditary prime ring but not a Dedekind prime ring.
($\exists\ M = \pi D + xR \supset P$, $M^2 + P = M$.)

(3) If $p = 4n + 3$, then R/P is not a hereditary prime ring.
($P \subseteq M^2$, where $M = (1 + x)R + 2R$.)
However, let $S = \{2^i |\, i = 0, 1, 2, \cdots\}$, a central multiplicative set in R. Then R_S/P_S is a Dedekind prime ring.

4. Questions

Let Λ be a hereditary prime P.I. ring with σ, an automorphism of Λ.

(Q1) Does $R = \Lambda[x; \sigma]$ have the similar properties as in the first step about the factor ring R/P for a prime ideal P of R?

We can get similar results except Proposition 4. We have an example that the center is different from the Dedekind domain case. In general, we have the following example:

Let Q be a simple Artinian ring with its center $\mathbb{Z}(Q) = K$. Suppose $[Q : K] < \infty$. $\sigma \in \mathrm{Aut}_K(Q)$ (i.e. $\sigma \in \mathrm{Aut}(Q)$ s.t. $\sigma(k) = k$ for all $k \in K$), then σ is an inner automorphism [9], that is, there exists $q \in U(Q)$ such that $\sigma(a) = q^{-1}aq$ for all $a \in Q$. Suppose $\sigma^n = 1$, $n > 1$. $Q[x; \sigma] \supseteq \mathbb{Z}(Q[x; \sigma]) \supsetneq K_\sigma[x^n]$ since there exists $c(x) = qx + b \in \mathbb{Z}(Q[x; \sigma]) - K_\sigma[x^n]$, where $b \in K_\sigma$.

Let D be a commutative Dedekind domain with σ, an automorphism of D and δ, a left σ-derivation of D ($\delta \neq 0$). Let $R = D[x; \sigma, \delta]$ be the Ore extension over D with

$$xa = \sigma(a)x + \delta(a) \ \text{ for all } \ a \in D.$$

We study the structure of R/P for any prime ideal P of R in terms of Goodearl's classification [10, (3.1)] on prime ideals as the following.

$R = D[x; \sigma, \delta]$ where D is a commutative Noetherian ring, σ is an automorphism of D and δ is a left σ-derivation of D ($\delta \neq 0$). Let P be a prime ideal of R, $\mathfrak{p} = P \cap D$. Then one of the following three cases must hold:

(1) $\mathfrak{p}$ is a (σ, δ)-prime ideal of D. In this case,
  (a) $\mathfrak{p}$ is a σ-prime ideal of D, or
  (b) $\mathfrak{p}$ is a δ-prime ideal of D and R/P has a unique associated prime ideal, which contains $(1 - \sigma)D$.

(2) $\mathfrak{p}$ is a prime ideal of D and $\sigma(\mathfrak{p}) \neq \mathfrak{p}$.

We assume that D is a commutative Dedekind domain.

When P is in the case (2), then P is not a minimal prime ideal. Hence R/P is a simple Artinian ring.

When P is in the case (1) (a). If $\mathfrak{p} \neq 0$, then $P = \mathfrak{p}[x; \sigma, \delta]$. Hence R/P is hereditary and R/P is Dedekind if and only if $\mathfrak{p} \in \mathrm{Spec}(D)$. If $\mathfrak{p} = 0$, then $P \in \mathrm{Spec}_0(R)$. We predict:

(Q2) R/P is hereditary if and only if $P \nsubseteq M^2$ for any maximal ideal M of R.

When P is in the case (1) (b). $\mathfrak{p} = \mathfrak{p}_o{}^e$ for some prime ideal $\mathfrak{p}_o$ of D and $P = \mathfrak{p}[x; \sigma, \delta]$.

(Q3) Whether gld (R/P) is finite or not?

References

[1] J.A. Hillman, *Polynomials determining Dedekind domains*, Bull. Austral. Math. Soc. **29** (1984), 167–175.

[2] J.K. Park amd K.W. Roggenkamp, *A note on hereditary rings*, Comm. in Algebra **5(8)** (1977), 783–794.

[3] H. Marubayashi, Y. Lee and J.K. Park, *Polynomials determining hereditary prime PI-rings*, Comm. in Algebra, **20(9)** (1992), 2503–2511.

[4] H. Marubayashi, *A Krull type generalization of HNP rings with enough invertible ideals*, Comm. in Algebra, **11** (1983), 469–499.

[5] H. Marubayashi, *Remarks on VHC-orders in a simple Artinian ring*, J. Pure and Applied Algebra **31** (1984), 109–118.

[6] I. Kaplansky, *Fields and Rings*, University of Chicago Press, Chicago, 1969.

[7] J.C. McConnell and J.C. Robson, *Noncommutative Noetherian Rings*, Wiley-Interscience, New York, 1987.

[8] N. Jacobson, *Pseudo-linear transformations*, Annals of Math. **38** (1937), 484–507.

[9] I. Reiner, *Maximal Order*, Academic Press, New York, 1975.

[10] K.R. Goodearl, *Prime ideals in skew polynomial rings and quantized Weyl algebras*, J. Algebra **150** (1992), 324–377.

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(θ, δ)-CODES WITH SKEW POLYNOMIAL RINGS

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ABSTRACT. In this paper we generalize coding theory of cyclic codes over finite fields to skew polynomial rings over finite rings. Codes that are principal ideals in quotient rings of skew polynomial rings by two sided ideals are studied. Next we consider skew codes of endomorphism type and derivation type. And we give some examples.

Key Words: Finite rings, (θ, δ)-codes, Skew polynomial rings.

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1. INTRODUCTION

Let $\mathbf{F}$ be a finite field. A linear $[n, k]$-code over $\mathbf{F}$ is a k-dimensional subspace C of the vector space $\mathbf{F}^n = \{(a_0, \cdots, a_{n-1}) \mid a_i \in \mathbf{F}\}$. We use polynomial representation of the code C, where we identify code words $(a_0, \cdots, a_{n-1}) \in C$ with coefficient tuples of polynomials $a_{n-1}X^{n-1} + \cdots + a_1X + a_0 \in \mathbf{F}[X]$. Those polynomials can also be seen as elements of a quotient ring $\mathbf{F}[X]/(f)$ where f is a polynomial of degree n.

D. Boucher, W. Geiselmann and F. Ulmer [3] generalized the notion of codes to skew polynomial rings. In [4], D. Boucher and P. Solé studied skew constacyclic codes. They considered skew polynomial rings over Galois rings. In this paper, we generalize the result of [4] to codes with (θ, δ)-type skew polynomial rings $R[X; \theta, \delta]$. We study mathematical aspects of coding theory with skew polynomial rings over finite rings.

Let R be a ring and θ be an endomorphism of R. A θ-derivation of R is an additive map $\delta : R \to R$ such that $\delta(rs) = \theta(r)\delta(s) + \delta(r)s$ for all r, $s \in R$. Throughout this paper, R represents a finite ring with $1 \neq 0$, θ an endomorphism of R with $\theta(1) = 1$ and δ a θ-derivation of R, unless otherwise stated.

We shall use the following conventions:

$Z(R[X; \theta, \delta])$ is the center of $R[X; \theta, \delta]$.
$(g)_l$ is the left ideal generated by $g \in R[X; \theta, \delta]$.
(g) is the two-sided ideal generated by $g \in R[X; \theta, \delta]$.
$R^\theta = \{r \in R \mid \theta(r) = r\}$.
$R^\delta = \{r \in R \mid \delta(r) = 0\}$, $Z^\delta = \{r \in Z \mid \delta(r) = 0\}$, where Z is the center of R.

2. SKEW (θ, δ)-CODES OVER FINITE RINGS

In this section, we define (θ, δ)-codes and study some properties of them.

Definition 1. Let R be a ring, θ be an endomorphism of R, δ be a θ-derivation of R. Suppose S is a free left R-module with basis 1, X, X^2, $\cdots$ and give a multiplication from

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the rules $X^iX^j = X^{i+j}$ and $Xr = \theta(r)X + \delta(r)$ for all $r \in R$. The ring S constructed in this way is denoted by $R[X; \theta, \delta]$ and is called a *skew polynomial ring.*

Proposition 2. *For any h, $g \in R[X; \theta, \delta]$, if the leading coefficients of g is invertible, then* $\deg(h{\cdot}g) = \deg(h) + \deg(g)$.

Proof. Straightforward. □

Proposition 3. *Let $h{\cdot}g \in Z(R[X; \theta, \delta])$. If the leading coefficient of g is invertible, then $h{\cdot}g = g{\cdot}h$ in $R[X; \theta, \delta]$.*

Proof. Straightforward. □

Proposition 4. *Let R be a ring, θ be an endomorphism of R, δ be a θ-derivation of R. For any f, $g \in R[X; \theta, \delta]$, if the leading coefficient of f is invertible, then there exist polynomials q and r such that $g = qf + r$ where* $\deg(r) < \deg(f)$.

Proof. By the induction on $\deg(g)$, it is proved. □

Definition 5. Let R be a finite ring, θ be an endomorphism of R, δ be a θ-derivation of R. Suppose $f \in R[X; \theta, \delta]$ is a nonzero polynomial with an invertible leading coefficient. Then, by Proposition 4, $R[X; \theta, \delta]/(f)$ is a finite ring and a left ideal of $R[X; \theta, \delta]/(f)$ is called a *skew (θ, δ)-code.*

A skew (θ, δ)-code is called an *$[n, k]$-code* if the degree of f and the rank of C as a free left R-module are n and k, respectively. If $f \in Z(R[X; \theta, \delta])$, then we call a skew (θ, δ)-code corresponding to a left ideal of $R[X; \theta, \delta]/(f)$ a *central (θ, δ)-code.*

We shall consider skew codes under the condition $R[X; \theta, \delta]f = fR[X; \theta, \delta]$, which is a weaker condition than $f \in Z(R[X; \theta, \delta])$.

Note that not all left ideals in $R[X; \theta, \delta]/(f)$ are principal, but in the following we will focus on those ideals.

Definition 6. A *(θ, δ)-principal code* is a skew (θ, δ)-code corresponding to a left ideal $(g)_l/(f)$ where $(g)_l$ is a left ideal generated by g and $hg = f$ for some h. A *(θ, δ)-cyclic code* is a (θ, δ)-principal code corresponding to a left ideal $(g)_l/(X^n - 1)$.

In what follows, for a code $C = (g)_l/(f)$, we assume that $n = \deg(f) \geq 2$.

Proposition 7. *If C is a (θ, δ)-cyclic code, then $(a_0, a_1, \cdots, a_{n-1}) \in C$ implies $(\theta(a_{n-1}) + \delta(a_0), \theta(a_0) + \delta(a_1), \theta(a_1) + \delta(a_2), \cdots, \theta(a_{n-2}) + \delta(a_{n-1})) \in C$.*

Proof. Straightforward. □

A ring R is said to be *Dedekind finite* if $ab = 1$ implies $ba = 1$ (a, $b \in R$). It is well-known that a finite ring is Dedekind finite.

Theorem 8. *Let $C = (g)_l/(f)$ be a skew code in $R[X; \theta, \delta]/(f)$ and $f = hg$. Suppose that the leading coefficients of f and g are invertible. If f satisfies the condition $R[X; \theta, \delta]f = fR[X; \theta, \delta]$, then C is a free left R-module and* $\operatorname{rank} C = \deg(f) - \deg(g)$.

Example 9. Let C be a (θ, δ)-code corresponding to a left ideal generated by g in $R[X; \theta, \delta]/(f)$ and $R[X; \theta, \delta]f = fR[X; \theta, \delta]$. Suppose that $\deg(f) = 4$ and $g = g_1X + g_0$. Then the generator matrix of C is given by

$$\begin{pmatrix} g_0 & g_1 & 0 & 0 \\ \delta(g_0) & \theta(g_0)+\delta(g_1) & \theta(g_1) & 0 \\ \delta^2(g_0) & (\theta\delta+\delta\theta)(g_0)+\delta^2(g_1) & \theta^2(g_0)+(\theta\delta+\delta\theta)(g_1) & \theta^2(g_1) \end{pmatrix}.$$

Lemma 10. *Let $C=(g)_l/(f)$ be a skew code in $R[X;\theta,\delta]/(f)$ and $f=hg=gh$. Suppose that the leading coefficient of h is invertible and $R[X;\theta,\delta]f=fR[X;\theta,\delta]$. Then $\overline{a}\in C$ if and only if $\overline{a}\,\overline{h}=0$ in $R[X;\theta,\delta]/(f)$.*

For any subset $T\subseteq R$, the left annihilator of T is the set

$$l.ann_R(T)=\{r\in R\mid rt=0 \text{ for all } t\in T\},$$

which is a left ideal of R. The right annihilator $r.ann_R(T)$ is defined, similarly.

Then we can get the following corollary.

Corollary 11. *Let $C=(g)_l/(f)$ be a skew code in $\overline{R}=R[X;\theta,\delta]/(f)$ and $f=hg=gh$. Suppose that the leading coefficient of h is invertible and $R[X;\theta,\delta]f=fR[X;\theta,\delta]$. Then we have $C=l.ann_{\overline{R}}\left(\overline{h}\right)$.*

3. Skew codes of endomorphism type and derivation type

First we study skew codes of endomorphism type, i.e., skew $(\theta,0)$-codes.

Proposition 12. *Let $C=(g)_l/(f)$ be a skew code in $R[X;\theta]/(f)$ and $f=hg$. Suppose that the leading coefficients of f and g are invertible and $R[X;\theta]f=fR[X;\theta]$. If $\deg(f)=n$ and $g=g_{n-k}X^{n-k}+g_{n-k-1}X^{n-k-1}+\cdots+g_1X+g_0$, then C is a free R-module and has the $k\times n$ generator matrix given by*

$$\begin{pmatrix} g_0 & g_1 & \cdots & g_{n-k} & 0 & \cdots & 0 \\ 0 & \theta(g_0) & \theta(g_1) & \cdots & \theta(g_{n-k}) & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & \theta^{k-1}(g_0) & \theta^{k-1}(g_1) & \cdots & \theta^{k-1}(g_{n-k}) \end{pmatrix}.$$

We study constacyclic codes and determine their parity check matrix.

Proposition 13. *Suppose that R is a finite commutative ring, $X^n-\alpha=f=h{\cdot}g\in Z(R[X;\theta])$ and the leading coefficient of g is invertible. Let C denote the $(\theta,0)$-code corresponding to the left ideal generated by g in $R[X;\theta]/(X^n-\alpha)$. Denote by $h=h_kX^k+h_{k-1}X^{k-1}+\cdots+h_1X+h_0$. If the dual code $C^\perp$ is a free R-module and* $\operatorname{rank} C^\perp=n-k$, *then C has the following $(n-k)\times n$ parity check matrix given by*

$$\begin{pmatrix} h_k & \cdots & \theta^{k-1}(h_1) & \theta^k(h_0) & 0 & \cdots & 0 \\ 0 & \theta(h_k) & \cdots & \theta^k(h_1) & \theta^{k+1}(h_0) & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & & \vdots \\ 0 & \cdots & 0 & \theta^{n-k-1}(h_k) & \cdots & \theta^{n-2}(h_1) & \theta^{n-1}(h_0) \end{pmatrix}.$$

Theorem 14. *Suppose that R is a finite commutative ring, θ is an automorphism and $X^n-\alpha=h{\cdot}g\in Z(R[X;\theta])$ with $\alpha\in R^\theta$ and $\alpha^2=1$. Let C denote the central $(\theta,0)$-code corresponding to the left ideal generated by g in $R[X;\theta]/(X^n-\alpha)$ where the leading* *coefficient of g is invertible. Denote by $h = h_kX^k + h_{k-1}X^{k-1} + \cdots + h_1X + h_0$. If the dual code $C^\perp$ is a free left R-module and $\operatorname{rank} C^\perp = n - k$, then the dual of the θ-constacyclic code $(g)/(X^n - \alpha)$ is the θ-constacyclic code $(g^\perp)/(X^n - \alpha)$ where*

$$g^\perp = h_k + \theta(h_{k-1})X + \cdots + \theta^k(h_0)X^k.$$

Next we consider skew codes of derivation type, i.e., skew $(1, \delta)$-codes, and give some examples.

Lemma 15. *Let f be in $R[X; \delta]$. Then the following conditions are equivalent:*
(1) *f satisfies the condition $R[X; \delta]f = fR[X; \delta]$.*
(2) *f is central, that is, $f \in Z(R[X; \delta])$.*

Proof. See the proof of [1, Lemma 1.6]. □

Lemma 16. *Assume that R is a finite ring of prime characteristic p and Z is the center of R. Let $f = X^p + aX + b$ be in $R[X; \delta]$. Then f satisfies the condition $R[X; \delta]f = fR[X; \delta]$ if and only if*
(a) *$a \in Z^\delta$ and $b \in R^\delta$,*
(b) *$\delta^p(r) + a\delta(r) = rb - br$ for any $r \in R$.*

Proof. See [2, Lemma 2.1]. □

In $R[X; \delta]$, we have $X^l r = \sum_{i=0}^{l} \binom{l}{i} \delta^{l-i}(r)X^i$ for $r \in R$. So we can calculate a generator matrix for a given polynomial $g = g_{n-k}X^{n-k} + g_{n-k-1}X^{n-k-1} + \cdots + g_1X + g_0$.

Now we give some examples of skew codes of derivation type $R[X; \delta]$. Let $Z_p = Z/pZ$ be a finite field of p elements and let

$$R_{(p)} = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \in M_2(Z_p) \;\middle|\; a, b \in Z_p \right\}.$$

Then $R_{(p)}$ is a finite commutative local ring with the unique maximal ideal

$$M = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \in M_2(Z_p) \;\middle|\; b \in Z_p \right\}.$$

Now we can define a derivation $\delta : R_{(p)} \to R_{(p)}$ by $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mapsto \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$. Therefore we can consider a skew polynomial ring of derivation type $R_{(p)}[X; \delta]$.

Example 17. We consider the skew polynomial ring of derivation type $R_{(3)}[X; \delta]$. Let $f = X^3 + 2X$. By Lemma 16, f satisfies the condition $R_{(3)}[X; \delta]f = fR_{(3)}[X; \delta]$. Put $g = X + 2\beta$ and $h = X^2 + \beta X + \alpha$, where $\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. We get the following factorizations:

$$X^3 + 2X = (X + 2\beta)(X^2 + \beta X + \alpha) = (X^2 + \beta X + \alpha)(X + 2\beta).$$

Then $(g)_l/(f)$ is a $[3, 2]$ skew δ-code.

Let $S_{(p)} = M_2(R_{(p)})$ and define a derivation $\Delta : M_2(R_{(p)}) \to M_2(R_{(p)})$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \delta(a) & \delta(b) \\ \delta(c) & \delta(d) \end{pmatrix}$. Then we can consider a skew polynomial ring of derivation type $S_{(p)}[Y; \Delta]$.

Example 18. We consider the skew polynomial ring of derivation type $S_{(3)}[Y;\Delta]$. Let $f = Y^3 + 2Y$. By Lemma 16, f satisfies the condition $S_{(3)}[Y;\Delta]f = fS_{(3)}[Y;\Delta]$. Put $g = Y + 2\beta$ and $h = Y^2 + \beta Y + \alpha$. We get the following factorizations:

$$Y^3 + 2Y = (Y + 2\beta)(Y^2 + \beta Y + \alpha) = (Y^2 + \beta Y + \alpha)(Y + 2\beta).$$

Then $(g)_l/(f)$ is a $[3,2]$ skew Δ-code. So the factorizations of $R_{(3)}[X;\delta]$ is lifted to $S_{(3)}[Y;\Delta]$.

References

[1] S. Ikehata, *On separable polynomials and Frobenius polynomials in skew polynomial rings*, Math. J. Okayama. Univ. **22** (1980), 115–129.

[2] S. Ikehata, *On H-separable and Galois polynomials of degree p in skew polynomial rings*, International Mathematical Forum **3**(32) (2008), 1581–1586.

[3] D. Boucher, W. Geiselmann and F. Ulmer, *Skew cyclic codes*, Applied Algebra in Engineering, Communication and Computing **18** (2007), 379–389.

[4] D. Boucher and P. Solé, *Skew constacyclic codes over Galois rings*, Advances in Mathematics of Communications **2**(3) (2008), 273–292.

[5] D. Boucher and F. Ulmer, *Codes as modules over skew polynomial rings*, Proceedings of the 12th IMA conference on Cryptography and Coding, Cirencester, Lecture Notes in Computer Science 5921, pp. 38–55, 2009.

[6] K. R. Goodearl and R. B. Warfield, Jr., *An Introduction to Noncommutative Noetherian Rings*, Cambridge University Press, Cambridge, 1989.

[7] B. R. McDonald, *Finite Rings with Identity*, Pure and Applied Mathematics 28, Marcel Dekker, Inc., New York, 1974.

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HOCHSCHILD COHOMOLOGY AND GORENSTEIN NAKAYAMA ALGEBRAS

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Abstract. Let A be a Nakayama algebra over an algebraically closed field k, $\mathrm{HH}(A)$ the Hochschild cohomology ring. We will study the condition when $\mathrm{HH}(A)$ is a finitely generated algebra and $\mathrm{Ext}^*_A(A/J, A/J)$ is a finitely generated $\mathrm{HH}(A)$-module, where J is the Jacobson radical of A. In [4], it is shown that if an algebra satisfies the both finiteness conditions, then the algebra is Gorenstein. We will investigate the Hochschild cohomology of Gorenstein Nakayama algebras and show that Gorenstein Nakayama algebras satisfy the both finiteness conditions above.

1. Introduction

Let A be a finite dimensional algebra over an algebraically closed field k and H a noetherian commutative graded subalgebra of the Hochschild cohomology algebra $HH(A)$ with $H^0 = HH^0(A)$. In [9], Snashall and Solberg defined the support variety of a finitely generated A-module M over H as the set of maximal ideals of H containing the annihilator $\mathrm{Ann}_H \mathrm{Ext}^*_A(M, M)$, where the H-action on $\mathrm{Ext}^*_A(M, M)$ is given by the graded algebra homomorphism

$$H \xrightarrow{incl.} \mathrm{HH}^*(A) \xrightarrow{-\otimes M} \mathrm{Ext}^*_A(M, M).$$

In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer showed that some geometric properties of the support variety and some representation theoretic properties are related if A satisfies the following finiteness condition:

$$\mathrm{Ext}^*_A(A/J, A/J) \text{ is a finitely generated } H\text{-module},$$

where J is the Jacobson radical of A. This finiteness condition holds for group algebras of finite groups and, in [4], various results for finite groups are generalized to those for the class of selfinjective algebras satisfying the finiteness condition. It is known that the condition holds for any block of a finite dimensional cocomutative Hopf algebra [6], for any complete intersection in commutative setting [7], and so on [5].

In this paper, we consider this finiteness condition in the case of Nakayama algebras. In [9], Hochschild cohomology rings of Nakayama algebras with a single relation are investigated and some of them do not satisfy the finiteness condition. On the other hand, in [4], it is shown that any algebra A is Gorenstein if A satisfies the finiteness condition. We are, therefore, interested in to determine when Gorenstein Nakayama algebras satisfy the finiteness condition. One of our main results, Theorem 9 answers to this question.

The detailed version of this paper will be submitted for publication elsewhere.

2. The finiteness condition (Fg)

In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer introduce some finiteness conditions (Fg1) and (Fg2) for an algebra A and a graded subalgebra H of $\mathrm{HH}(A)$. These conditions are the followings:

(Fg1) H is a commutative noetherian algebra with $H^0 = HH^0(A)$.
(Fg2) $\mathrm{Ext}^*_A(A/J, A/J)$ is a finitely generated H-module.

In [4], some geometric properties of the support variety and some representation theoretic properties are related if A satisfies the finiteness condition above. Moreover various results for finite groups are generalized to those for selfinjective algebras satisfying the finiteness conditions.

On the other hand, in [10], Solberg showed the following.

Proposition 1. *Let A be a finite dimensional algebra. Then there exists a graded subalgebra H of* $\mathrm{HH}(A)$ *such that A and H satisfy (Fg1) and (Fg2) if and only if* $\mathrm{HH}(A)$ *is a finitely generated algebra and* $\mathrm{Ext}^*_A(A/J, A/J)$ *is a finitely generated* $\mathrm{HH}(A)$*-module.*

Definition 2. We denote by (Fg) the latter condition in the proposition above.

3. Stratifying ideals

In this section, we will give some results on algebras with stratifying ideals. The stratifying ideal is defined as follows.

Definition 3. Let A be an algebra and $e = e^2$ an idempotent. The two-sided ideal AeA generated by e is called a *stratifying ideal* if the following conditions are satisfied:
(a) The multiplication map $Ae \otimes_{eAe} eA \to AeA$ is an isomorphism.
(b) $\mathrm{Tor}^{eAe}_n(Ae, eA) = 0$ for all $n > 0$.

The following lemma will be used to check if an ideal is stratifying [8].

Lemma 4. *Let e be an idempotent element in A. If AeA is projective as a right or left A-module, then AeA is stratifying.*

In [8], it is shown that there exist several long exact sequences relating Hochschild cohomology of algebras with a stratifying ideal. The followings are the sequences, which we will use to prove Proposition 6.

Theorem 5. *Let A be an algebra with a stratifying ideal AeA and B the factor algebra A/AeA. Then there are long exact sequences as follows:*
(1) $\to \mathrm{Ext}^n_{A^e}(A, AeA) \to \mathrm{HH}^n(A) \to \mathrm{HH}^n(B) \to \mathrm{Ext}^{n+1}_{A^e}(A, AeA) \to$;
(2) $\to \mathrm{Ext}^n_{A^e}(B, A) \to \mathrm{HH}^n(A) \to \mathrm{HH}^n(eAe) \to \mathrm{Ext}^{n+1}_{A^e}(B, A) \to$; *and*
(3) $\to \mathrm{Ext}^n_{A^e}(B, AeA) \to \mathrm{HH}^n(A) \to \mathrm{HH}^n(B) \oplus \mathrm{HH}^n(eAe) \to \mathrm{Ext}^{n+1}_{A^e}(B, AeA) \to$.
Moreover these sequences induce graded algebra homomorphisms between Hochschild cohomology algebras, especially, the second sequence is induced from the functor $eA \otimes_A - \otimes_A Ae$.

The following proposition is one of our main results and we will apply this for the class of Nakayama algebras in the next section.

Proposition 6. *Let A be an algebra with a stratifying ideal AeA. Suppose $\mathrm{pd}_{A^e} A/AeA < \infty$. Then we have*

(1) $\mathrm{HH}^{\geq n}(A) \cong \mathrm{HH}^{\geq n}(eAe)$ *as graded algebras, where* $n = \mathrm{pd}_{A^e} A/AeA + 1$,
(2) *A satisfies (Fg) if and only if so does eAe,*
(3) *A is Gorenstein if and only if so is eAe.*

Proof. By the second long exact sequence in theorem 5, the first assertion (1) holds.

For the proof of (2), applying the functor $\mathrm{Hom}_{A^e}(-, \mathrm{Hom}_k(A/J, A/J))$ to the short exact sequense $0 \to AeA \to A \to A/AeA \to 0$ we obtain the isomorphism

$$\mathrm{Ext}^n_{A^e}(A, \mathrm{Hom}_k(A/J, A/J)) \cong \mathrm{Ext}^n_{A^e}(AeA, \mathrm{Hom}_k(A/J, A/J))$$

for any $n \geq \mathrm{pd}_{A^e} A/AeA + 1$. This gives the following isomorphism

$$\mathrm{Ext}^n_A(A/J, A/J) \cong \mathrm{Ext}^n_{eAe}(eA/eJ, eA/eJ)$$

for any $n \geq \mathrm{pd}_{A^e} A/AeA + 1$, which is induced from the exact functor $eA \otimes_A -$. Then we have the following commutative diagram of graded algebra homomorphism,

$$\begin{tikzcd}
\mathrm{HH}(A) \arrow[r, "-\otimes_A A/J"] \arrow[d, "eA\otimes_A - \otimes_A Ae"] & \mathrm{Ext}^*_A(A/J, A/J) \arrow[d, "eA\otimes_A -"] \\
\mathrm{HH}(eAe) \arrow[r, "-\otimes_{eAe} eA/eJ"'] & \mathrm{Ext}^*_{eAe}(eA/eJ, eA/eJ),
\end{tikzcd}$$

both columns are isomorphic on all but finite degrees. Hence (2) holds.

For the proof of (3), applying the functor $\mathrm{Hom}_{A^e}(-, \mathrm{Hom}_k(X, A))$ to the short exact sequense $0 \to AeA \to A \to A/AeA \to 0$ we obtain the isomorphism

$$\mathrm{Ext}^n_{A^e}(A, \mathrm{Hom}_k(X, A)) \cong \mathrm{Ext}^n_{A^e}(AeA, \mathrm{Hom}_k(X, A))$$

for any $n \geq \mathrm{pd}_{A^e} A/AeA + 1$. This gives the following isomorphism

$$\mathrm{Ext}^n_A(X, A) \cong \mathrm{Ext}^n_{eAe}(eX, eA)$$

for any $n \geq \mathrm{pd}_{A^e} A/AeA + 1$. Therefore we have that $\mathrm{id}_A A < \infty$ if and only if $\mathrm{id}_{eAe} eA < \infty$. Hence if $\mathrm{id}_A A < \infty$ then $\mathrm{id}_{eAe} eAe < \infty$. On the other hand, since

$$\mathrm{Ext}^n_A(AeA, X) \cong \mathrm{Ext}^n_{eAe}(eA, eX)$$

for any i, we have that $\mathrm{pd}_A AeA = \mathrm{pd}_{eAe} eA$. By the assumption $\mathrm{pd}_{A^e} A/AeA < \infty$, it follows that $\mathrm{pd}_A AeA < \infty$. Hence if $\mathrm{id}_{eAe} eAe < \infty$ then $\mathrm{id}_{eAe} eA < \infty$, so that $\mathrm{id}_A A < \infty$.

Similarly we can show that $\mathrm{id}\, A_A < \infty$ if and only if $\mathrm{id}\, eAe_{eAe} < \infty$. Hence (3) holds. □

4. Nakayama algebras

Throughout this section, we assume that the algebras are basic for simplicity. Because Hochschild cohomology is a Morita-invariance, Theorem 9 holds for any algebra. In this section, we will prove our main theorem, which states that Gorenstein Nakayama algebras satisfy the finiteness condition (Fg). An algebra A is called *Nakayama* if the indecomposable projective right and left modules are uniserial. It is known that if the indecomposable projective modules over a Nakayama algebra have the same length, then the algebra is selfinjective (see [1, Proposition 3.8.]). Especially, any local Nakayama algebra is selfinjective. Using this fact, it is easy to show the following.

Lemma 7. *Let A be a Nakayama algebra. If A is not self injective, then there exists a primitive idempotent f in A such that Jf is a non-zero projective A-module.*

Lemma 8. *Let A be a Gorenstein Nakayama algebra. If A is not selfinjective, then there exists an idempotent $e \neq 1$ such that*

(1) *AeA is projective as left A-module;*
(2) $\mathrm{pd}_{A^e} A/AeA < \infty$*; and*
(3) *eAe is a Gorenstein Nakayama algebra.*

Proof. Assume that A is not selfinjective. By Lemma 7, there exists a primitive idempotent f in A such that Jf is a non-zero projective A-module, so that there exists a primitive idempotent $f' \neq f$ such that $Jf \cong Af'$. Put $e = 1 - f$. Since $f' \neq f$, $AfJf < AeJf$, so that $Jf = AfJf + AeJf = AeJf \leq AeAf < Af$. We obtain that $Jf = AeAf$ because Jf is a maximal submodule of Af. Since $Jf = AeAf$, $J \leq AeA$, so that $fJ \leq fAeA < fA$. Therefore we obtain that $fJ = fAeA$ because fJ is a maximal submodule of fA.

(1) Since $AeAf = Jf$, it follows that $AeA = AeAe \oplus AeAf = Ae \oplus Jf$, so that AeA is projective as left A-module.

(2) Since $(A/AeA)e = 0$, it follows that $A/AeA \cong Af/AeAf = Af/Jf$. Similarly we have that $A/AeA \cong fA/fJ$. Thus A/AeA is simple A^e-module and $A/AeA \cong Af/Jf \otimes_k fA/fJ$. Since Jf is projective, the left projective dimension of Af/Jf is finite and the right injective dimension of $fA/fJ \cong D(Af/Jf)$ is finite. Since A is Gorenstein, the right projective dimension of fA/fJ is finite. Hence $\mathrm{pd}_{A^e} A/AeA < \infty$.

(3)By Lemma 4, AeA is a stratifying ideal. By the assertion (2) above and Proposition 6, eAe is Gorenstein. It is clear that eAe is a Nakayama algebra. ☐

Theorem 9. *Let A be a Gorenstein Nakayama algebra. Then we have*

(1) *There exists a selfinjective Nakayama algebra B such that $\mathrm{HH}^{\geq n}(A) \cong \mathrm{HH}^{\geq n}(B)$ as graded algebras for some n,*
(2) *A satisfies the finiteness condition (Fg).*

Proof. By Proposion 6 and Lemma 8, if A is not selfinjective, then there exists an idempotent $e \neq 1$ such that $\mathrm{HH}^{\geq n}(A) \cong \mathrm{HH}^{\geq n}(eAe)$ as graded algebras for some n. Since the number of the simple modules of eAe is less than that of A and local Nakayama algebras are selfinjective, the assertion (1) holds.

By Proposion 6 and Lemma 8, if A is not selfinjective, then there exists an idempotent $e \neq 1$ such that A satisfies (Fg) if and only if so does eAe. By [2, Section 4], selfinjective Nakayama algebras satisfy (Fg). Hence assertion (2) holds. ☐

Corollary 10. *Let A be a Nakayama algebra. Then*

A is Gorenstein if and only if A satisfies the finiteness condition (Fg).

Proof. By [4] and [10], if an algebra satisfies the finiteness condition (Fg), then the algebra is Gorenstein. Hence, by Theorem 9, the assertion holds. ☐

This corollary gives us a way to check whether a given Nakayama algebra satisfies the finiteness condition (Fg) or not without computing Hochschild cohomology, because we can check whether a given Nakayama algebra is Gorenstein or not by using the Kupish series.

References

1. I. Assem, D. Simson and A. Skowronski: *Elements of the Representation Theory of Associative Algebras*, London Math. Soc. Student Texts **65** (2006).
2. P. A. Bergh: *On the vanishing cohomolgy in triangulated categories*, arviv 0811.2684.
3. H. Cartan and S. Eilenberg: *Homological algebra*, Princeton Landmarks in Mathematics (1973). Originally published 1956.
4. K. Erdmann, M. Holloway, N. Snashall, Ø. Solberg and R.Taillefer: *Support varieties for self-injective algebras*, K-theory. **33** (2004), no. 1, 67–87.
5. K. Erdmann, Ø. Solberg: *Radical cube zero weekly symmetric algebras and support varieties*, J. Pure Appl. Algebra. **215** (2011), no. 2, 185–200.
6. E. M. Friedlander and A. Suslin: *Cohomology of finite group schemes over a field*, Invent. Math. **127** (1997), no. 2, 209–270.
7. T. H. Gulliksen: *A change of ring theorem with applications to Poincaré series and intersection multiplicity*, Math. Scand., **34** (1974), 167–183.
8. S. Koenig and H. Nagase: *Hochschild cohomology and stratifying ideals*, J. Pure Appl. Algebra. **213** (2009), no. 5, 886–891.
9. N. Snashall and Ø. Solberg: *Support varieties and Hochschild cohomology rings*, Proc. London Math. Soc. (3) **88** (2004), no. 3, 705–732.
10. Ø. Solberg: *Support varieties for modules and complexes*, Trends in Representation Theory of Algebras and Related Topics, Contemporary Math. **406** Amer. Math. Soc. (2006), 239–270.

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THE FIRST HILBERT COEFFICIENTS OF PARAMETERS

KAZUHO OZEKI

ABSTRACT. The conjecture of Wolmer Vasconcelos [13] on the vanishing of the first Hilbert coefficient $\mathrm{e}_Q^1(A)$ is solved affirmatively, where Q is a parameter ideal in a commutative Noetherian local ring A. Basic properties of the rings for which $\mathrm{e}_Q^1(A)$ vanishes are derived. The invariance of $\mathrm{e}_Q^1(A)$ for parameter ideals Q and its relationship to Buchsbaum rings are studied.

Key Words: commutative algebra, Cohen-Macaulay local ring, Buchsbaum local ring, Hilbert coefficient.

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1. Introduction

This is based on [1, 5] a joint work with L. Ghezzi, J. Hong, T. T. Phuong, and W. V. Vasconcelos.

Let A be a commutative Noetherian local ring with the maximal ideal $\mathfrak{m}$ and the Krull dimension $d = \dim A > 0$. Let $\ell_A(M)$ denote, for an A-module M, the length of M. Then, for each $\mathfrak{m}$-primary ideal I in A, we have integers $\{\mathrm{e}_I^i(A)\}_{0\le i\le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d\mathrm{e}_I^d(A)$$

holds true for all integers $n \gg 0$, which we call the Hilbert coefficients of A with respect to I. We say that A is unmixed, if $\dim \widehat{A}/\mathfrak{p} = d$ for every $\mathfrak{p} \in \operatorname{Ass}\widehat{A}$, where $\widehat{A}$ denotes the $\mathfrak{m}$-adic completion of A.

With this notation Wolmer V. Vasconcelos posed, exploring the vanishing of the first Hilbert coefficient $\mathrm{e}_Q^1(A)$ for parameter ideals Q, in his lecture at the conference in Yokohama of March, 2008 the following conjecture.

Conjecture 1 ([13]). *Assume that A is unmixed. Then A is a Cohen-Macaulay local ring, once $\mathrm{e}_Q^1(A) = 0$ for some parameter ideal Q of A.*

In Section 2 of the present paper we shall settle Conjecture 1 affirmatively. Here we should note that Conjecture 1 is already solved partially by [2] and [7]. Let us call those local rings A with $\mathrm{e}_Q^1(A) = 0$ for some parameter ideals Q *Vasconcelos*. In Section 3 we shall explore basic properties of Vasconcelos rings. In Section 4 we will study the problem of when $\mathrm{e}_Q^1(A)$ is constant and independent of the choice of parameter ideals Q in A. We shall show that A is a Buchsbaum ring, if A is unmixed and $\mathrm{e}_Q^1(A)$ is constant (Theorem 12).

The detailed version of this paper has been submitted for publication elsewhere.

In what follows, unless otherwise specified, let A denote a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A$. Let $\{\mathrm{H}^i_{\mathfrak{m}}(*)\}_{i \in \mathbb{Z}}$ be the local cohomology functors of A with respect to the maximal ideal $\mathfrak{m}$.

Let $\operatorname{Assh} A = \{\mathfrak{p} \in \operatorname{Ass} A \mid \dim A/\mathfrak{p} = d\}$ and let $(0) = \bigcap_{\mathfrak{p} \in \operatorname{Ass} A} \mathrm{I}(\mathfrak{p})$ be a primary decomposition of (0) in A with $\mathfrak{p}$-primary ideals $\mathrm{I}(\mathfrak{p})$ in A. We put

$$\mathrm{U}_A(0) = \bigcap_{\mathfrak{p} \in \operatorname{Assh} A} \mathrm{I}(\mathfrak{p})$$

and call it the unmixed component of (0) in A.

2. Proof of the conjecture of Vasconcelos

The purpose of this section is to prove the following, which settles Conjecture 1 affirmatively. One of the main results of this paper is the following.

Theorem 2. *Let A be unmixed. Then the following four conditions are equivalent.*

(1) *A is a Cohen-Macaulay local ring.*
(2) *$\mathrm{e}^1_I(A) \geq 0$ for every $\mathfrak{m}$-primary ideal I in A .*
(3) *$\mathrm{e}^1_Q(A) \geq 0$ for some parameter ideal Q in A.*
(4) *$\mathrm{e}^1_Q(A) = 0$ for some parameter ideal Q in A.*

In our proof of Theorem 2 the following facts are the key. See [3, Section 3] for the proof.

Proposition 3 ([3])**.** *Let $(A, \mathfrak{m})$ be a Noetherian local ring with $d = \dim A \geq 2$, possessing the canonical module K_A. Assume that $\dim A/\mathfrak{p} = d$ for every $\mathfrak{p} \in \operatorname{Ass} A \setminus \{\mathfrak{m}\}$. Then the following assertions hold true.*

(1) *The local cohomology module $\mathrm{H}^1_{\mathfrak{m}}(A)$ is finitely generated.*
(2) *The set $\mathcal{F} = \{\mathfrak{p} \in \operatorname{Spec} A \mid \dim A_{\mathfrak{p}} > \operatorname{depth} A_{\mathfrak{p}} = 1\}$ is finite.*
(3) *Suppose that the residue class field $k = A/\mathfrak{m}$ of A is infinite and let I be an $\mathfrak{m}$-primary ideal in A. Then one can choose an element $a \in I \setminus \mathfrak{m}I$ so that a is superficial for I and $\dim A/\mathfrak{p} = d-1$ for every $\mathfrak{p} \in \operatorname{Ass}_A A/aA \setminus \{\mathfrak{m}\}$.*

Proof of Theorem 2. The implication (1) $\Rightarrow$ (2) is due to [8]. The implication (1) $\Rightarrow$ (4) is well known, and (4) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) are trivial. Thus we have only to check the implication (3) $\Rightarrow$ (1). Let $Q = (a_1, a_2, \cdots, a_d)$ with a system $a_1, a_2, \cdots, a_d$ of parameters in A. Enlarging the residue class field $A/\mathfrak{m}$ of A and passing to the $\mathfrak{m}$-adic completion of A, we may assume that the field $A/\mathfrak{m}$ is infinite and that A is complete. The assertion is obvious in the case where $d \leq 2$. Recall that for any Noetherian local ring $(A, \mathfrak{m})$ of dimension one, we have $\mathrm{e}^1_Q(A) = -\ell_A(\mathrm{H}^0_{\mathfrak{m}}(A))$; see [4, Lemma 2.4 (1)], and the two-dimensional case is readily deduced from this fact via the reduction modulo some superficial element $x = a_1$ of Q; see [4, Lemma 2.2] and notice that x is A-regular.

We may assume that $d \geq 3$ and that our assertion holds true for $d-1$. Then we are able to choose, thanks to Proposition 3 (3), the element $x = a_1$ so that x is a superficial element of the parameter ideal Q and (the ring A/xA is *not necessarily* unmixed but) the unmixed component $U = \mathrm{U}_B(0)$ of (0) in $B = A/xA$ has finite length, whence $U = \mathrm{H}^0_{\mathfrak{m}}(B)$. Then the $d-1$ dimensional local ring B/U is Cohen-Macaulay by the hypothesis of induction on d, because

$$\mathrm{e}^1_{Q\cdot(B/U)}(B/U) = \mathrm{e}^1_{QB}(B) = \mathrm{e}^1_Q(A) \geq 0$$

(cf. [4, Lemma 2.2]). Hence $\mathrm{H}^i_{\mathfrak{m}}(B) = (0)$ for all $i \neq 0, d-1$. We now look at the long exact sequence

$$\begin{array}{rcl} \cdots & \to & \mathrm{H}^1_{\mathfrak{m}}(A) \xrightarrow{x} \mathrm{H}^1_{\mathfrak{m}}(A) \to \mathrm{H}^1_{\mathfrak{m}}(B) \to \\ \cdots \to \mathrm{H}^{i-1}_{\mathfrak{m}}(B) & \to & \mathrm{H}^i_{\mathfrak{m}}(A) \xrightarrow{x} \mathrm{H}^i_{\mathfrak{m}}(A) \to \cdots \\ \cdots \to \mathrm{H}^{d-2}_{\mathfrak{m}}(B) & \to & \mathrm{H}^{d-1}_{\mathfrak{m}}(A) \xrightarrow{x} \mathrm{H}^{d-1}_{\mathfrak{m}}(A) \to \cdots \end{array}$$

of local cohomology modules, derived from the short exact sequence

$$0 \to A \xrightarrow{x} A \to B \to 0$$

of A-modules. We then have $\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $2 \leq i \leq d-1$, since $\mathrm{H}^i_{\mathfrak{m}}(B) = (0)$ for all $1 \leq i \leq d-2$, while $\mathrm{H}^1_{\mathfrak{m}}(A) = x\mathrm{H}^1_{\mathfrak{m}}(A)$, since $\mathrm{H}^1_{\mathfrak{m}}(B) = (0)$. Consequently $\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$, because the A-module $\mathrm{H}^1_{\mathfrak{m}}(A)$ is finitely generated by Proposition 2 (1). Thus A is a Cohen-Macaulay ring. $\square$

Let us give one consequence of Theorem 2.

Corollary 4 ([7])**.** *We have $\mathrm{e}^1_Q(A) \leq 0$ for every parameter ideals Q in A.*

3. Vasconcelos rings

The purpose of this section is to develop a theory of Vasconcelos rings. Let us begin with the definition.

Definition 5. We say that A is a *Vasconcelos* ring, if either $d = 0$, or $d > 0$ and $\mathrm{e}^1_Q(A) = 0$ for some parameter ideal Q in A.

Here is a basic characterization of Vasconcelos rings.

Theorem 6. *Suppose that $d = \dim A > 0$. Then the following four conditions are equivalent.*

(1) *A is a Vasconcelos ring.*
(2) *$\mathrm{e}^1_Q(A) = 0$ for every parameter ideal Q in A.*
(3) *$\widehat{A}/\mathrm{U}_{\widehat{A}}(0)$ is a Cohen-Macaulay ring and $\dim_{\widehat{A}} \mathrm{U}_{\widehat{A}}(0) \leq d-2$, where $\mathrm{U}_{\widehat{A}}(0)$ denotes the unmixed component of (0) in the $\mathfrak{m}$-adic completion $\widehat{A}$ of A.*
(4) *The $\mathfrak{m}$-adic completion $\widehat{A}$ of A contains an ideal $I \neq \widehat{A}$ such that $\widehat{A}/I$ is a Cohen-Macaulay ring and $\dim_{\widehat{A}} I \leq d-2$.*

When this is the case, $\widehat{A}$ is a Vasconcelos ring, $\mathrm{H}^{d-1}_{\mathfrak{m}}(A) = (0)$, and the canonical module $\mathrm{K}_{\widehat{A}}$ of $\widehat{A}$ is a Cohen-Macaulay $\widehat{A}$-module.

Proof. See [1, Theorem 3.3]. $\square$

Notice that condition (3) of Theorem 6 is free from parameters. Therefore, since $\mathrm{e}^1_Q(A) = 0$ for some parameter ideal, then $\mathrm{e}^1_Q(A) = 0$ for every parameter ideals Q in A. This is what the theorem says.

In the rest of this section, let us give some consequences of Theorem 6.

Corollary 7. *Let A be a Noetherian local ring with the maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$. Let Q be a parameter ideal in A. Assume that $\mathrm{e}_Q^i(A) = 0$ for all $1 \le i \le d$. Then A is a Cohen-Macaulay ring.*

Suppose that $d > 0$ and let Q be a parameter ideal in A. We denote by $R = \mathcal{R}(Q)$ (resp. $G = \mathrm{G}(Q)$) the Rees algebra (resp. the associated graded ring) of Q. Hence

$$R = A[Qt] \quad \text{and} \quad G = \mathcal{R}'(Q)/t^{-1}\mathcal{R}'(Q),$$

where t is an indeterminate over A and $\mathcal{R}'(Q) = A[Qt, t^{-1}]$. Let $\mathfrak{M} = \mathfrak{m}R + R_+$ be the graded maximal ideal in R. With this notation we have the following.

Corollary 8. *The following assertions hold true.*

(1) *A is a Vasconcelos ring if and only if $G_{\mathfrak{M}}$ is a Vasconcelos ring.*
(2) *Suppose that A is a homomorphic image of a Cohen-Macaulay ring. Then $R_{\mathfrak{M}}$ is a Vasconcelos ring, if A is a Vasconcelos ring.*

Thus Vasconcelos rings enjoy very nice properties.

4. Buchsbaumness in local rings possessing constant first Hilbert coefficients of parameters

In this section we study the problem of when $\mathrm{e}_Q^1(A)$ is constant and independent of the choice of parameter ideals Q in A.

Here let us briefly recall the definition of Buchsbaum local rings. The readers may consult the monumental book [11] of J. Stückrad and W. Vogel for a detailed theory, some of which we shall note here for the use in this paper.

We say that our local ring A is Buchsbaum, if the difference

$$\ell_A(A/Q) - \mathrm{e}_Q^0(A)$$

is independent of the choice of parameter ideals Q in A and is an invariant of A, which we denote by $\mathbb{I}(A)$. As is well-known, A is a Buchsbaum ring if and only if every system $a_1, a_2, \cdots, a_d$ of parameters in A forms a d-sequence in the sense of C. Huneke ([6]). When A is a Buchsbaum local ring, one has

$$\mathfrak{m}{\cdot}\mathrm{H}_{\mathfrak{m}}^i(A) = (0)$$

for all $i \ne d$, whence the local cohomology modules $\{\mathrm{H}_{\mathfrak{m}}^i(A)\}_{i \ne d}$ are finite-dimensional vector spaces over the field $A/\mathfrak{m}$, and the equality

$$\mathbb{I}(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}_{\mathfrak{m}}^i(A))$$

holds true.

We say that A is a generalized Cohen-Macaulay local ring, if all the local cohomology modules $\{\mathrm{H}_{\mathfrak{m}}^i(A)\}_{i \ne d}$ are finitely generated. Hence every Cohen-Macaulay local ring is Buchsbaum with $\mathbb{I}(A) = 0$ and Buchsbaum local rings are generalized Cohen-Macaulay. A given Noetherian local ring A with $d = \dim A > 0$ is a generalized Cohen-Macaulay local ring if and only if

$$\mathbb{I}(A) := \sup_Q \{\ell_A(A/Q) - \mathrm{e}_Q^0(A)\} < \infty,$$

where Q runs through parameter ideals in A ([10]). When this is the case, one has

$$\mathbb{I}(A) = \sum_{i=0}^{d-1} \binom{d-1}{i} \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A)).$$

Suppose that A is a generalized Cohen-Macaulay local ring and let Q be a parameter ideal in A. Then Q is called standard, if

$$\mathbb{I}(A) = \ell_A(A/Q) - \mathrm{e}^0_Q(A).$$

This condition is equivalent to saying that Q is generated by a system $a_1, a_2, \cdots, a_d$ of parameters which forms a strong d-sequence in any order ([10]).

Let

$$\Lambda = \Lambda(A) = \{\mathrm{e}^1_Q(A) \mid Q \text{ be a parameter ideal in } A\ \}.$$

Then we can ask the following questions.

Question 9. When is Λ a finite set or a singleton?

For example, our characterization of Vasconcelos rings says that $0 \in \Lambda$ if and only if $\Lambda = \{0\}$.

Let us summarize what is known about the questions, where we put $h^i(A) = \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$ for each $i \in \mathbb{Z}$.

Proposition 10 ([4, 9])**.** *Suppose that A is a generalized Cohen-Macaulay local ring and $d \geq 2$. Let Q be a parameter ideal in A. Then we have the following.*

(1) $\mathrm{e}^1_Q(A) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$.

(2) *We have* $\mathrm{e}^1_Q(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$, *if Q is standard.*

Thanks to Proposition 10 (1) and Corollary 4, if A is a generalized Cohen-Macaulay ring then we have

$$0 \geq \mathrm{e}^1_Q(A) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$$

for every parameter ideal Q in A. Hence Λ is finite. If A is a Buchsbaum ring then, since all parameter ideals in A are standard, we have

$$\mathrm{e}^1_Q(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$$

for every parameter ideal Q in A. Thus, we have

$$\Lambda = \left\{ -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A) \right\},$$

so that Λ is a singleton. It is natural to guess the converse is also true.

Our answer is the following.

Theorem 11. *Suppose that $d \geq 2$ and A is unmixed. Assume that Λ is a finite set and put $\ell = -\min \Lambda$. Then $\mathfrak{m}^\ell \mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for every $i \neq d$. Hence $\mathrm{H}^i_{\mathfrak{m}}(A)$ is a finitely generated A-module for every $i \neq d$, so that A is a generalized Cohen-Macaulay local ring.*

The main result of this section is stated as follows.

Theorem 12. *Suppose that $d = \dim A \geq 2$ and A is unmixed. Then the following two conditions are equivalent.*

(1) *A is a Buchsbaum local ring.*
(2) *The first Hilbert coefficients $\mathrm{e}_Q^1(A)$ of A are constant and independent of the choice of parameter ideals Q in A.*

When this is the case, one has the equality

$$\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$$

for every parameter ideal Q in A.

Thus Buchsbaum rings are characterized in terms of consistency of the first Hilbert coefficients of parameters. This is a new characterization of Buchsbaum rings.

The following result is a key for the proof of Theorem 12.

Theorem 13. *Suppose that A is a generalized Cohen-Macaulay local ring with $d = \dim A \geq 2$ and $\operatorname{depth} A > 0$. Let Q be a parameter ideal in A. Then the following two conditions are equivalent.*

(1) *Q is a standard parameter ideal in A.*
(2) $\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$.

In our proof of Theorem 13 we need the following result. Let

$$\mathrm{U}(a) = \bigcup_{n \geq 0} [(a) :_A \mathfrak{m}^n]$$

for each $a \in A$.

Proposition 14. *Suppose that A is a generalized Cohen-Macaulay local ring with $d = \dim A \geq 3$ and $\operatorname{depth} A > 0$. Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Assume that $(a_1, a_d)\mathrm{H}_\mathfrak{m}^1(A) = (0)$ and that the parameter ideal $(a_1, a_2, \cdots, a_{d-1}){\cdot}[A/\mathrm{U}(a_d)]$ is standard in the generalized Cohen-Macaulay local ring $A/\mathrm{U}(a_d)$. Then*

$$\mathrm{U}(a_1) \cap Q = (a_1).$$

Proof. Since $\mathrm{U}(a_1) \cap Q = (a_1) + [\mathrm{U}(a_1) \cap (a_2, a_3, \cdots, a_d)]$, we have only to show

$$\mathrm{U}(a_1) \cap (a_2, a_3, \cdots, a_d) \subseteq (a_1).$$

Let $x \in \mathrm{U}(a_1) \cap (a_2, a_3, \cdots, a_d)$ and put $\overline{A} = A/\mathrm{U}(a_d)$. Let $\overline{x}$ and $\overline{a_i}$ respectively denote the images of x and a_i in $\overline{A}$. Then we have

$$\overline{x} \in \mathrm{U}(\overline{a_1}) \cap (\overline{a_2}, \overline{a_3}, \cdots, \overline{a_{d-1}}) \subseteq (\overline{a_1}),$$

because $\mathrm{U}(\overline{a_1}) = (\overline{a_1}) :_{\overline{A}} \overline{a_2}$ and $\overline{a_2}, \overline{a_3}, \cdots, \overline{a_{d-1}}$ forms a d-sequence in $\overline{A}$ (recall that by our assumption $(\overline{a_2}, \overline{a_3}, \cdots, \overline{a_{d-1}})$ is a standard parameter ideal in the generalized Cohen-Macaulay local ring $\overline{A}$). Hence

$$x \in [(a_1) + \mathrm{U}(a_d)] \cap \mathrm{U}(a_1) = (a_1) + [\mathrm{U}(a_1) \cap \mathrm{U}(a_d)].$$

Let $x = y + z$ with $y \in (a_1)$ and $z \in \mathrm{U}(a_1) \cap \mathrm{U}(a_d)$. We will show that $z \in (a_1)$.

Since $a_1\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$ and a_1 is A-regular, we have

$$\mathrm{H}^1_{\mathfrak{m}}(A) \cong \mathrm{H}^0_{\mathfrak{m}}(A/(a_1)) = \mathrm{U}(a_1)/(a_1),$$

whence $a_d\mathrm{U}(a_1) \subseteq (a_1)$, because $a_d\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$ by our assumption. By the same argument applied to a_d we get $a_1\mathrm{U}(a_d) \subseteq (a_d)$. Hence $a_1 z \in (a_d)$ and $a_d z \in (a_1)$. Let us now write

$$a_1 z = a_d u \quad \text{and} \quad a_d z = a_1 v \quad \text{with} \quad u, v \in A.$$

Then, since $a_1 a_d z = a_d^2 u = a_1^2 v$, we have $u \in \mathrm{U}(a_1^2)$. Notice that

$$\mathrm{H}^1_{\mathfrak{m}}(A) \cong \mathrm{H}^0_{\mathfrak{m}}(A/(a_1^2)) = \mathrm{U}(a_1^2)/(a_1^2),$$

since $a_1^2\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$ and a_1^2 is A-regular. Therefore $a_d\mathrm{U}(a_1^2) \subseteq (a_1^2)$, because $a_d\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$. Hence $a_1 a_d z = a_d{\cdot}a_d u \in (a_1^2 a_d)$, so that $z \in (a_1)$. Thus $x = y + z \in (a_1)$, as is claimed. $\square$

To prove Theorem 13 we also need the following lemma.

Lemma 15 ([1, Lemma 4.5])**.** *Suppose that A is a generalized Cohen-Macaulay local ring with $d = \dim A \geq 2$ and $\operatorname{depth} A > 0$. Let Q be a parameter ideal in A and assume that $\mathrm{e}^1_Q(A) = -\sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(A)$. Then $Q\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $1 \leq i \leq d-1$.*

We are now in a position to prove Theorem 13.

Proof of Theorem 13. Enlarging the residue class field $A/\mathfrak{m}$ of A if necessary, we may assume that the field $A/\mathfrak{m}$ is infinite. Let $Q = (a_1, a_2, \cdots, a_d)$, where each a_j is superficial for the ideal Q. Recall that $Q\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for all $1 \leq i \leq d-1$ by Lemma 15. Hence Q is standard, if $d = 2$ ([12, Corollary 3.7]).

Assume that $d \geq 3$ and that our assertion holds true for $d-1$. Let $B = A/(a_j)$ with $1 \leq j \leq d$ and put $\overline{A} = B/\mathrm{H}^0_{\mathfrak{m}}(B)$ $(= A/\mathrm{U}(a_j))$. Then $\mathrm{H}^i_{\mathfrak{m}}(\overline{A}) \cong \mathrm{H}^i_{\mathfrak{m}}(B)$ for all $i \geq 1$. On the other hand, since $a_j\mathrm{H}^i_{\mathfrak{m}}(A) = (0)$ for $1 \leq i \leq d-1$ and a_j is A-regular, we get for each $0 \leq i \leq d-2$ the short exact sequence

$$0 \to \mathrm{H}^i_{\mathfrak{m}}(A) \to \mathrm{H}^i_{\mathfrak{m}}(B) \to \mathrm{H}^{i+1}_{\mathfrak{m}}(A) \to 0$$

of local cohomology modules. Consequently we get $\mathbb{I}(A) = \mathbb{I}(B)$ and

$$\begin{aligned}
\mathrm{e}^1_Q(A) = \mathrm{e}^1_{QB}(B) &= \mathrm{e}^1_{Q\overline{A}}(\overline{A}) \\
&\geq -\sum_{i=1}^{d-2}\binom{d-3}{i-1}h^i(\overline{A}) \\
&= -\sum_{i=1}^{d-2}\binom{d-3}{i-1}h^i(B) \\
&= -\sum_{i=1}^{d-2}\binom{d-3}{i-1}[h^i(A) + h^{i+1}(A)] \\
&= -\sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(A) \\
&= \mathrm{e}^1_Q(A).
\end{aligned}$$

Hence the equality

$$\mathrm{e}^1_{Q\overline{A}}(\overline{A}) = -\sum_{i=1}^{d-2}\binom{d-3}{i-1}h^i(\overline{A})$$

holds true for the parameter ideal $Q\overline{A}$ in the generalized Cohen-Macaulay local ring $\overline{A}$. Thus the hypothesis of induction on d yields that $Q{\cdot}[A/\mathrm{U}(a_j)]$ is a standard parameter ideal in $A/\mathrm{U}(a_j)$ for every $1 \le j \le d$. Therefore $\mathrm{U}(a_1) \cap Q = (a_1)$ by Proposition 14, so that $Q{\cdot}[A/(a_1)]$ is a standard parameter ideal in $A/(a_1)$ ([12, Corollary 2.3]), since $Q{\cdot}[A/\mathrm{U}(a_1)]$ is a standard parameter ideal for the local ring $A/\mathrm{U}(a_1)$. Thus Q is a standard parameter ideal in A ([12, Corollary 2.1]), since $\mathbb{I}(A) = \mathbb{I}(A/(a_1))$. $\square$

We are now ready to prove Theorem 12.

Proof of Theorem 12. We have only to show the implication (2) $\Rightarrow$ (1). Since $\sharp\Lambda = 1$, by Theorem 11, A is a generalized Cohen-Macaulay local ring, so that

$$\Lambda = \left\{-\sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(A)\right\}$$

by Proposition 10 (2). Hence by Theorem 13 every parameter ideal Q is standard in A, because $\mathrm{e}^1_Q(A) = -\sum_{i=1}^{d-1}\binom{d-2}{i-1}h^i(A)$, so that A is a Buchsbaum local ring. $\square$

Unless A is unmixed, Theorem 12 is no more true, even if $\mathrm{e}^1_Q(A) = 0$ for every parameter ideal Q in A (cf. [1, Theorem 2.7]). Let us note one example.

Example 16. Let R be a regular local ring with the maximal ideal n and $d = \dim R \ge 3$. Let $X_1, X_2, \cdots, X_d$ be a regular system of parameters of R. We put $\mathfrak{p} = (X_1, X_2, \cdots, X_{d-1})$ and $D = R/\mathfrak{p}$. Then D is a DVR. Let $A = R \ltimes D$ denote the idealization of D over R. Then A is a Noetherian local ring with the maximal ideal $\mathfrak{m} = \mathrm{n} \times D$ and $\dim A = d$. Let Q be a parameter ideal in A and put $\mathfrak{q} = \varphi(Q)$, where $\varphi : A \to R, \varphi(a, x) = a$ denotes the projection map. We then have

$$\begin{aligned}
\ell_A(A/Q^{n+1}) &= \ell_R(R/\mathfrak{q}^{n+1}) + \ell_D(D/\mathfrak{q}^{n+1}D)\\
&= \ell_R(R/\mathfrak{q}){\cdot}\binom{n+d}{d} + \ell_D(D/\mathfrak{q}D){\cdot}\binom{n+1}{1}\\
&= \mathrm{e}^0_{\mathfrak{q}}(R)\binom{n+d}{d} + \mathrm{e}^0_{\mathfrak{q}D}(D)\binom{n+1}{1}
\end{aligned}$$

for all integers $n \ge 0$, so that $\mathrm{e}^0_Q(A) = \mathrm{e}^0_{\mathfrak{q}}(R), \mathrm{e}^{d-1}_Q(A) = (-1)^{d-1}\mathrm{e}^0_{\mathfrak{q}D}(D)$, and $\mathrm{e}^i_Q(A) = 0$ if $i \ne 0, d-1$. Hence $\mathrm{e}^1_Q(A)$ is constant but A is not even a generalized Cohen-Macaulay local ring, because $\mathrm{H}^1_{\mathfrak{m}}(A)$ $(\cong \mathrm{H}^1_{\mathrm{n}}(D))$ is not a finitely generated A-module. The local ring A is not unmixed, although $\operatorname{depth} A = 1$.

We close this paper with a characterization of Noetherian local rings A possessing $\sharp\Lambda = 1$. Let us note the following.

Proposition 17 ([1, Proposition 4.7])**.** *Suppose that $d = \dim A \ge 2$ and let U be the unmixed component of the ideal* (0) *in A. Assume that there exists an integer $t \ge 0$*

such that $\mathrm{e}_Q^1(A) = -t$ *for every parameter ideal* Q *in* A. *Then* $\dim_A U \le d-2$ *and* $\mathrm{e}_{\mathfrak{q}}^1(A/U) = -t$ *for every parameter ideal* $\mathfrak{q}$ *in* A/U.

The goal of this paper is the following.

Theorem 18. *Suppose that* $d = \dim A \ge 2$. *Then the following two conditions are equivalent.*

(1) $\sharp\Lambda = 1$.

(2) *Let* $U = \mathrm{U}_{\widehat{A}}(0)$ *be the unmixed component of the ideal* (0) *in the* $\mathfrak{m}$*-adic completion* $\widehat{A}$ *of* A. *Then* $\dim_{\widehat{A}} U \le d-2$ *and* $\widehat{A}/U$ *is a Buchsbaum local ring.*

When this is the case, one has the equality

$$\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(\widehat{A}/U)$$

for every parameter ideal Q *in* A.

Proof. (1) $\Rightarrow$ (2) For every parameter ideal $\mathfrak{q}$ of $\widehat{A}$ we have $\mathfrak{q} = (\mathfrak{q} \cap A)\widehat{A}$, so that $\mathfrak{q} \cap A$ is a parameter ideal in A. Hence $\Lambda(\widehat{A}) = \Lambda$ and so the implication follows from Theorem 12 and Proposition 17.

(2) $\Rightarrow$ (1) Since $\dim_{\widehat{A}} U \le d-2$ and $\widehat{A}/U$ is a Buchsbaum local ring, we get $\#\Lambda(\widehat{A}) = 1$ by [1, Lemma 2.4 (c)], whence $\#\Lambda = 1$.

See Proposition 10 (2) and 17 for the last assertion. $\square$

References

[1] L. Ghezzi, S. Goto, J. Hong, K. Ozeki, T. T. Phuong, and W. V. Vasconcelos, *Cohen-Macaulayness versus the vanishing of the first Hilbert coefficient of parameters*, J. London Math. Soc., **81** (2010), 679–695.

[2] L. Ghezzi, J.-Y. Hong, W. V. Vasconcelos, *The signature of the Chern coefficients of local rings*, Math. Ress. Lett. **16** (2009), no. 2, 279–289.

[3] S. Goto and Y. Nakamura, *Multiplicities and Tight closures of parameters*, J. Algebra, **244** (2001), 302–311.

[4] S. Goto and K. Nishida, *Hilbert coefficients and Buchsbaumness of associated graded rings*, J. Pure and Appl. Algebra, Vol **181** (2003), 61–74.

[5] S. Goto and K. Ozeki, *Buchsbaumness in local rings possessing constant first Hilbert coefficients of parameters*, Nagoya Math. J., **199** (2010), 95–105.

[6] C. Huneke, *On the Symmetric and Rees Algebra of an Ideal Generated by a d-sequence*, J. Algebra., **62** (1980), 268–275.

[7] M. Mandal and J. K. Verma, *On the Chern number of an ideal*, Proc. Amer. Math. Soc., **138** (2010), 1995–1999.

[8] D. G. Northcott, *A note on the coefficients of the abstract Hilbert function*, J. London Math. Soc. **35** (1960) 209–214.

[9] P. Schenzel, *Multiplizitäten in verallgemeinerten Cohen-Macaulay-Moduln*, Math. Nachr., **88** (1979), 295–306.

[10] P. Schenzel, N. V. Trung, and N. T. Cuong, *Verallgemeinerte Cohen-Macaulay-Moduln*, Math. Nachr., 85 (1978), 57–73.

[11] J. Stückrad and W. Vogel, *Toward a theory of Buchsbaum singularities*, Amer. J. Math., **100** (1978), no. 4, 727–746.

[12] N. V. Trung, *Toward a theory of generalized Cohen-Macaulay modules*, Nagoya Math. J., **102**, 1986, 1–49.

[13] W. V. Vasconcelos, *The Chern coefficients of local rings*, Michigan Math. J., **57** (2008), 725–743.

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ARTINIAN RINGS WITH INDECOMPOSABLE RIGHT MODULES UNIFORM

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Abstract. It is well known that any indecomposable module over a generalized uniserial ring is uniserial, therefore it is local as well as uniform. This motivated Tachikawa (1959) to study rings satisfying the following conditions. A ring R is said to satisfy condition (*) if it is artinian and every finitely generated indecomposable right R-module is local. A ring R is said to satisfy condition (**), if it is artinian and every finitely generated indecomposable right R-module is uniform. He had given a characterisation of condition (**). If a ring R satisfies (*), it admits a finitely generated injective co-generator. Consider any artinian ring R such that *mod*-R admits a finitely generated injective co-generator M, Let $Q = End(M)$ acting on left. By Tachikawa, every finitely generated indecomposable right R-module is local if and only if every finitely generated indecomposable left Q-module is uniform. In the present note, we give a characterisation of condition (**) in terms of the structure of the right ideals of the given ring. The approach in the present paper is quite different from that followed by Tachikawa. Let M be a uniform module of finte composition length, $D = End(soc(M))$ and D' the subdivision ring of D consisting of those $\sigma \in D$, which have some extensions in $End(M)$. Then the pair (D, D') is called *division ring pair associate* (in short *drpa*) of M. An outline of the proof the following result is given. A ring R with Jacobson radical J satisfies (**) if and only if it satisfies the following conditions: (1) R is a both sided artinian, right serial ring; (2) for any three indecomposable idempotents e, f, $g \in R$ with eJ, fJ, gJ non-zero the following hold: (i) If (D, D') is the *drpa* of $\frac{eR}{eJ^2}$, then the left dimension and the right dimensions of D over D' both are less than or equal to 2; (ii) if e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2}$, then $eJ^2 = 0$ or $fJ^2 = 0$; (iii) if e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2} \cong \frac{gJ}{gJ^2}$, then g is isomorphic to e or f; (iv) if $\frac{eR}{eJ^2}$ is not quasi-injective, then $eJ^2 = 0$ and $\frac{eJ}{eJ^2} \ncong \frac{fJ}{fJ^2}$, whenever e is not isomorphic to f. First step in the proof is to develop some techniques of construction of indecomposable modules which may be uniform or may not be uniform. There after a theorem involving lifting of an isomorphisms between simple homomorphic images of two finitely generated uniform modules is established, which is used to give the proof of the main theorem.

Key Words: Right serial rings, uniserial modules, quasi-injective, quasi-projective modules.

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Introduction

We consider the following conditions on a ring R. (**) R is a both sided artinian ring such that every finitely generated indecomposable right R-module is uniform. And its

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dual condition (*) R is both sided artinian such that every finitely generated indecomposable right module is local. These conditions have been studied by Tachikawa [7]. In [7, Theorem 5.3], a characterization of a ring satisfying (**) on the left is given. Here we discuss another approach to the study of rings satisfying these conditions and give a characterization of rings satisfying (**) in terms of the structure of its right ideals. The main purpose is to outline the proof of the main theorem, therefore. The main steps in the proof of the main theorem are given detail, but are stated without proof. In the process we also determine the structure of indecomposable modules over a ring satisfying (**). Throughout R is an artinian ring. In Section 1, some concepts and results proved in [5] are collected, in particular the concept of division ring pair associate of a uniform module of finite composition length is given in Definition 1.2. In Section 2, the ring of endomorphisms of a finite direct sum of uniform modules of finite composition lengths is investigated. These results can be of independent interest. The study of condition (**) is started in Section 3. To start with a lifting property of isomorphism between simple homomorphic images of uniform modules over a ring R satisfying (**) is proved. If an artinian ring R satisfies this lifting property, then R is said to satisfy condition weak (**). In Proposition 3.6, it is proved that any ring satisfying weak (**) is right serial. The concept of a critical uniserial submodule of a uniform module over an artinian ring is given in Definition 3.3. In Proposition 3.4, it is proved that if a ring R satisfies weak (**), then a uniform right R-module is either uniserial or its critical uniserial submodule is simple. In Proposition 3.9 and Theorem 3.10, some properties and relations between indecomposable summand of R_R, where R satisfies weak (**) are proved, which form a basis for giving a characterization of rings satisfying (**). In Section 4, a condition (***) motivated by results in Section 3, is introduced, which is satisfied by any ring satisfying (**). The structure of indecomposable modules over a ring R satisfying (**) is given in Theorems 4.6 and 4.7. The main result is given in Theorem 4.12. The whole paper depends on various constructions of indecomposable, non-uniform modules.

1. Preliminaries

All the modules considered here are unitary right modules, unless otherwise stated. For any ring R, its Jacobson radical is denoted by $J(R)$ (or simply by J). For any module M, $E(M)$, $End(M), d(M)$, $J(M)$ denote its injective hull, ring of endomorphisms, composition length, radical of M respectively. By a summand of a module M, we shall mean a summand other than 0, M. If a module $M = A \oplus B$, the resulting projection of M on A will be sometime denoted by π_A. The symbols $A \leqslant B$ $(A < B)$ will mean that A is a submodule of a module B (A is a submodule of a module B, but $A \neq B$). A non-zero element x of a module M_R is called a *local* (*uniform*) element, if xR is a local (uniform) module. A ring R is said to be *artinian*, if it is right artinian as well as left artinian. Let S, T be two simple modules over a ring R. Then T is called a *predecessor* of S and S is called a *successor* of T, if there exists a uniserial module A_R such that $d(A)$ $= 2$, and for the maximal submodule B of A , $S \cong B$, $T \cong \frac{A}{B}$. A module M is said to be *uniserial*, if the family of its submodules is linearly ordered under inclusion. If a ring R is such that R_R is a finite direct sum of uniserial modules, then R is called a *right serial* *ring.* An artinian ring that is right and left serial is called a *generalized uniserial ring.* For various concepts on rings and modules one may consult [1] or [8].

The following is a modification of [5, Lemma 2.1]. For this see also [7, Lemma 1.3, Proposition 2.3].

Lemma 1.1. *Let A, B be two uniform modules over a right artinian ring R, and S be the simple submodule of A. Let there exist a monomorphism $\sigma : S \to B$, $L = \{(a, -\sigma(a)) : a \in S\}$ and $M = \frac{A \times B}{L}$. If $\overline{(x, y)}R$ is a simple submodule of M other than $T = \{\overline{(s, 0)} : s \in S\}$, then $f : xR \to yR$, $f(xr) = yr$, $r \in R$ defines a homomorphism extending $-\sigma$. Further M is uniform if and only if there is no module C_R with $S < C \leqslant A$ for which there exists a homomorphism $f : C \to B$ extending σ.*

Definition 1.2. Let A_R be a uniform module of finite composition length, $S = soc(A)$, $D = End(S)$, and D' be the division subring of D consisting of those $\sigma \in D$ that can be extended to some endomorphisms of A. Then the pair (D, D') is called the *division ring pair associate* (in short the *drpa*) of A.

For any subdivision ring D' of a division ring D, $[D, D']_l$ ($[D, D']_r$) will denote the dimesion of D as a left (right) vector space over D'.

Lemma 1.3. *Let A_R be a uniserial quasi-projective module with $d(A) = 2$ and $S = soc(A)$. Let (D, D') be the drpa of A.*

(i) [5, Lemma 2.2]. *Let ω_1 $(= I)$, ω_2, ..., ω_n any n non-zero members of the D. Then $M = \frac{A^{(n)}}{L}$, where $L = \{(\omega_1 x_1, \omega_2 x_2, .., \omega_n x_n) : x_i \in S, \Sigma_i x_i = 0\}$ is uniform if and only if $\omega_1^{-1}, \omega_2^{-1}, ..., \omega_n^{-1}$ are right linearly independent over D'.*

(ii) [5, Lemmas 2.3, 2.4]. *Let $E = E(A)$, λ_i be automorphisms of E for $1 \leq i \leq n$, with $\lambda_1 = I$, where n is some positive integer. Let $K = A_1 + A_2 + ... + A_n$, where each $A_i = \lambda_i(A)$, and let $\omega_i = \lambda_i \mid S$. Then $A_j \not\subseteq \Sigma_{i \neq j} A_i$ for any j if and only if $\omega_1, \omega_2, ..., \omega_n$ are right linearly independent over D'. Further, if $A_j \not\subseteq \Sigma_{i \neq j} A_i$ for every j, for $A = A_1$, $\frac{A^{(n)}}{L} \cong K$, where $L = \{(\omega_1^{-1} x_1, \omega_2^{-1} x_2, ..., \omega_n^{-1} x_n) : x_i \in S, \ \Sigma_i x_i = 0\}$, and this isomorphism is induced by the epimorphism $\lambda : A^{(n)} \to K$, $\lambda(a_1, a_2, ..., a_n) = \lambda_1(a_1) + \lambda_2(a_2) + ... + \lambda_n(a_n)$.*

Lemma 1.4. *Let A_R be a uniserial module with $d(A) = 2$, $S = soc(A)$ and $E = E(A)$.*

(i) *If λ is an automorphism of E such that $\lambda \mid S$ is identity on S, then $\lambda(A) = A$.*

(ii) *If two automorphisms σ, η of E are equal on S, then $\sigma(A) = \eta(A)$.*

Proof. (i) Suppose $A \neq \lambda(A)$. Then $S = A \cap \lambda(A)$. Let (D, D') be the *drpa* of A. Consider the mapping $\mu : A \times A \to A + \lambda(A)$, $\mu(a, b) = a + \lambda(b)$. Here $a + \lambda(b) = 0$ gives $a \in S$, therefore $ker\, \mu$ is $L = \{(a, -\lambda^{-1}(a)) : a \in S\}$. Therefore $M = \frac{A \times A}{L}$ is uniform, and by (1.3), I , ω $(= \lambda \mid S)$ are right linearly independent over D', which is a contradiction. Hence $\lambda(A) = A$.

(ii) is immediate from (i). $\square$

The following is from Lemmas 2.6 and 2.7 in [5]

Lemma 1.5. *Let K_R be a non-simple uniform module of finite composition length, $S = soc(K)$ and (D, D') drpa of K. Let ω_1 $(= I)$, $\omega_2, ..., \omega_n$ be any n non-zero members of D,* *and $L = \{(\omega_1 x, \omega_2 x, ..., \omega_n x) : x \in S\}$. Then L is not contained in a summand of $K^{(n)}$ if and only if ω_1, $\omega_2, ..., \omega_n$ are left linearly independent over D'. If L is not contained in a summand of $K^{(n)}$ and K is quasi-projective then $M = \frac{K^{(n)}}{L}$ is indecomposable; if in addition $n > 2$, then M is not uniform.*

2. Endomorphism rings

Theorem 2.1. *Let A_1, A_2, ..., A_n be any finitely many uniform right modules of finite composition lengths, over a ring R, $M = A_1 \oplus A_2 \oplus ... \oplus A_n$ and $K = End(M)$. Then $J(K)$ is the set of all those $n \times n$-matrices $[\sigma_{ij}]$, where no $\sigma_{ij} : A_j \to A_i$ is an isomorphism.*

Proof. Let A, B, C be any three non-zero uniform right modules of finite composition lengths, over R. Let σ, $\eta : A \to B$ be two homomorphisms, which are not isomorphisms. If one of σ, η is a monomorphism, then $d(A) < d(B)$, therefore $\sigma + \eta$ is not an isomorphism. If neither of σ, η is a monomorphism, then both of them are zero on the $soc(A)$, therefore again $\sigma + \eta$ is not an isomorphism. After this it can be seen that the set N of all $[\sigma_{ij}] \in K$, in which no entry is an isomorphism, is an ideal of K.

Now suppose that $d(A) \leq d(B)$. Let $\lambda : B \to C$ be a homomorphism which is not an isomorphism, but $d(C) \leq d(B)$. As λ is not a monomorphism, it is zero on $soc(B)$. Let $0 \neq L \leqslant A$ and $\sigma : A \to B$ be a homomorphism. If $\sigma(L) \neq 0$, then $soc(B) \leqslant \sigma(L)$, $d(\lambda\sigma(L)) < d(\sigma(L))$. Therefore $d(\lambda\sigma(L)) < d(L)$. Let $\mu : C \to A$ be an homomorphism which is not an isomorphism, and $d(C) \geq d(A)$. Then μ is zero on $soc(C)$. Therefore for any non-zero submodule L of C, $d(\mu(L)) < d(L)$, $d(\sigma\mu(L)) < d(L)$. If we take an admissible product of a sequence of up to $n+1$ entries of $[\sigma_{ij}] \in N$, it results in some homomorphisms $\eta_{ji} : A_i \to A_j$, $\eta_{kj} : A_j \to A_k$ where the situation is similar to the one discussed above in the sense that either $d(A_i) \leq d(A_j) \geq d(A_k)$ or $d(A_i) \geq d(A_j) \leq d(A_k)$. Using this we find that each member of N is nilpotent. Hence $N \subseteq J(K)$. If a $[\sigma_{ij}] \in J(K)$, it can be easily seen that no entry in $[\sigma_{ij}]$ is an isomorphism. Hence $N = J(K)$. $\square$

Theorem 2.2. *Let A_1, A_2, ..., A_n be any finitely many uniform right modules of finite composition lengths, over a ring R, such that they have isomorphic socles. Let C_1, C_2, ..., C_t be the isomorphism classes of A_1, A_2, ..., A_n. arranged in such a way that for any $i < t$, if some $A_k \in C_i$ and $A_l \in C_{i+1}$, then $d(A_k) \leq d(A_l)$. Let A_1, A_2, ..., A_n be re-indexed such that if an $A_k \in C_i$ and an $A_l \in C_{i+1}$, then $k < l$. Let $S = soc(A_i)$ for $1 \leq i \leq n$, $\omega_1 (= I)$, ω_2, ..., ω_n be any n non-zero members of $D = End(S_R)$. Let $0 \neq x_1 \in S$ and $x_i = \omega_i x_1$ for $1 \leq i \leq n$. Then $x = (x_1, x_2, ..., x_n) \in M = A_1 \times A_2 \times .... \times A_n$ is contained in a summand of M if and only if for some $1 < j \leq n$, there exist homomorphisms $\eta_{jk} : A_k \to A_j$ for $1 \leq k < j$ such that $\omega_j = \mu_{j1}\omega_1 + \mu_{j2}\omega_2 + ... + \mu_{jj-1}\omega_{j-1}$, where each $\mu_{jk} = \eta_{jk} \mid S$.*

Proof. For $1 \leq j \leq t$, let B_j be the direct sum of those A_i's that are in C_j. Suppose the cardinality of C_j is k_j. Now $M = B_1 \oplus B_2 \oplus .... \oplus B_t$. Any $\sigma \in T = End(M)$ can be represented as a block matrix $[H_{ij}]$, where each H_{ij} is a $k_i \times k_j$-matrix representing an R-homomorphism from $B_j \to B_i$. Now x is contained in a summand of M if and only if there exists a non-zero idempotent $\sigma \in T$, satisfying $\sigma y = 0$, where y is the transpose of the row matrix $[x_1, x_2, ..., x_n]$. Consider any $j > i$. For any $A_k \in C_i$, $A_l \in C_j$, as

$d(A_k) \leq d(A_l)$, any homomorphism $\theta_{kl} : A_l \to A_k$ is zero on $soc(A_l)$. That means that the effect of H_{ij} on the corresponding block in y is zero. Thus the lower triangular block matrix $\eta = [G_{ij}]$ such that $G_{ij} = H_{ij}$ for $i \geq j$, and $G_{ij} = 0$ for $j > i$ has same effect on y as of σ on y. But $\eta \equiv \sigma \ (mod\ J(T))$. Suppose no entry of any G_{ii} is an isomorphism, then the matrix of the diagonal block of η is in $J(T)$, from which it follows that η is nilpotent. Consequently σ is nilpotent, which is a contradiction. Hence there exists smallest positive integer k such that some entry of G_{kk} is an isomorphism. Write $x = [z_1, z_2, ..., z_t]$, where each z_i is a block with k_i entries and let u_i be the transpose of z_i . As $\sigma y = 0$, $\sum\limits_{i=1}^{k} H_{ki} z_i = 0$. Now H_{kk} has an entry, say σ_{rs} which is an isomorphism. For this, we choose s to be largest with respect to the fixed r. At the same time write $\sigma = [\sigma_{ij}]$ where each $\sigma_{ij} : A_j \to A_i$. Let $\sigma_{rs}^{-1} e_{sr}$ be the $n \times n$ matrix whose (s, r)-th entry is σ_{rs}^{-1} and its other entries are zero. Then $(\sigma_{rs}^{-1} e_{sr})\sigma y = 0$. This gives $\sum\limits_{i=1}^{s} \lambda_{si} x_i = 0$, for some $\lambda_{si} : A_i \to A_s$, $\lambda_{ss} = I$, the identity map on A_s. As $x_i = \omega_i x_1$, we get $\omega_s = \sum\limits_{i=1}^{s-1} \mu_{si}\omega_i$, where each $\mu_{si} = -\lambda_{si} \mid S$.

Conversely, let $\omega_s = \sum\limits_{i=1}^{s-1} \mu_{si}\omega_i$ for some $s > 1$, such that each $-\mu_{si}$ is the restriction to S of some homomorphism $\eta_{si} : A_i \to A_s$. Let $\psi = [\psi_{ij}]$, where $\psi_{ij} = 0$ for $i \neq s$, $\psi_{si} = -\mu_{si}$ for $1 \leq i \leq s-1, \psi_{ss} = I$, $\psi_{sj} = 0$ for $j > s$. Then ψ is a non-zero idempotent such that $\psi y = 0$. □

3. Condition weak (**)

We start with the following condition. (**) R is an artinian ring such that every finitely generated indecomposable right R-module is uniform.

Following is a lifting property for condition (**).

Theorem 3.1. *Let R be a ring satisfying (**). Let M, N be two finitely generated uniform right R-modules. If for some maximal submodules M', N' of M, N respectively, there exists an isomorphism $\sigma : \frac{M}{M'} \to \frac{N}{N'}$, then σ or σ^{-1} can be lifted to a homomorphism η from M to N or from N to M respectively.*

Proof. Let $T = \{(a, b)$: $a \in M$, $b \in N$, $\sigma(\overline{a}) = \overline{b}\}$. Then T is a submodule of $M \times N$ containing $M' \times N'$ such that if an $(a, b) \in T$ with $a \notin M'$, then $T = (a, b)R + M' \times N'$. Therefore $M' \times N'$ is maximal in T and T is maximal in $M \times N$. For the projections $\pi_1 : M \times N \to M$, $\pi_2 : M \times N \to N$, $\pi_1(T) = M$, $\pi_2(T) = N$. As $d(soc(T)) = 2$, $T = C \oplus D$ for some uniform submodules C, D. Let $0 \neq s \in soc(M)$. Then $(s, 0) = c + u$, $c \in soc(C)$, $u \in soc(D)$.

We take $d(M) \geqslant d(N)$, $c = (s_1, s_2)$, $u = (u_1, u_2)$ for some s_1, $u_1 \in M$, u_2, $s_2 \in N$. Now $c \neq 0$ or $u \neq 0$. Suppose $c \neq 0$. Then M' embeds in C under π_C. therefore $d(C) = d(M)$ or $d(C) = d(M) - 1$. Suppose $d(C) = d(M)$. Then $d(D) = d(N) - 1 < d(C)$. Now Suppose $s_1 = 0$, then C embeds in N under π_2, therefore $d(C) = d(N)$, $\pi_2(C) = N$. Thus there exists $(a', b') \in C$ with $b' \notin N'$. Now $\sigma^{-1}(\overline{b'}) = \overline{a'}$. Let $y \in N$. As $\pi_2 \mid C$ is an isomorphism, there exists unique $(x, y) \in C$, with $x \in M$. We get a homomorphism $\eta : N \to M$ for which $\eta(y) = x$, this homomorphism lifts σ^{-1}. Suppose $s_1 \neq 0$. Then

C is isomorphic to M under π_1. Let $x \in M$. Then there exists unique $y \in N$ such that $(x, y) \in C$. This gives a homomorphism $\eta : M \to N$ for which $\eta(x) = y$, which lifts σ. We shall be using similar arguments for some other situations.

Now suppose $d(C) = d(M) - 1$. Then $d(D) = d(N)$. Suppose $u = 0$, then $soc(C) = soc(M)$, $soc(D) \neq soc(M)$, and $D \cong \pi_2(D) = N$; as before, we get a homomorphism $\eta : N \to M$ lifting σ^{-1}. Now suppose $u \neq 0$. Suppose $u_2 \neq 0$. Then $D \cong \pi_2(D) = N$. We get a lifting of σ^{-1}. Suppose $u_2 = 0$. Then $s_2 = 0$, $u_1 \neq 0$ as $(s, 0) = c + u$. Then $C \cap D = 0$, gives $c = 0$, $C \cong \pi_2(C) = N$. This gives a lifting of σ^{-1}. □

The above result is a partial dual of [6, Proposition 2.2]. It is not known, whether the converse of the above result holds

The above theorem motivates the following condition.

Definition 3.2. A ring R is said to satisfy condition weak (**) if it is artinian and it has the following property: Let M, N be any two finitely generated, uniform right R-modules and M', N' be any maximal submodules of M, N respectively. If there exists an isomorphism $\sigma : \frac{M}{M'} \to \frac{N}{N'}$, then there exists a homomorphism η from M to N or from N to M, lifting σ or σ^{-1} respectively.

Suppose A, B are two non-simple uniserial modules over a ring R satisfying weak (**), such that $d(A) = d(B)$, and there exists an isomorphism $\sigma : \frac{A}{AJ} \to \frac{B}{BJ}$. By the definition, we can fix σ such that it has a lifting $\eta : A \to B$. Then $\eta(A) \not\subseteq BJ$, therefore η is an isomorphism. Thus A, B are isomorphic.

Similar arguments shows that if A_R, B_R are two uniform modules such that $d(A) = d(B)$, $\frac{A}{soc(A)}$, $\frac{B}{soc(B)}$ are semi-simple and some simple module embeds in both $\frac{A}{soc(A)}$, $\frac{B}{soc(B)}$. Then $A \cong B$. Using this result, one can easily prove the following. Let M_R be a uniform module, and $\frac{M}{soc(M)}$ be semi-simple, then either $\frac{M}{soc(M)}$ is homogeneous or it has only two homogeneous component, and each of them is simple, in other words, either $\frac{M}{soc(M)}$ is homogeneous or $d(M) = 3$.

Let R be any right artinian ring, A_R a uniform modules. If k is a positive integer and $soc^k(A)$ is uniserial, then for any $x \in A \backslash soc^k(M)$. $soc^k(M) < xR$. If k is maximal such that $soc^{k-1}(M) < soc^k(M) \neq soc^{k+1}(M)$, then $\frac{soc^{k+1}(M)}{soc^k(M)}$ is not simple. This motivates the following.

Definition 3.3. Let R be any right artinian ring, and K_R a non-zero uniform module. Then a uniserial submodule N of K is called the *critical uniserial submodule* of K, if for some $k > 0$, $N = soc^k(K)$, but $\frac{soc^{k+1}(K)}{soc^k(K)}$ is not simple, whenever it is non-zero.

The critical uniserial submodule of a uniform module over a right artinian ring is uniquely determined.

Proposition 3.4. *Let R be a ring satisfying weak (**), K_R a uniform module. and N its critical uniserial submodule . If $N = soc^k(K)$ and $\frac{soc^{k+1}(K)}{N}$ is non-zero , then $k = 1$. The module K is either uniserial or the critical uniserial submodule of K is simple.*

Proof. Suppose $k \geqslant 2$. As $\frac{soc^{k+1}(K)}{N}$ is not simple, we get two uniserial submodules U, V of E such that $U \cap V = N$, $d(V) = k+1 = d(U)$. Let $M = U + V$. Let S be the simple submodule of N and $B = \frac{V}{S}$, Clearly, B is uniserial. Now $\frac{M}{U} \cong \frac{B}{\overline{N}}$, where $\overline{N} = \frac{N}{S}$.Therefore there exists a homomorphism $\sigma : M \to B$ such that $\sigma(U) \subseteq \overline{N}$ and $\sigma(V)+\overline{N} = B$, or $\sigma : B \to M$ such that $\sigma(\overline{N}) \subseteq U$, $\sigma(B) \nsubseteq U$.

Suppose $\sigma : M \to B$. Let $L = ker\,\sigma$. Then $\frac{M}{L}$ is uniserial. But $\frac{M}{N}$ is not uniform and $d(\frac{M}{N}) = 2$. Thus $L \nsubseteq N$, therefore $N < L$ and $d(\frac{M}{L}) = 1$. Thus $d(\sigma(M)) = 1$. But $\sigma(V) \nsubseteq \overline{N}$, gives $d(\sigma(M)) \geqslant 2$, which is a contradiction. Thus $\sigma : B \to M$. Then $\sigma(B) \nsubseteq U$ gives $N < \sigma(B)$, $d(\sigma(B)) \geq k+1$. But $d(\sigma(B)) \leqslant d(B) = k$, which gives a contradiction. Hence $k = 1$. Now the last part is immediate. □

Lemma 3.5. *Let R be a ring satisfying weak (**) and A_R, B_R be two uniserial modules with $d(B) \leqslant d(A)$. Then A is B-projective. Any uniserial right R-module is quasi-projective. If A is quasi-injective, then any homomorphic image of A is quasi-projective.*

Proof. Let $\sigma : A \to \frac{B}{C}$ be a non-zero homomorphism. Without loss of generality, we take σ an epimorphism. Let $d(\frac{B}{C}) = n$. We apply induction on n. Let $ker\,\sigma = L$. If $n = 1$, then L is maximal in A. As R satisfies weak (**), there exists an epimorphism $\eta : A \to B$, lifting $\overline{\sigma} : \frac{A}{L} \to \frac{B}{C}$. Then η lifts σ. Hence the result holds for $n = 1$.

For some $k \geqslant 1$, let the result hold for $n = k$. Suppose $n = k+1$. We get $C < C' < B$ with $d(\frac{C'}{C}) = 1$. Let $\pi : \frac{B}{C} \to \frac{B}{C'}$ be the natural mapping. Then $\sigma' = \pi\sigma : A \to \frac{B}{C'}$ is an epimorphism. Then $L = ker\,\sigma$, $L' = ker\,\sigma'$ are such that $d(\frac{A}{L}) = k+1$, $d(\frac{A}{L'}) = k$, therefore $d(\frac{L'}{L}) = 1$. By the induction hypothesis, σ' lifts to a homomorphism $\beta : A \to B$. If β lifts σ, we finish. Otherwise we get induced non-zero mapping $\sigma - \overline{\beta} : A \to \frac{B}{C}$with $Im(\sigma - \overline{\beta}) = \frac{C'}{C}$, where $\overline{\beta} : A \to \frac{B}{C}$ is induced by β. As the result holds for $n = 1$, we get a homomorphism $\mu : A \to B$ lifting $\sigma - \overline{\beta}$.Then $\eta = \beta + \mu$ lifts σ. Hence A is B-projective. It also follows that A is quasi-projective, After this the second part is obvious. □

Proposition 3.6. *Let R be a ring satisfying weak (**), then R is right serial and any local right R-module is uniserial.*

Proof. Let e be an indecomposable idempotent in R. It is enough to show that $\frac{eR}{eJ^2}$is uniserial. Therefore we take $J^2 = 0$. Suppose $eJ \neq 0$, then $eJ = A \oplus B$, where A, B are right ideals with A a minimal right ideal. Now $M = \frac{eR}{B}$ is uniserial, and by (3.4) it is quasi-projective. Let $T = ann(M)$. Any quasi-projective module H over an artinian ring Q is projective as a $\frac{Q}{ann(H)}$-module [3]. Thus M is a projective $\frac{R}{T}$-module. Now eR embeds in a finite direct sum of uniserial modules , each of composition length two and a homomorphic image of eR, by (3.3), these uniserial modules are isomorphic, therefore $T = ann(eR)$, $eT = 0$. Consequently $M \cong eR$, eR is uniserial. Hence R is right serial. The last part is obvious. □

Let e, f, g be three non-isomorphic, indecomposable idempotents in a ring R satisfying (**), such that eJ, fJ, gJ are non-zero and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2} \cong \frac{gJ}{gJ^2}$. Then for the simple module $S = \frac{eJ}{eJ^2}$, $E = E(S)$, $\frac{soc^2(E)}{S}$ has more than two homogeneous components, which is a contradiction.

The following result will lead us to the structure of indecomposable modules over rings satisfying (**). This is also a dual of [6, Lemma 2.7].

Proposition 3.7. *Let R be a ring satisfying weak (**), M_R a uniform R-module of finite composition length and $S = soc(M)$. Then $\frac{M}{S}$ has no uniform submodule which is not uniserial. If R satisfies (**), then $\frac{M}{S}$ is a direct sum of uniserial modules.*

Proof. Let $L \leqslant M$ be such that $\frac{L}{S}$ is a non-zero uniform module. Let $\frac{T}{S} = soc(\frac{L}{S})$. Then T in uniserial and $d(T) = 2$. Let K be a submodule of L not contained in T. Then $T < K$. Thus the critical uniserial submodule of L is not simple. By (3.6), L is uniserial. The second part is immediate from the definition of condition (**). □

Theorem 3.8. (i) *Let R be a local ring satisfying* (**) *and $J = J(R)$. Then either $J^2 = 0$ or R is both sided serial.*

(ii) *Let R be an indecomposable ring satisfying* (**)*, for which there exists a simple module S_R as its own successor. If $E = E(S)$ is such that $\frac{soc^2(E)}{S}$ is non-zero and homogeneous, then $J^2 = 0$ and R is a full matrix ring over a local ring.*

(iii) *Let R be a ring satisfying* (**)*. If $e \in R$ is an indecomposable idempotent such that $\frac{eR}{eJ^2}$ is not quasi-injective, then $eJ^2 = 0$.*

Proof. R is both sided serial iff $\frac{R}{J^2}$ is right self-injective. Suppose $\frac{R}{J^2}$ is not right self-injective and $J^2 \neq 0$. We take $J^3 = 0$. Now $A_R = \frac{R}{J^2}$ is not quasi-injective, therefore its injective hull over $\frac{R}{J^2}$ is not uniserial. Let $E = E(R_R)$. Set $S = J^2$, $\overline{E} = \frac{E}{S}$. Let $\sigma : \frac{J}{J^2} \to \overline{E}$ be a non-zero homomorphism. It induces homomorphism $\sigma' : J \to \overline{E}$. Then $(range\, \sigma') = \frac{L}{S}$ with $d(L) = 2$. As J is a projective $\frac{R}{J^2}$-module, σ' lifts a homomorphism $\eta : J \to E$. However, E is injective, therefore η extends to a homomorphism $\lambda : R \to E$. We get induced map $\overline{\lambda} : \frac{R}{J^2} \to \overline{E}$. This proves that $\overline{E}$ is an injective $\frac{R}{J^2}$-module. Hence $\overline{E}$ is a direct sum of uniform modules, none of which is uniserial. This contradicts (3.7).

(ii) The hypothesis gives that S is its only predecessor. Now $S \cong \frac{eR}{eJ}$, then all composition factors of eR are isomorphic and $fRe = 0$ for any indecomposable idempotent f not isomorphic to e. Therefore R is a matrix ring over a local ring R', which by (i) is such that $J(R')^2 = 0$.

(iii) Suppose $eJ^2 \neq 0$. Set $M = \frac{eR}{eJ^3}$, and $S = \frac{eJ}{eJ^2}$ As A is not quasi-injective, there exists an $\omega \in End(S)$ which cannot be extended in $End(A)$. As $\frac{eJ}{eJ^3}$ is quasi-projective, ω lifts to a $\mu \in End(\frac{eJ}{eJ^3})$. Set $\sigma = \mu \mid \frac{eJ^2}{eJ^3}$, $N = \frac{M \times M}{L}$, where $L = \{(x, -\sigma x) : x \in \frac{eJ^2}{eJ^3}\}$. As μ is an extension of σ, N is not uniform, so it has a summand. Therefore there exists an extension $\lambda \in End(M)$ of σ. Then λ is not an extension of μ for otherwise, we get an extension of ω in $End(\frac{eR}{eJ^2})$. Set $\lambda_1 = \lambda \mid \frac{eJ}{eJ^3}$. Then $(\lambda_1 - \mu)\frac{eJ}{eJ^3} = \frac{eJ^2}{eJ^3}$, which proves that the successor of S is also S. By (ii) $J^2 = 0$. This proves the result. □

Proposition 3.9. *Let R be a ring satisfying weak* (**)*, e, f two non-isomorphic indecomposable idempotents such that $eJ^2 \neq 0 \neq fJ^2$ and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2}$. Then there exists an indecomposable right R-module of finite composition length, that is neither uniform nor local. If R satisfies* (**)*, then $eJ^2 = 0$ or $fJ^2 = 0$.*

Proof. Let $S = \frac{eJ^2}{eJ^3}$, $E = E(S)$. The hypothesis gives two submodules A, B of E such that $A \cong \frac{eR}{eJ^3}$, $B \cong \frac{fR}{fJ^3}$, $d(A \cap B) = 2$. As the critical uniserial submodule of E is S, there exists a uniserial submodule L of E such that $d(L) = 2$, $L \cap K = S$, where $K = A \cap B$. Set $M = L \oplus A \oplus B$.

Case 1. $\frac{L}{S} \ncong \frac{K}{S}$. Then there is no monomorphism from any of L, A, B into the other. Let $0 \neq u \in S$. Then $T = (u, u, u)R$ is a simple submodule of M, which by (2.2) is not contained in any summand of M. We prove that $\overline{M} = \frac{M}{T}$ is indecomposable. Suppose otherwise, then $M = C + N$ for some C, $N < M$ such that $T < C$, $T < N$ and $T = C \cap N$. As $d(\overline{M}) = 7$, we take $d(C) \leq 4$. No summand of C contains T, and no summand of C is contained in MJ.

Subcase 1. $d(C) = 2$. Then C is uniserial, thus $C \subseteq L \oplus AJ \oplus BJ$. Then $\pi_L(C) = L$, for otherwise, $C \subset MJ$. Thus C is a summand of M, which is a contradiction.

Subcase 2. $d(C) = 3$. If C is uniform, then $\pi_A(C) = A$ or $\pi_B(C) = B$, therefore C is a summand of M, which is a contradiction. It follows that C is not uniform, $C \subseteq L \oplus AJ \oplus BJ$ and $\pi_L(C) = L$. However, R is right serial, therefore L is a projective $\frac{R}{J^2}$-module. As $L \oplus AJ \oplus BJ$ is an $\frac{R}{J^2}$-module, we get that C has a simple summand, which is a contradiction.

Subcase 3. $d(C) = 4$. As M is an $\frac{R}{J^3}$-module, A, B are projective $\frac{R}{J^3}$-module, C cannot project on A or B, for otherwise C will have a simple summand. Thus $C \subseteq L \oplus AJ \oplus BJ$. Then $C = C_1 \oplus C_2$ with $d(C_1) = 2$, $\pi_L(C_1) = L$, $C_2 = C \cap (AL \oplus BL) \subseteq MJ$, which is a contradiction.

Case 2. $\frac{L}{S} \cong \frac{K}{S}$. Then L and K are not quasi-injective, but they are isomorphic. So there exists an $\omega \in End(S)$ that cannot be extended to a homomorphism from L into K. For a fixed $0 \neq u \in S$, $T = (u, \omega u, \omega u)R$ is a simple submodule of M, which is not contained in any summand of M. Now follow the arguments as in Case 1.

This proves that $\overline{M}$ is indecomposable. Clearly $\overline{M}$ is neither uniform nor local. After this the last part is obvious. ☐

Theorem 3.10. *Let R be a ring satisfying weak (**).*

(i) *If there exists a uniserial module A_R such that $d(A) = 2$, and its drpa (D, D') satisfies $[D : D']_r > 2$, then there exists an indecomposable, non-uniform, non-local right R-module of finite composition length.*

(ii) *If R satisfies (**), the drpa (D, D') of a uniserial module A_R with $d(A) = 2$ satisfies $[D : D']_r \leq 2$.*

Proof. (i) Let $E = E(A)$, $S = soc(A)$. We get ω_1 $(= I)$, ω_2, $\omega_3 \in D$, which are right linearly independent over D'. Let $\lambda_1 (= I)$, λ_2, λ_3 be extensions of ω_1, ω_2, ω_3 respectively in $End(E)$ Let $A_i = \lambda_i(A)$. Set $B_1 = A_1$, $B_2 = B_1$, $B_3 = A_1 + A_2$. Then B_1, B_2, B_3 are of composition lengths 2, 2, 3 respectively. Fix an $x_1 \neq 0$ in S. Let $M = B_1 \oplus B_2 \oplus B_3$ and $\pi_i : M \to B_i$ be the associated projections. Let $x_i = \omega_i x_1$, then $T = (x_1, x_2, x_3)R$ is a simple submodule of M. Suppose T is contained in a summand of M. By (2.2), we have following possibilities. (1) $\omega_2 = \eta_{21}\omega_1$ for some homomorphism $\eta_{21} : B_1 \to B_2$. Then η_{21} is an automorphism of A. If $\lambda \in End(E)$ is an extension of η_{21}, Then $\mu = \lambda\lambda_1$ is an extension of ω_2, therefore by (1.5), $\mu(A_1) = A_2$. But $\lambda\lambda_1(A_1) = A_1$, which is a contradiction. (2) $\omega_3 = \eta_{31}\omega_1 + \eta_{32}\omega_2$ for some homomorphisms $\eta_{31} : B_1 \to B_3$, $\eta_{32} : B_2 \to B_3$. Let λ, $\mu \in End(E)$ be extensions of η_{31}, η_{32} respectively. Then $\rho = \lambda\lambda_1 + \mu\lambda_2$ is an extension of ω_3, therefore $\rho(A) = A_3$. However $(\lambda\lambda_1 + \mu\lambda_2)(A) \subseteq B_3$ and by (1.3)(ii) $A_3 \nsubseteq B_3$, which is a contradiction. Hence T is not contained in any summand of M.

We now prove that $\overline{M} = \frac{M}{T}$ is indecomposable. Now $d(\overline{M}) = 6$. Suppose $\overline{M}$ has a summand. We get a summand $\overline{C}$ with $d(\overline{C}) \leq 3$. Now $\overline{C} = \frac{C}{T}$ for some $C < M$ containing T. For some $N < M$, $M = C + N$, $T = C \cap N$. As $soc(M)$ is small in M, C has no semi-simple summand. In particular, T is not a summand of C. Indeed no summand of C contains T. As $soc^2(E)$ is a module over $\frac{R}{J^2}$, we take $J^2 = 0$. In that case every uniserial module of composition length 2 is projective. Let $x = (x_1, x_2, x_3)$

Case 1. $d(\overline{C}) = 1$. Then $d(C) = 2$, and C is uniserial, $x \in C$ has projection $x_1 \neq 0$ in B_1. Therefore C projects onto B_1. Thus C is a summand of M, which is a contradiction.

Case 2. $d(\overline{C}) = 2$. Then $d(C) = 3$. The projection of C in B_3 is non-zero, as x has non-zero projection in B_3. If C projects onto B_3, then $C \cong B_3$, therefore C is a summand of M, which is a contradiction. If the image of C in B_3 has composition length 2, then this image being projective, gives that C has a simple summand, which is also a contradiction. Suppose Image of C in B_3 is simple, then $C = T \oplus (C \cap (B_1 + B_2))$, which is also a contradiction.

Case 3. $d(\overline{C}) = 3$. Then $d(C) = 4$. If C projects onto $B_1 \oplus B_2$, then C is a summand of M, which is a contradiction. So $C \cap B_3 \neq 0$. We are left with the situation in which we also have $N \cap B_3 \neq 0$. In this case $C \cap N$ contains $T + soc(B_3)$, which is a contradiction.

Hence $\overline{M}$ is indecomposable. Clearly $\overline{M}$ is neither uniform nor local.

Now (ii) is immediate from (i). □

4. Condition (***)

Definition 4.1. A ring R is said to satisfy condition (***) if R is artinian, right serial, and for any three indecomposable idempotents e, f, $g \in R$ with eJ, fJ, gJ non-zero, the following hold.

(i) The *drpa* (D, D') of $A = \frac{eR}{eJ^2}$ is such that $[D : D']_r \leq 2$, $[D : D']_l \leq 2$.

(ii) If e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2}$, then $eJ^2 = 0$ or $fJ^2 = 0$.

(iii) If e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2} \cong \frac{gJ}{eJ^2}$, then g is isomorphic to e or f.

(iv) If $A = \frac{eR}{eJ^2}$ is not quasi-injective, then $eJ^2 = 0$ and $\frac{eJ}{eJ^2} \ncong \frac{fJ}{fJ^2}$ whenever e is not isomorphic to f.

Suppose R is a ring satisfying (**). By (3.6) R is right serial. By (1.5) and (3.10) condition (i) in (4.1) is satisfied by R. By (3.9) condition (ii) in (4.1) holds. Condition (iii) in (4'1) follows from remarks following (3.6). Condition (iv) from remarks after (3.2). We are going to prove that conditions (**) and (***) are equivalent.

Suppose R is a ring satisfying (***). Let S_R be a simple module, $E = E(S)$ and M any submodule of E. Suppose M is not uniserial and N is its critical uniserial submodule. Then for some $k > 0$, $N = soc^k(M)$, $d(\frac{soc^{k+1}(M)}{N}) > 1$. Let $G = \frac{soc^{k+1}(M)}{NJ}$ is uniform, $\frac{G}{soc(G)} \cong \frac{soc^{k+1}(M)}{N}$. By using conditions (iii) and (iv) in (4.1), we see that $d(\frac{G}{soc(G)}) = 2$, $G = C + H$ for some uniserial submodules C, H such that $d(C) = 2 = d(H)$ and C or H is projective. Let A, B in $soc^{k+1}(M)$ of C, H respectively, then they are uniserial, $d(A) = d(B) = k + 1$. Now $A \cong C$ or $B \cong H$, therefore $k = 1$. Thus any uniform R-module is either uniserial or its critical uniserial submodule is simple. In the later case $soc^2(M) = A+B = soc^2(E)$, where A, B are uniserial submodules such that $d(A) = 2 = d(B)$ and A or B is projective; in case $\frac{soc^2(M)}{soc(M)}$ is not homogeneous, A, B are uniquely determined and both are quasi-injective. In case $\frac{soc^2(M)}{soc(M)}$ is homogeneous, $A \cong B$ and they are projective.

Proposition 4.2. *Let R be a ring satisfying (***), S_R a simple module, and $E = E(S)$ be such that it is not uniserial and $soc^2(E) = A+B$, where A, B are uniserial submodules with $S = A \cap B$ and $d(A) = d(B)$.*

(a) *Let H, K are two uniserial submodules of E such that $H \nsubseteq K$, $K \nsubseteq H$, $A \nsubseteq H \cap K$ and $S < H \cap K$. Then A is not projective, B is projective, $B \subseteq H \cap K$, uniserial submodules of E containing A is linearly ordered under inclusion and there exists a unique uniserial submodule of E of maximum composition length that contains A. Let $H' \leqslant H$ such that $H'J = H \cap K$, then there exists a homomorphism from H' onto A. If a uniserial submodule G of E is such that $d(G) = 2$, $S = C \cap G$, for some projective uniserial submodule C with $d(C) = 2$, then the family of those uniserial submodules of E that contain G is linearly ordered under inclusion.*

(b) *If E is not uniserial and every uniserial submodule of E of composition length 2 is projective, then E is a sum of two uniserial modules whose intersection is S; in particular this holds if $\frac{soc^2(E)}{S}$ is homogeneous. In addition, if $\frac{soc^2(E)}{S}$ is homogeneous, then E is a sum of two isomorphic uniserial submodules whose intersection is S.*

Proof. Now $soc^2(E) = A + B$ for uniserial submodules A, B such that $d(A) = 2 = d(B)$, $S = A \cap B$.

(a) Suppose $S < H \cap K$ and $A \nsubseteq H \cap K$. Then $soc^2(H + K) = soc^2(E)$. We consider any uniserial submodule $xR \leq soc^2(M)$ such that $xR \nsubseteq H \cap K$. There exist $H' \leqslant H$, $K' \leqslant K$ such that $H'J = H \cap K = K'J$. For some indecomposable idempotent $e \in R$, we can take $x = xe$. For some $u \in H'$, $v \in K'$, $ue = u \notin H \cap K$, $ve = v \notin H \cap K$, we have $x = u + v$. We get epimorphism $\sigma : H' \to xR$. If xR is projective, we get $d(H') = 2$, which is a contradiction. Hence xR is not projective, so xR is quasi-injective. In particular, A is not projective. Then B is projective, therefore $B \subseteq H \cap K$. The second part is now obvious.

(b) Suppose every uniserial submodule of $soc^2(E)$ of composition length 2 is projective, this property holds in case $\frac{E}{S}$ is homogeneous. Now $soc^2(E) = A + B$ for some uniserial submodules A, B with $d(A) = d(B)$, $S = A \cap B$. By using (a), we get uniquely determined uniserial submodules H, K of E of maximum composition lengths such that $A \subseteq H$, $B \subseteq K$.

Case 1. $\frac{soc^2(E)}{S}$ is homogeneous. Then there exists a $\lambda \in End(E)$ such that $\lambda(A) = B$. Then $B \subseteq K \cap \lambda(H)$, therefore $\lambda(H) \subseteq K$, and $d(H) \leq d(K)$. We get $d(H) = d(K)$ and $H \cong K$. Set $M = H + K$. Suppose $E \neq M$. Then there exists a uniserial submodule L of E such that $L \nsubseteq M$. Then for $C = soc^2(E) \cap L$, $d(C) = 2$. For the *drpa* (D, D') of A, $[D : D']_r = 2$. There exists a $\sigma \in D$ which has extension $\mu \in End(E)$ such that $\mu(A) = C$. By considering μ^{-1},we get $d(L) \leq d(H)$. Let $\omega = \lambda \mid S$ and $\omega' = \mu \mid S$. Then $\omega' = \alpha + \omega\beta$ for some α, $\beta \in D'$. Let η_1, η_2 be extensions in $End(E)$ of α, β respectively. Then $\mu' = \eta_1 + \lambda\eta_2 \in End(E)$ is an extension of ω'. By using (1.5), we get $L \subseteq \mu(H) = \mu'(H) \subseteq H + K$, which is a contradiction. Hence $E = H + K$.

Case 2. $\frac{soc^2(E)}{S}$ is not homogeneous, Then any uniserial submodule L of E with $d(L) \geq 2$ contains A or B, therefore by (a) $L \subseteq H$ or $L \subseteq K$. Hence $E = H + K$. $\square$

Lemma 4.3. *Let R be a ring satisfying (***), and S_R a simple module such that $E = E(S)$ is not uniserial, but $\frac{E}{S}$ is homogeneous. If S is its own predecessor, then R is matrix ring over a local ring and $J^2 = 0$.*

Its proof is similar to that of (3.8). $\square$

If R is an artinian ring which is right serial ring, and A_R is a uniserial, projective module, then any uniserial module B_R containing A is projective.

Lemma 4.4. *Let a ring R satisfy (***). If uniserial module A_R is not quasi-injective, then it is projective.*

Proof. Set $B = soc^2(A)$ and $E = E(A)$. Suppose A is not projective, then B is not projective, therefor B is quasi-injective. Thus $\frac{soc^2(E)}{S}$ is not homogeneous and $soc^2(E) = B + C$ for some uniserial submodule C with $d(C) = 2$. Then C is projective. Now there exists a $\sigma \in End(E)$ for which $\sigma(A) \nsubseteq A$. If $B \subseteq \sigma(A)$, then by (4.2), C is not projective, which is a contradiction. Thus $C \subseteq \sigma(A)$, $\sigma(A)$ is projective. This gives $B \cong \sigma(B) = C$, therefore B is projective. Hence A is projective. $\square$

Theorem 4.5. *Let R be a ring satisfying (***) and E_R an indecomposable injective module that is not uniserial. Let $soc^2(E) = A + B$, where A, B are uniserial submodules of E with $d(A) = 2 = d(B)$, $soc(E) = A \cap B$. If P, Q are uniserial submodules of E of maximum composition lengths containing A, B respectively, then $E = P + Q$.*

Proof. Set $S = soc(E)$. If $\frac{soc^2(E)}{S}$ is homogeneous, the result follows from (4.2)(b). Suppose $\frac{soc^2(E)}{S}$ is not homogeneous. Then A, B are uniquely determined, and one of them say A is projective. Then Q is uniquely determined and it is quasi-injective. Let K be any uniserial submodule of E with $d(K) > 2$. If $B \subseteq K$, then $K \subseteq Q$. Suppose $B \nsubseteq K$. Then $A \subset K$. Every submodule of P containing A is projective. Therefore no two composition factors of $\frac{P}{S}$ are isomorphic. Also, by (4.4) $\frac{P}{S}$ is quasi-injective. Suppose $K \nsubseteq P$, then set $F = K \cap P$. Let K', P' be the submodules of K, P respectively, such that $K'J = F = Q'J$. We have epimorphism $\sigma : K' \to B$, which extends to a homomorphism $\eta : K \to Q$ with $ker\,\eta = FJ$. Thus $\frac{K}{FJ}$ embeds in Q. Similarly $\frac{P}{FJ}$ also embeds in Q. But $d(\frac{K}{FJ}) \leq d(\frac{P}{FJ})$. Hence $\frac{P}{FJ}$ is $\frac{K}{FJ}$-injective. We have a monomorphism $\lambda : \frac{K}{FJ} \to \frac{P}{FJ}$, which is identity on $\frac{F}{FJ}$. As K is projective, we get a monomorphism $\mu : K \to P$ lifting λ. If μ is identity on F, then $P + K = P \oplus W$ for some $W \leqslant P + K$, which is a contradiction. Thus μ is not identity on F. Let $\mu_1 = \mu \mid F$. As every submodule of K containing A is projective, no two composition factors of $\frac{K}{S}$ are isomorphic, therefore $(\mu_1 - I)F = S$, $\overline{\mu} : \frac{K}{S} \to \frac{P}{S}$ is identity on $\frac{F}{S}$, which gives $\frac{P}{S} + \frac{K}{S} = \frac{P}{S} \oplus \frac{V}{S}$ for some uniserial submodule V containing S. As $soc(\frac{V}{S}) \cong \frac{B}{S}$, we get $V \subseteq Q$. Hence $K \subseteq P + Q$, which proves that $E = P + Q$. $\square$

By using (4.2) and (4.5), one can prove the following.

Theorem 4.6. *Let R be a ring satisfying (***), S_R a simple module and $E = E(S)$ not a uniserial module. Let A, B be any two uniserial submodule of E such that $d(A) = d(B) = 2$ and $soc(E) = A + B$.*

(i) *If P is a uniserial submodule of E maximal with respect to containing A, then it is of maximum composition length among the uniserial submodules containing A.*

(ii) *If P, Q are any two uniserial submodules of E which are maximal with respect to containing A, B respectively, then $E = P + Q$.*

The above theorem gives the following.

Theorem 4.7. *Let R be a ring satisfying (***) and M_R a uniform module which is not uniserial and $E = E(M)$. Then $\frac{M}{soc(M)}$ is a direct sum of two uniserial submodules, there exist uniserial submodules P, Q of E such that $soc(M) = P \cap Q$, $E = P + Q$ and $M = G + H$, where $G = P \cap M$, $H = Q \cap M$. If $k = min\{d(G), d(H)\}$, then $soc^i(M) = soc^i(E)$ for $1 \leq i \leq k$.*

Lemma 4.8. (i) *Let R be a right artinian ring, K_R a quasi-projective uniserial module such that $d(K) = 3$, K is not homogeneous, KJ is homogeneous and $\frac{K}{KJ^2}$, KJ are quasi-injective. Then any endomorphism σ of $soc(K)$ is uniquely extendable to an endomorphism of K.*

(ii) *Let R be a ring satisfying (***), S_R a simple module and $E = E(S)$ not a uniserial module. Let A, B be any two uniserial submodule of E such that $d(A) = d(B) = 2$ and $soc(E) = A + B$. Let H be a uniserial submodule of E such that $d(H) \geq 3$. Then any endomorphism σ of $soc(H)$ can be extended to an endomorphism of H.*

Proof. (i) Let $0 \neq \sigma \in D = End(soc(K))$. As KJ is quasi-injective, there exists a $\sigma' \in End(KJ)$ extending σ. As $\frac{K}{KJ^2}$ is quasi-injective, and K is quasi-projective, there exists $\eta \in End(K)$ such that $\overline{\eta} \in End(\frac{K}{KJ^2})$ induced by η is an extension of $\overline{\sigma'} \in End(\frac{KJ}{KJ^2})$ induced by σ'. Let $\lambda = \eta \mid KJ$. Then $\overline{\lambda} - \overline{\sigma'} = 0$ gives $(\lambda - \sigma')KJ \subseteq soc(K)$. Hence λ is an extension of σ. That λ is uniquely determined by σ follows from the hypothesis that K is not homogeneous, but KJ is homogeneous.

(ii) Let L, M be uniserial submodules of maximum composition lengths containing A, B respectively. Then $E = L + M$.

We take $A = soc^2(E) \cap H$ and by using (4.6), we also take $H \subseteq L$. Let $\sigma \neq 0$. If B is projective, then L is uniquely determined, therefore the result holds. Suppose B is not projective. Then A is quasi-injective as well projective. Therefore there exists $\eta \in End(E)$ such that it extends σ and $A = \eta(A)$. If $\eta(H) \subseteq H$, we finish. Suppose $\eta(H) \nsubseteq H$. Now H is also projective, for some $u \in H$ and some indecomposable idempotent $e \in R$, $H = uR$, $ue = u$. We write $\eta(u) = x + y$ for some $x \in L$, $y \in M$ such that $xe = x$, $ye = y$. As $x \notin S$, xR is projective. By using the fact that $\frac{E}{S} = \frac{L}{S} \oplus \frac{M}{S}$, we get an automorphism $\lambda \in End(E)$ such that $\lambda(\eta(u)) = x$. Now $A = usR$ for some $s \in R$. Let $\mu = \lambda\eta$. Then $\eta(us) = xs + ys \in A$. This gives $ys \in S$, $xs \in A$. We also have homomorphism $\rho : uR \to yR$, $\rho(u) = y$.

Case 1. $ys = 0$. Then $us = xs$, $\eta(us) = xs = \mu(us)$. It follows that μ is an extension of σ.

Case 2. $ys \neq 0$. Then A is homogeneous. Now $B \cong \frac{fR}{fJ^2}$ for some indecomposable idempotent $f \in R$. As B is not projective, $fJ^2 \neq 0$. Therefore $\frac{fJ}{fJ^3} \cong A$, $fJ^3 = 0$. Set $K = \rho^{-1}(B)$. Then $K \cong fR$, K is not homogeneous, KJ is quasi-injective. As $\frac{K}{KJ^2}$ is isomorphic to B, it is also quasi-injective. Further K is projective as $A \subset K$ and A is projective. Therefore by (i), there exists a $\tau \in End(K)$ extending σ. Let $\varphi \in End(E)$ be an extension of τ. Suppose $\varphi(H) \nsubseteq H$. Set $F = H \cap \varphi(H)$. Then $K \subseteq F$. By (4.2), we get $F < H' \leqslant H$, such that $\frac{H'}{F} \cong \frac{C}{S} \cong \frac{K}{A}$, thus $\frac{H}{A}$ have two isomorphic composition factors. However as every submodule of H containing A is projective, no two composition factors of $\frac{H}{A}$ can be isomorphic, which is a contradiction. Hence $\varphi(H) = H$, which proves the result. □

Lemma 4.9. *Let R be an artinian ring, $N_R = N_1 \oplus N_2 \oplus ..... \oplus N_t$ be such that for some simple module S, $soc(N_i) = S$ and let T be a simple submodule of N generated by an element $(x_1, x_2, ..., x_t)$ with $x_i \neq 0$ for every i. If for some $i \neq j$, there exists a homomorphism $\lambda : N_i \to N_j$ such that $\lambda(x_i) = x_j$, then T is contained in a summand of N.*

Proof. By re-indexing, we take $i = 1$, $j = 2$. Let $C_1 = \{(x, \lambda x) : x \in N_1\}$. Then $N = N_2 \oplus (C_1 \oplus N_3 \oplus N_4 \oplus ... \oplus N_t)$ and $T \subseteq C_1 \oplus N_3 \oplus N_4 \oplus ... \oplus N_t$, a summand of N. □

Lemma 4.10. *Let R be an artinian ring, M_R an indecomposable module of finite composition length and $M = K_1 + K_2 + K_3 + ... + K_n$ for some uniform modules $K_i \nsubseteq MJ$, such that $n > 1$, $N = K_2 + K_3 + ... + K_n = K_2 \oplus K_3 \oplus .... \oplus K_n$, $K_1 \cap N = soc(K_1)$. Then the following hold.*
(i) *$T = xR = soc(K_1)$ is not contained in a summand of N.*
(ii) *For any $1 \leq i \leq n$, $N_i = \sum\limits_{j \neq i} K_j = \oplus \sum\limits_{j \neq i} K_j$, $K_i \cap N_i = soc(K_i)$.*

Proof. (i) Suppose $N = A \oplus B$ for some non-zero submodules A, B and $T \subseteq A$. Then $M = (K_1 + A) \oplus B$, which is a contradiction.

(ii) Now $x = x_1 + x_2 + ... + x_n$, $x_i \in K_i$. By (i) $x_i \neq 0$, and $soc(K_i) = x_i R$ for any $1 \leq i \leq n$. Clearly $x_i \in K_i \cap N_i$. Suppose for some $i > 1$, $0 \neq z_i \in K_i \cap N_i$, then $z_i = \sum\limits_{j \neq i} z_j$ for some $z_j \in K_j$. Then $z_1 = u_2 + ... + u_n$, where $u_i = z_i$, $u_j = -z_j$ for $j \neq i$, therefore $0 \neq z_1 \in K_1 \cap N_1 = soc(K_1)$. This gives that $z_i \in soc(K_i)$. Hence $K_i \cap N_i = soc(K_i)$. It also gives $N_i = \oplus \sum\limits_{j \neq i} K_j$. □

Lemma 4.11. *Let R be a ring satisfying (***), S_R a simple module such that $E = E(S)$ is not uniserial but $\frac{soc^2(E)}{S}$ is homogeneous, and L a uniserial submodule of E with $d(L) \geq 2$. If $A = soc^2(E) \cap L$, and (D, D') is drpa of L, then (D, D') is also drpa of A.*

Proof. Let (D_1, D'_1) be the *drpa* of A. By definition $D = End(S) = D_1$. Let $\sigma \in End(L)$. Then $\sigma \mid S = (\sigma \mid A) \mid S \in D'_1$, therefore $D' \subseteq D'_1$. Let $\eta \in End(A)$. As every submodule of L containing A is projective, no two composition factors of $\frac{L}{S}$ are isomorphic. Let $\overline{\eta} \in End(\frac{A}{S})$ be induced by η. As $\frac{L}{S}$ is quasi-injective and L is quasi-projective, there exists a $\lambda \in End(L)$ that induces $\overline{\eta}$. Suppose λ does not extend η, then $\lambda_1 = \lambda \mid A$ is such that $(\lambda_1 - \eta)(A) = S$, which gives that $\frac{A}{AJ} \cong S$ and S its only predecessor. By (4.3), $J^2 = 0$, therefore $d(L) = 2$, which is a contradiction. Thus λ is an extension of η, which gives $D'_1 \subseteq D'$. Hence (D, D') is *drpa* of A. □

We now prove the main theorem. We prove that conditions (**) and (***) are equivalent.

Theorem 4.12. *Let R be an artinian ring. Then every finitely generated indecomposable right R-module is uniform if and only if it satisfies the following.*

(α) *R is right serial.*

(β) *Let e, f, g be any three indecomposable idempotents of R with eJ, fJ, gJ non-zero. Then the following hold.*

(i) *The drpa (D, D') of $A = \frac{eR}{eJ^2}$ is such that $[D : D']_r \leq 2$, $[D : D']_l \leq 2$.*

(ii) *If e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2}$, then $eJ^2 = 0$ or $fJ^2 = 0$.*

(iii) *If e, f are non-isomorphic and $\frac{eJ}{eJ^2} \cong \frac{fJ}{fJ^2} \cong \frac{gJ}{eJ^2}$, then g is isomorphic to e or f.*

(iv) *If $A = \frac{eR}{eJ^2}$ is not quasi-injective, then $eJ^2 = 0$, $\frac{eJ}{eJ^2} \ncong \frac{fJ}{fJ^2}$ whenever e is not isomorphic to f.*

Proof. If every finitely generated indecomposable right R-module is uniform, as seen before, R satisfies the given conditions, i.e R satisfies (***).

Conversely, let R satisfy (***). Suppose the contrary. We get an indecomposable module M_R of smallest composition length, which is not uniform. Then $soc(M) \subseteq MJ$. Firstly, we prove that $M = G + N$ for some uniserial submodule $G \nsubseteq MJ$, $N < M$ such that $soc(G) = G \cap N$. Let S be a simple submodule of M. As $\frac{M}{S}$ is a direct sum of uniform modules, we get two submodules K, N of M such that $M = K + N$, $S = K \cap N$ and $\frac{K}{S}$ is a non-zero uniform module. If $\frac{K}{S}$ is uniserial, then K is uniserial and we finish. Suppose $\frac{K}{S}$ is not uniserial.

Case 1. $M = K$. As M is not uniform and $\frac{M}{S}$ is uniform, $soc(M) = S \oplus S'$ for some simple submodule S'. As $\frac{M}{S}$ is not uniserial, its critical uniserial submodule is $soc(\frac{M}{S})$. By (4.7), $\frac{M}{S+S'}$ is a direct sum of two uniserial modules. Therefore there exist non simple uniserial submodules A, B of M such that $M = A+B$, $(A+S) \cap (B+S) = S+S'$. Then one of A, B say A does not contain S. But $S + S' = S + A \cap (B + S)$. Thus $A \cap (B + S)$ $= soc(A)$ and $M = A + (B + S)$.

Case 2. $M \neq K$. Then K is a direct sum of uniform modules. So there exists a uniform summand L of K. Now $K = L \oplus W$ for some $W \leqslant K$. Then $M = L + (W + N)$ with $L \cap (W + N) = soc(L)$. If L is uniserial, we finish. Otherwise, by (4.7) $L = A + B$ for some non-simple uniserial submodules A, B such that $soc(L) = A \cap B$. Now neither of A, B is contained in MJ. Then $M = A + (B + W + N)$, $A \cap (B + W + N) = soc(A)$.

We get a uniserial submodule A of M of minimum composition length such that $A \nsubseteq MJ$, and for some $N < M$, $M = A + N$, $soc(A) = A \cap N$. Set $S = soc(A) = xR$. Now $N = K_1 \oplus K_2 \oplus .... \oplus K_t$ for some uniform submodules K_i. Suppose $t \geq 2$.

Suppose some K_i say K_1 is not uniserial. Then $K_1 = A_1 + B_1$ for some uniserial submodules A_1, B_1 such that $d(A_1) \geq 2$, $d(B_1) \geq 2$, $A_1 \cap B_1 = S_1 = soc(K_1)$. Now M $= A_1 + N_1$, where $N_1 = A + B_1 + K_2 + ... + K_t$. As $A_1 \cap N_1 = soc(A_1)$, the choice of A implies $d(A) \leq d(A_1)$. Similarly, $d(A) \leq d(B_1)$. Let $k = min\{d(A_1), d(B_1)\}$. Then A embeds in $soc^k(E) = soc^k(K_1)$, where $E = E(K_1)$. Hence K_1 is A-injective. Now M $= K_2 + H_2$, where $H_2 = A \oplus K_1 \oplus K_3 \oplus .... \oplus K_t$. and $soc(K_2) = K_2 \cap H_2$. Let $yR =$ $soc(K_2)$. Then $y = a + y_1 + y_3 + ... + y_t$ for some $a \in A$, $y_i \in K_i$, $i \neq 2$. We get a monomorphism $\sigma : A \to K_1$ for which $\sigma(a) = y_1$. Then by (4.9), yR is contained in a summand of H_2, which contradicts (4.10). Hence every K_i is uniserial and $d(A) \leq d(K_i)$. Set $A = K_0$. Arrange K_i's in a such way that $d(K_i) \leq d(K_{i+1})$ for $i > 0$. Fix an $x_0 \neq 0$

in $soc(K_0)$. Then $x_0 = x_1 + x_2 + ... + x_t$ for some uniquely determined non-zero $x_i \in K_i$. Now $M \cong \frac{K_0 \times K_1 \times .... \times K_t}{L}$, where $L = (x_0, -x_1, -x_2, ..., -x_t)R$ is a simple submodule not contained in any summand of $K_0 \times K_1 \times .... \times K_t$. Let $E = E(K_0)$, $S = soc(E)$. Then every K_i embeds in E. If E is uniserial, then K_1 is K_0-injective, and we get an embedding $\sigma : K_0 \to K_1$ such that $\sigma(x_0) = -x_1$, which gives that L is contained in a summand of $K_0 \times K_1 \times .... \times K_t$, therefore M is decomposable, which is a contradiction. Hence E is not uniserial.

Case 1. $\frac{soc^2(E)}{S}$ is homogeneous. Then given any two uniserial submodules V, W of E with $d(V) \leq d(W)$, there exists an automorphism of E that maps V into W. Thus if $K = K_t$, we take every $K_i \subseteq K$ Let (D, D') be the *drpa* of $B = A \cap soc^2(E)$, therefore $[D : D']_l = 2$. It can be seen that (D, D') is also *drpa* of K. Now $M \cong \frac{K_0 \times K_1 \times .... \times K_t}{L} \subseteq \frac{K^{(t+1)}}{L}$, where for some non-zero ω_i, $1 \leq i \leq t$ in D, $-x_i = \omega_i x_0$. But I, ω_1, ω_2, ..., ω_t are left linearly dependent over D'. Therefore for some $1 \leq i \leq t$, $\omega_i = \mu_0 I + \mu_1 \omega_1 + ... + \mu_{i-1}\omega_{i-1}$, where $\omega_0 = I$ and each μ_j is the restriction to S of some $\rho_j \in End(K)$. Let $\rho_{tj} : K_j \to K_t$ $(= K) = \rho_j \mid K_j$. Then for $\mu_{tj} = (\rho_{tj} \mid S) = \mu_j$, $\omega_i = \mu_{i0} I + \mu_{i1}\omega_1 + ... + \mu_{ii-1}\omega_{i-1}$. By (2.2), T is contained in a summand of $K_0 \times K_1 \times .... \times K_t$, which is a contradiction.

Case 2. $\frac{soc^2(E)}{S}$ is not homogeneous. Then $E = F + H$, for some uniserial submodules F, H such that $d(F) \geq 2$, $d(H) \geq 2$ and $S = F \cap H$. Let G, H be the intersection of F, H respectively with $soc^2(E)$. Then both G, H are quasi-injective, one of them say G is projective, and any uniserial submodule L of E of composition length at least 2 contains G or H. Once again, we suppose that all $K_i \subseteq E$. Suppose the number of K_i that contain H is more than one, say $H \subseteq K_1 \cap K_2$. Consider $W = \frac{K_1 \times K_2}{T'}$, where $T' = (x_1, x_2)R$. We know that there is no homomorphism $\sigma : K_1 \to K_2$ for which $\sigma(x_1) = x_2$, therefore W is indecomposable. However as H is quasi-injective, there exists a homomorphism $\eta : H \to H$ for which $\eta(x_1) = x_2$. By (1.1), W is not uniform, but $d(W) < d(M)$, which gives a contradiction to the choice of M. Thus there is only one K_i containing H. Similarly there is only one K_i containing G. Thus $t = 1$.

In any case $t = 1$, $M \cong \frac{K_0 \times K_1}{L}$, where $L = (x_0, -x_1)R$, and K_0 is uniserial. As argued earlier, K_1 is also uniserial. We regard K_0, $K_1 \subseteq E$, then for some $\omega \in End(S))$, $\omega x_0 = x_1$. Let $A = K_0 \cap soc^2(E)$, $B = K_1 \cap soc^2(E)$. As M is not uniform, by (1.1) ω extends to an isomorphism $\sigma : A \to B$.

Case 1. $\frac{soc^2(E)}{S}$ is homogeneous. Then for any extension $\lambda \in End(E)$ of σ, $\lambda(K_0) \subseteq K_1$, which proves that M is decomposable, which is a contradiction.

Case 2. $\frac{soc^2(E)}{S}$ is not homogeneous. Then $\sigma(A) = B$ gives $A = B$. Suppose A is not projective. Then $K_0 \subseteq K_1$. As there is unique maximal uniserial submodule P of E containing A, for any extension $\lambda \in End(E)$ of $\sigma, \lambda(K_0) \subseteq K_1$. Thus M is decomposable, which is a contradiction. This shows that A is projective, $soc^2(E) = A + C$ for some uniquely determined uniserial submodule C with $d(C) = 2$, $A \cap C = S$. Then there exists unique maximal uniserial submodule Q of E containing C. Let P be a maximal uniserial submodule of E containing K_1. By (4.9)(ii), there exists an $\eta \in End(E)$ which extend σ and $\eta(K_0) = K_0$. Now $K_0 = xR$ for some $x \in K_0$, such that for some indecomposable idempotent $e \in R$, $xe = x$. Then $\eta(x) = a + b$, for some $a \in P$, $b \in Q$ with $ae = a$, $be = b$. As $\frac{E}{S} = \frac{P}{S} \oplus \frac{Q}{S}$, and K_0, aR are projective, we get isomorphism $\rho : K_0 \to aR$ for which $\rho(x) = a$. As $d(K_0) \leq d(K_1)$, it follows that $a \in K_1$. Now $A = xsR$. Then $\eta(xs) \in A$, as $\eta(xs) = as + bs$, $as \in A$, $bs \in S$. We also have homomorphism $\lambda : K_0 \to Q$, $\lambda(x) = b$. It follows that if $xsr \in S$, then $bsr = 0$, therefore $\rho(xsr) = asr = \sigma(xrs)$. Hence $\rho : K_0 \to K_1$ extends σ, which is a contradiction. This proves the result. $\square$

It follows from the above theorem that any balanced ring, as discussed in [2], and which is right serial satisfies (**)

Definition 4.13. [6]. Let M be a local module, $D = End(\frac{M}{J(M)})$ and D' the division subring of D consisting of those $\sigma \in D$ which can be lifted to some endomorphisms of M. Then the pair (D, D') is called the dual division ring pair associate (in short *ddpa*) of M.

By suitable dualization of the arguments involved in proving the above theorem, we can prove the following dual of the above theorem.

Theorem 4.14. *Let R be an artinian ring. Then every finitely generated indecomposable right R-module is local if and only if the following hold.*
(α) *Any uniform right R-module is uniserial.*
(β) *For any three uniserial right R-modules A, B, C with $d(A) = d(B) = d(C) = 2$, the following hold.*
(i) *The ddpa (D, D') is such that $[D, D']_r \leq 2$, $[D, D]_l \leq 2$;*
(ii) *if A, B are not isomorphic and $\frac{A}{AJ} \cong \frac{B}{BJ}$, then A is injective or B is injective.*
(iii) *if A, B are not isomorphic and $\frac{A}{AJ} \cong \frac{B}{BJ} \cong \frac{C}{CJ}$, then $C \cong A$ or $C \cong B$;*
(iv) *if A is not quasi-projective, then A is injective and $\frac{A}{AJ} \ncong \frac{B}{BJ}$, whenever $A \ncong B$.*

Examples of rings satisfying (**) or (*) can be easily constructed.

Example 4.15. Let D be a division ring having a subdivision ring D' such that $[D, D']_r = [D : D']_l = 2$. Let $R = \begin{bmatrix} D' & D \\ 0 & D \end{bmatrix}$. Then R is right serial but not left serial, and its radical J satisfies $J^2 = 0$. Only uniserial right R-module with composition length 2 is $A = e_{11}R$. Its *drpa* is (D, D'). It follows from (4.12) that R satisfies (**). To within isomorphism, R admits only one uniserial module $A = \frac{Re_{22}}{D'e_{12}}$, it is injective and its *ddpa* is (D, D'). By (4.14), every finitely generated indecomposable left module is local. Now consider the ring $R' = \begin{bmatrix} D' & D' & D \\ 0 & D' & D \\ 0 & 0 & D \end{bmatrix}$. Then R' is right serial and $J^2 \neq 0$. There are only two uniserial right R-modules of composition length 2, viz $A = \frac{e_{11}R}{e_{11}J^2}$, $B = e_{22}R$. Here A is injective. As seen for R the *drpa* of B is (D, D'). By (4.12), R' satisfies (**). R' is also such that every finitely generated indecomposable left module is local.

Example 4.16. Let D be a division ring, and $R = \begin{bmatrix} D & 0 & D \\ 0 & D & D \\ 0 & 0 & D \end{bmatrix}$. Then R is right serial, but not left serial. Here $J^2 = 0$. It admits only two uniserial right modules of composition length 2, viz $A = e_{11}R$, $B = e_{22}R$. Both A, B are quasi-injective, and $soc(A) \cong soc(B) \cong e_{33}R$. It follows from (4.12) that R satisfies (**). R admits two uniserial left modules of composition length 2, viz modules $M = \frac{Re_{33}}{De_{23}}$, $N = \frac{Re_{33}}{De_{13}}$, both of them are quasi-projective and injective. Once again, by (4.14). every finitely generated indecomposable left R-module is local.

Example 4.17. Let D be a division ring admitting a division subring D' such that $[D, D']_r = 2$, $[D, D']_l > 2$. Such division rings exist [4]. Then $R = \begin{bmatrix} D' & D \\ 0 & D \end{bmatrix}$ is right serial, but it does not satisfy (**).

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References

[1] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules*, Graduate Texts in Mathematics, Vol. 13, Springer-Verlag, 1974.

[2] V. Dlab, C. M. Ringel, *The structure of balanced rings*, Proc. Lond. Math. Soc. (3) **26** (1973), 446–462.

[3] K. R. Fuller, D. A. Hill, *On quasi-projective modules via relative projectivity*, Arch. Math. **21** (1970), 369–373.

[4] A. H. Schoefield, *Representations of rings over skew fields*, Lon. Math. Soc. Lecture Notes Series, Vol. 92, Cambridge University Press, 1985.

[5] S. Singh, *Indecomposable modules over right artinian rings*, Advances in Ring Theory, Edit. S. K. Jain, S. Tariq Rizvi, pp. 295–304, Trends in Mathematics, Birkhäuser, Boston 1997.

[6] S. Singh, Hind Al-Bleehed, *Rings with indecomposable modules local*, Beiträge zur Algebra und Geometrie **45** (2005), 239–251.

[7] H. Tachikawa, *On rings for which every indecomposable right module has a unique maximal submodule*, Math. Z. **71** (1959), 200–222.

[8] R. Wisbauer, *Foundation of modules and ring theory*, Gordon and Breach Science Publishers, Algebra, Logic and Applications. Vol. 3, 1991.

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ON A GENERALIZATION OF STABLE TORSION THEORY

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Abstract. Throughout this paper R is a ring with a unit element, every right R-module is unital and Mod-R is the category of right R-modules. A subfunctor of the identity functor of Mod-R is called a preradical. A torsion theory $(\mathcal{T}, \mathcal{F})$ is called stable if $\mathcal{T}$ is closed under taking injective hulls. We denote $E(M)$ the injective hull of a module M. For a preradical σ, we denote $E_\sigma(M)$ the σ-injective hull of a module M, where $E_\sigma(M)$ is defined by $E_\sigma(M)/M := \sigma(E(M)/M)$. For a preradical σ we call a torsion theory $(\mathcal{T}, \mathcal{F})$ is σ-stable if $\mathcal{T}$ is closed under taking σ-injective hulls. In this note, we characterize σ-stable torsion theories and give some related facts.

0. Fundamental facts of torsion theory

For a preradical t it hold that $t(N) \subseteq t(M)$ and $t(M/N) \supseteq (t(M) + N)/N$ for any $M \in$ Mod-R and its submodule N. A preradical t is called idempotent (radical) if $t(t(M)) = t(M)$ $(t(M/t(M)) = 0)$ for any module M, respectively. For a preradical σ, $\mathcal{T}_\sigma := \{M \in \text{Mod-}R \mid \sigma(M) = M\}$ is the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma := \{M \in \text{Mod-}R \mid \sigma(M) = 0\}$ is the class of σ-torsionfree right R-modules. For a subclass $\mathcal{C}$ of Mod-R, it is said that $\mathcal{C}$ is closed under taking extensions if: if $N, M/N \in \mathcal{C}$ then $M \in \mathcal{C}$ for any $M \in$ Mod-R and its submodule N. A preradical t is called left exact if $t(N) = N \cap t(M)$ for any submodule N of a module M. It is also well known that a preradical t is idempotent and $\mathcal{T}_t$ is closed under taking submodules if and only if t is left exact. A right R-module M is called σ-injective if the functor $\text{Hom}_R(-, M)$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{T}_\sigma$. For a preradical σ a submodule N of a module M is called σ-dense in M if M/N is σ-torsion, and N is called σ-essential in M if N is σ-dense and essential in M. It holds that a module M is σ-injective if and only if M has no proper σ-essential extension.

Let σ be an idempotent radical. If X is minimal in $\{X \mid X \text{ is } \sigma\text{-injective and } X \supseteq M\}$, X is called to be a minimal σ-injective extension of M. If Y is maximal in $\{Y \mid Y \supseteq M \text{ and } M \text{ is } \sigma\text{-essential in } Y\}$, Y is called to be a maximal σ-essential extension of M. If $X \supseteq M$ and X is σ-injective and M is σ-essential in X, X is called to be a σ-injective σ-essential extension of M. For any module M a σ-injective σ-essential extension of M exists and is unique to within isomorphism. The σ-injective σ-essential extension of M coincides with the minimal σ-injective extension of M and the maximal σ-essential extension of M and is called to be the σ-injective hull of M. We put $\sigma(E(M)/M) = E_\sigma(M)/M$. For an idempotent radical σ, the σ-injective hull of M is isomorphic to $E_\sigma(M)$. But even if a preradical σ is not an idempotent radical, we call $E_\sigma(M)$ the σ-injective hull of a module M.

The detailed version of this paper will be submitted for publication elsewhere.

Let $\mathcal{C}$ be a subclass of Mod-R. A torsion theory for $\mathcal{C}$ is a pair $(\mathcal{T}, \mathcal{F})$ of classes of objects of $\mathcal{C}$ such that

(i) $\mathrm{Hom}_R(T, F) = 0$ for all $T \in \mathcal{T}$, $F \in \mathcal{F}$

(ii) If $\mathrm{Hom}_R(M, F) = 0$ for all $F \in \mathcal{F}$, then $M \in \mathcal{T}$

(iii) If $\mathrm{Hom}_R(T, N) = 0$ for all $T \in \mathcal{T}$, then $N \in \mathcal{F}$.

We put $t(M) = \sum\limits_{\mathcal{T} \ni N \subset M} N$ $(= \bigcap\limits_{M/N \in \mathcal{F}} N)$, then $\mathcal{T} = \mathcal{T}_t$ and $\mathcal{F} = \mathcal{F}_t$ hold and t is an idempotent radical. Conversely if t is an idempotent radical, then $(\mathcal{T}_t, \mathcal{F}_t)$ is a torsion theory.

1. A stable torsion theory relative to torsion theories

P. Gabriel studied a hereditary stable torsion theory in [3] (Or see p. 152 in [12]). We generalize hereditary stable torsion theory. First we generalize left exact preradicals. For preradicals σ and t, we call t a σ-left exact preradical if $t(N) = N \cap t(M)$ holds for any σ-dense submodule N of a module M.

Lemma 1. *If σ is a radical, then $E_\sigma(M)$ is σ-injective for any module M.*

Lemma 2. *For a preradical σ, the following hold.*

(1) *If σ is idempotent, then $\mathcal{F}_\sigma$ is closed under taking extensions. Conversely if σ is a radical and $\mathcal{F}_\sigma$ is closed under taking extensions, then σ is idempotent.*

(2) *If σ is a radical, then $\mathcal{T}_\sigma$ is closed under taking extensions. Conversely if σ is idempotent and $\mathcal{T}_\sigma$ is closed under taking extensions, then σ is a radical.*

In [14] we generalized hereditary torsion theories. For the sake of reader's convenience, we state the following propositions.

Proposition 3. *For a left exact preradical σ and an idempotent preradical t, t is σ-left exact if and only if $\mathcal{T}_t$ is closed under taking σ-dense submodules.*

Proof. $(\rightarrow)$: Let N be a σ-dense submodule of a module $M \in \mathcal{T}_t$. Then $t(N) = N \cap t(M) = N \cap M = N$, as desired.

$(\leftarrow)$: Let N be a σ-dense submodule of a module M. Since $t(M)/(N \cap t(M)) \simeq (N + t(M))/N \subseteq M/N \in \mathcal{T}_\sigma$ and $t(M) \in \mathcal{T}_t$, $N \cap t(M) \in \mathcal{T}_t$. Then it holds that $N \cap t(M) = t(N \cap t(M)) \subseteq t(N)$. Since it is clear that $N \cap t(M) \supseteq t(N)$, $N \cap t(M) = t(N)$ holds. $\square$

Proposition 4. *For an idempotent radical σ and a radical t, t is σ-left exact if and only if $\mathcal{F}_t$ is closed under taking σ-injective hulls.*

Proof. $(\rightarrow)$: Let M be in $\mathcal{F}_t$. Then $0 = t(M) = M \cap t(E_\sigma(M))$, and so $t(E_\sigma(M)) = 0$, as desired.

$(\leftarrow)$: Let N be a σ-dense submodule of a module $M \in \mathcal{T}_t$. Consider the following diagram.

$$\begin{array}{ccccccc} 0 \to & N & \xrightarrow{g} & M & \to M/N \to 0 \\ & \downarrow_j & & \downarrow_f & \\ 0 \to & N/t(N) & \xrightarrow{i} & E_\sigma(N/t(N)), & \end{array}$$

where g and i are the inclusion maps, j is the canonical epimorphism and f is a homomorphism determined by the σ-injectivity of $E_\sigma(N/t(N))$. Since t is a radical, $E_\sigma(N/t(N)) \in \mathcal{F}_t$ by the assumption. Since $f(t(M)) \subseteq t(E_\sigma(N/t(N))) = 0$, it holds that $t(M) \subseteq \ker f$. Let $f|_N$ be a restriction map of f to N. Then it follows that $t(N) = \ker j = \ker f|_N = N \cap \ker f \supseteq N \cap t(M) \supseteq t(N)$, and so $t(N) = N \cap t(M)$, as desired. $\square$

Lemma 5. *Let σ be an idempotent radical. If M is a σ-essential extension of a module N, then $E_\sigma(M) = E_\sigma(N)$ holds. Conversely if σ is a left exact radical, $N \subseteq M$ and $E_\sigma(M) = E_\sigma(N)$, then M is a σ-essential extension of N.*

Lemma 6. *Let σ be a left exact radical and L a submodule of a module M. Then the following are equivalent.*

(1) *$L = E_\sigma(L) \cap M$.*
(2) *L is σ-essentially closed in M, that is, if L is σ-essential in X such that $L \subseteq X \subseteq M$, then $L = X$.*

Lemma 7. *Let σ be an idempotent radical and M a module. Then M is σ-injective if and only if $E_\sigma(M) = M$.*

A preradical t is called stable if $\mathcal{T}_t$ is closed under taking injective hulls. Next we generalize stable torsion theory. We call a preradical t σ-stable if $\mathcal{T}_t$ is closed under taking σ-injective hulls for a preradical σ. We put $\mathcal{X}_t(M) := \{X : M/X \in \mathcal{T}_t\}$ and $N \cap \mathcal{X}_t(M) := \{N \cap X : X \in \mathcal{X}_t(M)\}$. The following theorem generalize Proposition 7.1 in [12] and (i) and (ii) of Theorem 2.8 in [2].

Theorem 8. *Let t be an idempotent preradical and σ an idempotent radical. Then the following conditions (1), (2) and (3) are equivalent.*

Assume that t is an idempotent radical and $\mathcal{T}_t$ is closed under taking σ-dense submodules and σ is a left exact radical, then all conditions (1)˜(10) except (6) are equivalent. Moreover if t is left exact, then all conditions are equivalent.

(1) *t is σ-stable, that is, $\mathcal{T}_t$ is closed under taking σ-injective hulls.*
(2) *The class of σ-injective modules are closed under taking $t(-)$, that is, $t(E)$ is σ-injective for any σ-injective module E.*
(3) *$E_\sigma(t(M)) \subseteq t(E_\sigma(M))$ holds for any module M.*
(4) *$\mathcal{T}_t$ is closed under taking σ-essential extensions.*
(5) *If M/N is σ-torsion, then $N \cap \mathcal{X}_t(M) = \mathcal{X}_t(N)$ holds.*
(6) *Every module $M \notin \mathcal{T}_t$ with $M/t(M) \in \mathcal{T}_\sigma$ contains a nonzero submodule $N \in \mathcal{F}_t$.*
(7) *For any module M, $t(M) = E_\sigma(t(M)) \cap M$ holds.*
(8) *For any module M, $t(M)$ is σ-essentially closed in M.*
(9) *For any σ-injective module E with $E/t(E) \in \mathcal{T}_\sigma$, $t(E)$ is a direct summand of E.*
(10) *$E_\sigma(t(M)) = t(E_\sigma(M))$ holds for any module M.*

Proof. (1)→(3): Let t be an idempotent preradical and $M \in$ Mod-R. Then $t(M) \in \mathcal{T}_t$, and by assumption $E_\sigma(t(M)) \in \mathcal{T}_t$. Since $E_\sigma(t(M)) \subseteq E_\sigma(M)$, it follows that $E_\sigma(t(M)) = t(E_\sigma(t(M))) \subseteq t(E_\sigma(M))$, as desired.

(3)→(2): Let σ be an idempotent radical and X be a σ-injective module, and then we have $E_\sigma(X) = X$ by Lemma 1. Then it follows that $E_\sigma(t(X)) \subseteq t(E_\sigma(X)) = t(X)$ by the assumption. Since $E_\sigma(t(X)) \supseteq t(X)$ holds clearly, it follows that $E_\sigma(t(X)) = t(X)$, and so $t(X)$ is σ-injective by Lemma 1, as desired.

(2)$\rightarrow$(1): Let σ be a radical and $M \in \mathcal{T}_t$. By the assumption, $t(E_\sigma(M))$ is σ-injective. Since $t(E_\sigma(M)) \supseteq t(M) = M$, $E_\sigma(M)/t(E_\sigma(M))$ is an epimorphic image of $E_\sigma(M)/M$, and so $E_\sigma(M)/t(E_\sigma(M)) \in \mathcal{T}_\sigma$. Thus the exact sequence $(0 \rightarrow t(E_\sigma(M)) \rightarrow E_\sigma(M) \rightarrow E_\sigma(M)/t(E_\sigma(M)) \rightarrow 0)$ splits. Then there exists a submodule K of $E_\sigma(M)$ such that $E_\sigma(M) = t(E_\sigma(M)) \oplus K$, and so $0 = K \cap t(E_\sigma(M)) \supseteq K \cap M$. Since M is essential in $E_\sigma(M)$, it follows that $K = 0$, and so $E_\sigma(M) = t(E_\sigma(M))$, as desired.

(1)$\rightarrow$(4): Assume that σ is an idempotent radical and $\mathcal{T}_t$ is closed under taking σ-dense submodules. Let $M \in \mathcal{T}_t$ be σ-essential in a module X. By the assumption it follows that $E_\sigma(M) \in \mathcal{T}_t$. By Lemma 5 $E_\sigma(M) = E_\sigma(X)$. Thus $E_\sigma(X) \in \mathcal{T}_t$. Since X is a σ-dense submodule of $E_\sigma(X)$, it follows that $X \in \mathcal{T}_t$, as desired.

(4)$\rightarrow$(1): It is clear.

(3)$\rightarrow$(7): Let t be a σ-left exact preradical. By the assumption it follows that $t(M) \subseteq M \cap E_\sigma(t(M)) \subseteq M \cap t(E_\sigma(M)) = t(M)$. Thus $t(M) = M \cap E_\sigma(t(M))$.

(7)$\rightarrow$(9): Let σ be an idempotent radical, E be σ-injective and $E/t(E) \in \mathcal{T}_\sigma$. Then it follows that $t(E) = E_\sigma(t(E)) \cap E$ and $E_\sigma(t(E)) \subseteq E_\sigma(E) = E$, and so $t(E) = E_\sigma(t(E))$. Hence $t(E)$ is σ-injective. Thus the sequence $0 \rightarrow t(E) \rightarrow E \rightarrow E/t(E) \rightarrow 0$ splits, as desired.

(9)$\rightarrow$(1): Let σ be an idempotent radical and t be an idempotent preradical and $M \in \mathcal{T}_t$, then it follows that $M = t(M) \subseteq t(E_\sigma(M))$. Thus $E_\sigma(M)/t(E_\sigma(M))$ is a factor module of $E_\sigma(M)/M \in \mathcal{T}_\sigma$. By the assumption there exists a submodule K of $E_\sigma(M)$ such that $E_\sigma(M) = K \oplus t(E_\sigma(M))$. Thus it follows that $0 = K \cap t(E_\sigma(M)) \supseteq K \cap M$, and so $K = 0$. Hence $E_\sigma(M) = t(E_\sigma(M)) \in \mathcal{T}_t$.

(10)$\rightarrow$(2): It is clear.

(3)$\rightarrow$(10): Here we assume that σ is a left exact radical and t is a σ-left exact preradical.

First we claim that $t(M)$ is σ-essential in $t(E_\sigma(M))$. Suppose that $L \cap t(M) = 0$ for a submodule L of $t(E_\sigma(M))$. Then it follows that $0 = L \cap t(M) = L \cap M \cap t(E_\sigma(M)) = L \cap M$. Since M is essential in $E_\sigma(M)$, $L = 0$, and so $t(M)$ is essential in $t(E_\sigma(M))$. It is clear that $t(M)$ is a σ-dense submodule of $t(E_\sigma(M))$ since $t(E_\sigma(M))/t(M) = t(E_\sigma(M))/(M \cap t(E_\sigma(M))) \simeq (M + t(E_\sigma(M)))/M \subseteq E_\sigma(M)/M \in \mathcal{T}_\sigma$.

Thus $t(M)$ is σ-essential in $t(E_\sigma(M))$, and so by Lemma 5 $E_\sigma(t(M)) = E_\sigma(t(E_\sigma(M))) \supseteq t(E_\sigma(M))$. By the assumption $E_\sigma(t(M)) \subseteq t(E_\sigma(M))$, and so $E_\sigma(t(M)) = t(E_\sigma(M))$, as desired.

(4)$\rightarrow$(5): Assume that $\mathcal{T}_t$ is closed under taking σ-dense submodules. Let N be a σ-dense submodule of a module M.

First we claim that $N \cap \mathcal{X}_t(M) \supseteq \mathcal{X}_t(N)$. Let $N_0 \in \mathcal{X}_t(N)$. Then $N/N_0 \in \mathcal{T}_t$. We put $\Gamma = \{M_i/N_0 \subseteq M/N_0 : (M_i/N_0) \cap (N/N_0) = 0\}$. Then by Zorn's argument, Γ has a maximal element M_0/N_0 which is a complement of N/N_0 in M/N_0, and then $M_0 \cap N = N_0$. Hence $(M_0/N_0) \oplus (N/N_0)$ is essential in M/N_0, and so $[(M_0/N_0) \oplus (N/N_0)]/[M_0/N_0]$ is essential in $[M/N_0]/[M_0/N_0]$. Therefore $(M_0+N)/M_0$ is essential in M/M_0. Since $M/N \in \mathcal{T}_\sigma$, it follows that $M/(M_0 + N_0) \in \mathcal{T}_\sigma$. Thus $\mathcal{T}_t \ni N/N_0 = N/(M_0 \cap N) \simeq (N + M_0)/M_0$. So $(N + M_0)/M_0$ is σ-essential in M/M_0. By the assumption it follows that $M/M_0 \in \mathcal{T}_t$. Since $M_0 \cap N = N_0$, it conclude that $N \cap \mathcal{X}_t(M) \supseteq \mathcal{X}_t(N)$.

Next we will show that $N \cap \mathcal{X}_t(M) \subseteq \mathcal{X}_t(N)$. Let $M_1 \in \mathcal{X}_t(M)$, and then $M/M_1 \in \mathcal{T}_t$. Simce $N/(N \cap M_1) \simeq (N + M_1)/M_1 \subseteq M/M_1 \in \mathcal{T}_t$ and $\mathcal{T}_\sigma \ni M/N \to M/(N + M_1) \to 0$, it follows that $N/(N \cap M_1) \in \mathcal{T}_t$ by the assumption, and so $N \cap M_1 \in \mathcal{X}_t(N)$.

(5)$\to$(1): Let σ be an idempotent preradical and M be in $\mathcal{T}_t$. Since $E_\sigma(M)/M \in \mathcal{T}_\sigma$, $\mathcal{X}_t(E_\sigma(M)) \cap M = \mathcal{X}_t(M) \ni 0$ for $M \in \mathcal{T}_t$. Thus there exists a submodule X of $E_\sigma(M)$ such that $E_\sigma(M)/X \in \mathcal{T}_t$ and $X \cap M = 0$. Since M is essential in $E_\sigma(M)$, it follows that $X = 0$, and so $E_\sigma(M) \in \mathcal{T}_t$.

(1)$\to$(6): Let $M \notin \mathcal{T}_t$ with $M/t(M) \in \mathcal{T}_\sigma$. Suppose that any nonzero submodule N of M is not t-torsionfree. Since $0 \neq t(N) \subseteq N \cap t(M)$, $N \cap t(M) \neq 0$ holds for any nonzero submodule N of M, and so $t(M)$ is essential in M. By the assumption it follows that $t(M)$ is σ-essential in M. By Lemma 5, $E_\sigma(t(M)) = E_\sigma(M)$ holds. Since t is an idempotent preradical, it follows that $t(M) \in \mathcal{T}_t$ and so $E_\sigma(t(M)) \in \mathcal{T}_t$ by the assumption. Thus $E_\sigma(M) \in \mathcal{T}_t$. Then $t(M) = M \cap t(E_\sigma(M)) = M \cap E_\sigma(M) = M$, and so $M \in \mathcal{T}_t$. This is a contradiction, and so $M \notin \mathcal{T}_t$ with $M/t(M) \in \mathcal{T}_\sigma$ contains a nonzero submodule $N \in \mathcal{F}_t$.

(6)$\to$(1): Let $M \in \mathcal{T}_t$, then $t(E_\sigma(M)) \supseteq t(M) = M$. Suppose that $E_\sigma(M) \notin \mathcal{T}_t$. Since $E_\sigma(M)/M \to E_\sigma(M)/t(E_\sigma(M)) \to 0$, it follows that $0 \neq E_\sigma(M)/t(E_\sigma(M)) \in \mathcal{T}_\sigma$. By the assumption there exists a nonzero submodule $N \in \mathcal{F}_t$ of $E_\sigma(M)$. Since M is essential in $E(M)$, it follows that $M \cap N \neq 0$, and so $\mathcal{F}_t \ni N \supseteq N \cap M \subseteq M \in \mathcal{T}_t$. As t is left exact, $N \cap M \in \mathcal{F}_t \cap \mathcal{T}_t = \{0\}$. This is a contradiction. Thus it follows that $E_\sigma(M) \in \mathcal{T}_t$, as desired. $\square$

2. Some applications of σ-stable torsion theory

If R is right noetherian, t is stable if and only if every indecomposable injective module is t-torsion or t-torsionfree by Proposition 11.3 in [6]. We will generalize this. First we need the following torsion theoretic generalization of Matlis Papp's theorem in Theorem 1 in [10].

For a left exact radical σ, we denote $\mathcal{L}_\sigma := \{I \subseteq R; R/I \in \mathcal{T}_\sigma\}$

[10, Theorem 1] *Let σ be a left exact radical. Then $\mathcal{L}_\sigma$ satisfies ascending chain conditions if and only if every σ-injective σ-torsion R-module is a direct sum of σ-injective σ-torsion indecomposable submodules.*

The following theorem generalizes [6, Proposition 11.3].

Theorem 9. *Assume that t is an idempotent radical, σ is a left exact radical and $\mathcal{T}_t$ is closed under taking σ-dense submodules. Then the following hold.*

1. *If t is σ-stable, then $(*)$ every indecomposable σ-injective module E with $E/t(E) \in \mathcal{T}_\sigma$ is either t-torsion or t-torsionfree.*
2. *If the ring R satisfies the condition $(*)$ and $\mathcal{L}_\sigma$ satisfies ascending chain conditions, then $\mathcal{T}_t \cap \mathcal{T}_\sigma$ is closed under taking σ-injective hulls.*

Proof of (1): Let E be an indecomposable σ-injective module with $E/t(E) \in \mathcal{T}_\sigma$. By (9) in Theorem 8, $t(E)$ is a direct summand of E. As E is indecomposable, $t(E) = 0$ or $t(E) = E$, as desired.

Proof of (2): Let M be in $\mathcal{T}_t \cap \mathcal{T}_\sigma$. Since $\mathcal{T}_\sigma$ is closed under taking extensions, $E_\sigma(M)$ is σ-torsion. As $E_\sigma(M)$ is σ-injective and σ-torsion, it follows that $E_\sigma(M) = \sum_{i \in I} \oplus E_i$

by [10, Theorem 1], where I is an index set and E_i is a nonzero σ-injective σ-torsion indecomposable submodule of $E_\sigma(M)$. As $E_i \subseteq E_\sigma(M) \in \mathcal{T}_\sigma$, it follows that $E_i \in \mathcal{T}_\sigma$, and so $E_i/t(E_i) \in \mathcal{T}_\sigma$, it follows that E_i is t-torsion or t-torsionfree. Since M is essential in $E_\sigma(M)$, it follows that $M \cap E_i \neq 0$. Since $M \in \mathcal{T}_\sigma$, $M/(M \cap E_i) \in \mathcal{T}_\sigma$. As $M \in \mathcal{T}_t$ and $\mathcal{T}_t$ is closed under taking σ-dense submodules, $M \cap E_i \in \mathcal{T}_t$. Thus $t(E_i) \supseteq t(M \cap E_i) = M \cap E_i \neq 0$, and so $t(E_i) \neq 0$. Hence $t(E_i) = E_i$ holds for all i. Since every preradical preserves direct sums, it follows that $t(E_\sigma(M)) = t(\sum_{i\in I} \oplus E_i) = \sum_{i\in I} \oplus\, t(E_i) = \sum_{i\in I} \oplus E_i = E_\sigma(M)$, and so $E_\sigma(M) \in \mathcal{T}_t$. □

The following proposition generalizes [7, Proposition 1.2].

Proposition 10. *Let $(\mathcal{T}_t, \mathcal{F}_t)$ be a σ-hereditary σ-stable torsion theory, that is, t is an idempotent radical and $\mathcal{T}_t$ is closed under taking σ-injective hulls and σ-dense submodules, where σ is a left exact radical. Then there exists an isomorphism: $E_\sigma(M/t(M)) \simeq E_\sigma(M)/E_\sigma(t(M))$, if $M/t(M) \in \mathcal{T}_\sigma$.*

Proof. For a module M consider the following commutative diagram.

$$\begin{array}{ccccc} 0 \to & M & \xrightarrow{j} & E_\sigma(M) & \\ & \downarrow g & & \downarrow f & \\ 0 \to & M/t(M) & \underset{i}{\to} & E_\sigma(M/t(M)), & \end{array}$$

where i and j are inclusions and g is a canonical epimorphism and f is an induced morphism by σ-injectivity of $E_\sigma(M/t(M))$. By the above diagram, $t(M) = \ker g = \ker(f|_M) = \ker f \cap M$, and so $t(M) = \ker f \cap M$ follows. Since $M/t(M) \in \mathcal{F}_t$ and $\mathcal{F}_t$ is closed under taking σ-injective hulls and σ is a left exact preradical, it follows that $E_\sigma(M)/\ker f \subseteq E_\sigma(M/t(M)) \in \mathcal{F}_t$. Thus it follows that $t(E_\sigma(M)) \subseteq \ker f$. Since $\mathcal{T}_\sigma$ is closed under taking extensions and $M/t(M) \in \mathcal{T}_\sigma$ and $E_\sigma(M)/M \in \mathcal{T}_\sigma$, it follows that $E_\sigma(M)/t(M) \in \mathcal{T}_\sigma$. Since $E_\sigma(M)/t(E_\sigma(M))$ is an epimorphic image of $E_\sigma(M)/t(M)$, it follows that $E_\sigma(M)/t(E_\sigma(M)) \in \mathcal{T}_\sigma$. Since σ is left exact preradical and $\ker f/t(E_\sigma(M)) \subseteq E_\sigma(M)/t(E_\sigma(M)) \in \mathcal{T}_\sigma$, it follows that $\ker f/t(E_\sigma(M)) \in \mathcal{T}_\sigma$. By the assumption $t(E_\sigma(M))$ is σ-injective. Then the exact sequence $(0 \to t(E_\sigma(M)) \to \ker f \to \ker f/t(E_\sigma(M)) \to 0)$ splits. Then there exists a submodule S of $\ker f$ such that $\ker f = S \oplus t(E_\sigma(M))$. Then since $0 = S \cap t(E_\sigma(M)) \supseteq S \cap t(M)$, it follows that $0 = S \cap t(M) = S \cap \ker f \cap M$. As M is essential in $E_\sigma(M)$, it follows that $0 = S \cap \ker f = S$. Thus it follows that $t(E_\sigma(M)) = \ker f$. So $f(E_\sigma(M)) \simeq E_\sigma(M)/\ker f = E_\sigma(M)/t(E_\sigma(M)) \in \mathcal{T}_\sigma$. Thus the exact sequence $0 \to t(E_\sigma(M)) \to E_\sigma(M) \to f(E_\sigma(M)) \to 0$ splits as $t(E_\sigma(M))$ is σ-injective. Thus $f(E_\sigma(M))$ is a direct summand of σ-injective module $E_\sigma(M)$, and so $f(E_\sigma(M))$ is also σ-injective. Since $E_\sigma(M/t(M)) \supseteq f(E_\sigma(M)) \supseteq g(M) \supseteq M/t(M)$, it follows that $E_\sigma(M/t(M))/f(E_\sigma(M)) \in \mathcal{T}_\sigma$. Thus the exact sequence $0 \to f(E_\sigma(M)) \to E_\sigma(M/t(M)) \to E_\sigma(M/t(M))/f(E_\sigma(M)) \to 0$ splits. So there exists a submodule K of $E_\sigma(M/t(M))$ such that $E_\sigma(M/t(M)) = K \oplus f(E_\sigma(M))$. Since $f(E_\sigma(M)) \supseteq M/t(M)$, it follows that $K \cap (M/t(M)) = 0$. But $M/t(M)$ is essential in $E_\sigma(M/t(M))$, and so $K = 0$. Thus $E_\sigma(M/t(M)) = f(E_\sigma(M)) \simeq E_\sigma(M)/\ker f = E_\sigma(M)/t(E_\sigma(M))$, as desired. □

Hereafter we omit the proof of the following propositions.

We call A σ-M-injective if $\mathrm{Hom}_R(-, A)$ preserves the exactness for any exact sequence $0 \to N \to M \to M/N \to 0$, where $M/N \in \mathcal{T}_\sigma$. The following proposition is a generalization of Theorem 15 in [16].

Proposition 11. *A is σ-M-injective if and only if $f(M) \subseteq A$ for any $f \in \mathrm{Hom}_R(E_\sigma(M), E_\sigma(A))$.*

We obtain the following corollary as a torsion theoretic generalization of the Johnson Wong theorem by putting $M = A$ in Proposition 11. We call a module A σ-quasi-injective if A is σ-A-injective.

Corollary 12. *A is σ-quasi-injective if and only if $f(A) \subseteq A$ for any $f \in \mathrm{Hom}_R(E_\sigma(A), E_\sigma(A))$.*

The following lemma generalizes Proposition 2.3 in [17].

Lemma 13. *If A is σ-quasi-injective and $E_\sigma(A) = M \oplus N$, then $A = (M \cap A) \oplus (N \cap A)$.*

Now we can generalize [1, Theorem 2.3]

Theorem 14. *Assume that σ is a left exact radical and $\mathcal{T}_t$ is closed under taking σ-injective hulls, then every σ-quasi-injective R-module A with $A/t(A) \in \mathcal{T}_\sigma$ splits, that is, $A = t(A) \oplus N$ where $N \in \mathcal{F}_t$, and then if $t(A)$ is σ-torsion, then N is σ-quasi-injective.*

The following corollary generalizes Corollary 2.15 in [5].

Corollary 15. *Let M be a σ-quasi-injective module. Then any σ-essentially closed and σ-dense submodule of M is a direct summand of M, and any direct summand is σ-quasi-injective.*

References

[1] E. P. Armendariz, *Quasi-injective modules and stable torsion classes*, Pacific J. Math. **31** (1969), 227–280.

[2] L. Bican, P. Jambor, T. Kepka and P. Nemec, *Stable and costable preradicals*, Acta Universitatis Carolinae-Mathematicaet Physica, **16(2)** (1975), 63–69.

[3] P. Gabriel, *Des categories abeliennes*, Bull. Soc. Math. France **90** (1962), 323–448.

[4] J. L. Gómez Pardo, *Spectral Gabriel topologies and relative singular functors*, Comm. in Algebra **13(1)** (1985), 21–57.

[5] K. R. Goodearl, *Ring Theory*, Marcel Dekker, New York, 1976.

[6] J. S. Golan, *Localization of Noncommutative Rings*, Marcel Dekker, New York, 1976.

[7] J. S. Golan, *Structure Sheaves over a Noncommutative Rings*, Marcel Dekker, 1980.

[8] J. S. Golan, *Torsion Theories*, Longman Scientific & Technical, 1986.

[9] J. N. Manocha, *Singular submodule relative to a kernel functor*, Comm. in Algebra, **10(5)** (1982), 474–491.

[10] K. Masaike and T. Horigome, *Direct sums of τ-injective modules*, Tsukuba J. Math. **4** (1980), 77–81.

[11] R. A. Rubin, *Semi-simplicity relative to kernel functors*, Can. J. Math. **26** (1974), 1405–1411.

[12] Bo Stenström, *Rings of Quotients*, Springer-Verlag, Berlin, 1975.

[13] Y. Takehana, *On generalization of CQF-3′ modules and cohereditary torsion theories*, to appear in Math. J. Okayama Univ.

[14] Y. Takehana, *On generalization of QF-3′ modules and hereditary torsion theories*, to appear in Math. J. Okayama Univ.

[15] Z. Papp, *Semi-stability and topologies on R-sp*, Comm. in Algebra **4** (1976), 793–809.
[16] G. Azumaya, *M-projective and M-injectives*, unpublished.
[17] M. Harada, *Note on quasi-injective modules*, Osaka J. Math. **2** (1965), 351–356.

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ON GRADED MORITA EQUIVALENCES FOR AS-REGULAR ALGEBRAS

KENTA UEYAMA

ABSTRACT. One of the most active projects in noncommutative algebraic geometry is to classify AS-regular algebras. The motivation of this article is to find a nice criterion of graded Morita equivalence for AS-regular algebras. In this article, we associate to a geometric AS-regular algebra A a new algebra $\overline{A}$, and it is proved that $\overline{A}$ is isomorphic to $\overline{A'}$ as graded algebras if A is graded Morita equivalent to A'. In particular, if A, A' are generic geometric 3-dimensional AS-regular algebras, then $\overline{A}$ is isomorphic to $\overline{A'}$ as graded algebras if and only if A is graded Morita equivalent to A'.

Key Words: graded Morita equivalence, AS-regular algebra, generalized Nakayama automorphism.

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1. Introduction

This is based on a joint work with Izuru Mori.

In noncommutative algebraic geometry, classification of AS-regular algebras has been one of the major projects since its beginning. In fact, AS-regular algebras (of finite GK-dimension) up to dimension 3 were classified (cf. [1], [2], [9], [10]). Since classifying 4-dimensional AS-regular algebras up to isomorphism of graded algebras is difficult, it is natural to try to classify them up to something weaker than graded isomorphism such as graded Morita equivalence. In general, it is difficult to check whether two graded algebras are graded Morita equivalent. The main result of this article (Theorem 8) gives a new criterion of graded Morita equivalences for geometric AS-regular algebras.

2. Preliminaries

Throughout this paper, we fix an algebraically closed field k. Let A be a graded k-algebra. We denote by $\operatorname{GrMod} A$ the category of graded right A-modules and right A-module homomorphisms preserving degree. We say that two graded algebras A and A' are graded Morita equivalent if there exists an equivalence of categories between $\operatorname{GrMod} A$ and $\operatorname{GrMod} A'$. For $M \in \operatorname{GrMod} A$ and $n \in \mathbb{Z}$, the shift of M, denoted by $M(n)$, is the graded right A-module such that $M(n)_i = M_{i+n}$. For $M, N \in \operatorname{GrMod} A$, we define the

The detailed version of this paper will be submitted for publication elsewhere.

graded k-vector spaces

$$\underline{\mathrm{Hom}}_A(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_{\mathrm{GrMod}\,A}(M,N(n)), \text{and}$$
$$\underline{\mathrm{Ext}}^i_A(M,N) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Ext}^i_{\mathrm{GrMod}\,A}(M,N(n)).$$

We say A is connected if $A_i = 0$ for all $i < 0$, and $A_0 = k$.

Definition 1. Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be a connected graded algebra such that $\dim_k A_i < \infty$ for all i. We define the Gelfand-Kirillov dimension (GK-dimension) of A by

$$\mathrm{GKdim}\, A = \limsup_{n\to\infty} \frac{\log(\sum_{i=0}^n \dim_k A_i)}{\log n}$$

If A is a commutative algebra, then $\mathrm{GKdim}\, A = \mathrm{Kdim}\, A$, the Krull dimension of A.

An AS-regular algebra defined below is one of the first classes of algebras studied in noncommutative algebraic geometry.

Definition 2. Let A be a connected graded k-algebra. Then A is called a d-dimensional AS-regular (resp. AS-Gorenstein) algebra of Gorenstein parametar ℓ if it satisfies the following conditions:

- $\mathrm{gldim}\, A = d < \infty$ (resp. $\mathrm{id}(A) = d < \infty$), and
- (Gorenstein condition)

$$\underline{\mathrm{Ext}}^i_A(k,A) \cong \begin{cases} 0 & \text{if } i \neq d, \\ k(\ell) & \text{if } i = d. \end{cases}$$

We do not assume that $\mathrm{GKdim}\, A < \infty$ in the definition.

Every 1-dimensional AS-regular algebra of Gorenstein parameter ℓ is isomorphic to a polynomial algebra $k[x]$ with $\deg x = \ell$.

The classification of 2-dimensional AS-regular algebras were completed by Zhang [13].

We now focus on 3-dimensional AS-regular algebras generated in degree 1 of finite GK-dimension. These algebras were completely classified by Artin, Tate and Van den Bergh [2] using geometric techniques. In this article, we will use their classification only in the quadratic case.

Let $T(V)$ be the tensor algebra on V over k where V is a finite dimensional vector space. We say that A is a quadratic algebra if A is a graded algebra of the form $T(V)/(R)$ where $R \subseteq V \otimes_k V$ is a subspace and (R) is the ideal of $T(V)$ generated by R. For a quadratic algebra $A = T(V)/(R)$, we define

$$\Gamma_2 := \{(p,q) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid f(p,q) = 0 \text{ for all } f \in R\}.$$

Definition 3. [5] A quadratic algebra $A = T(V)/(R)$ is called geometric if there exists a geometric pair (E,σ) where $E \subseteq \mathbb{P}(V^*)$ is a closed k-subscheme and σ is a k-automorphism of E such that

(G1) $\Gamma_2 = \{(p,\sigma(p)) \in \mathbb{P}(V^*) \times \mathbb{P}(V^*) \mid p \in E\}$, and
(G2) $R = \{f \in V \otimes_k V \mid f(p,\sigma(p)) = 0 \text{ for all } p \in E\}$.

Let $A = T(V)/(R)$ be a quadratic algebra. If A satisfies the condition **(G1)**, then A determines a geometric pair (E, σ). If A satisfies the condition **(G2)**, then A is determined by a geometric pair (E, σ), so we will write $A = \mathcal{A}(E, \sigma)$.

If A is a 3-dimensional quadratic AS-regular algebra of finite GK-dimension, then $A = \mathcal{A}(E, \sigma)$ is geometric, and E is either $\mathbb{P}^2$ or a cubic curve in $\mathbb{P}^2$. Artin, Tate and Van den Bergh [2] gave a list of geometric pairs (E, σ) for "generic" 3-dimensional quadratic AS-regular algebras. In their generic classification, E is one of the following:

(1) a triangle.
(2) a union of a line and a conic meeting at two points.
(3) an elliptic curve.

Example 4. Let

$$A = k\langle x, y, z\rangle/(\alpha yz + \beta zy + \gamma x^2, \alpha zx + \beta xz + \gamma y^2, \alpha xy + \beta yx + \gamma z^2)$$

where $\alpha, \beta, \gamma \in k$. Unless $\alpha^3 = \beta^3 = \gamma^3$, or two of $\{\alpha, \beta, \gamma\}$ are zero, $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional quadratic AS-regular algebra of GK-dimension 3 such that

$$E = \mathcal{V}(\alpha\beta\gamma(x^3 + y^3 + z^3) - (\alpha^3 + \beta^3 + \gamma^3)xyz) \subset \mathbb{P}^2$$

is an elliptic curve, and $\sigma \in \operatorname{Aut}_k E$ is given by the translation automorphism by a fixed point $(\alpha, \beta, \gamma) \in E$. In this case, A is called a 3-dimensional Sklyanin algebra.

For the purpose of this article, we define the Types of some geometric pairs (E, σ) of 3-dimensional quadratic AS-regular algebras as follows:

- Type $\mathbb{P}^2$: E is $\mathbb{P}^2$, and $\sigma \in \operatorname{Aut}_k \mathbb{P}^2$.
- Type S_1: E is a triangle, and σ stabilizes each component.
- Type S_2: E is a triangle, and σ interchanges two components.
- Type S_3: E is a triangle, and σ circulates three components.
- Type S_1': E is a union of a line and a conic meeting at two points, and σ stabilizes each component and two intersection points.
- Type S_2': E is a union of a line and a conic meeting at two points, and σ stabilizes each component and interchanges two intersection points.

Remark 5. If E is a union of a line and a conic meeting at two points, and σ interchanges these two components, then $\mathcal{A}(E, \sigma)$ is not an AS-regular algebra [2, Proposition 4.11]. Thus the above types completely cover the generic singular cases and $E = \mathbb{P}^2$.

Recall that the Hilbert series of A is defined by

$$H_A(t) = \sum_{i=-\infty}^{\infty} (\dim_k A_i)t^i \quad \in \mathbb{Z}[[t, t^{-1}]].$$

If A is a 3-dimensional quadratic AS-regular algebra of finite GK-dimension, then A is a noetherian Koszul domain and $H_A(t) = (1-t)^{-3}$. In particular, the Gorenstein parameter of A is equal to 3.

At the end of this section, we prepare the definition of the generalized Nakayama automorphism to state our theorem.

Let A be a d-dimensional AS-Gorenstein algebra, and $\mathfrak{m} := A_{\geq 1}$ the unique maximal homogeneous ideal of A. We define the graded A-A bimodule ω_A by

$$\omega_A := \mathrm{H}^d_{\mathfrak{m}}(A)^* = \underline{\mathrm{Hom}}_k(\lim_{n\to\infty} \underline{\mathrm{Ext}}^d_A(A/A_{\geq n}, A), k).$$

It is known that $\omega_A \cong {}_{\nu}A(-\ell)$ as graded A-A bimodules for some graded k-algebra automorphism $\nu \in \mathrm{Aut}_k A$, where ${}_{\nu}A$ is the graded A-A bimodule defined by ${}_{\nu}A = A$ as a graded k-vector space with a new action $a * x * b := \nu(a)xb$ (see [3, Theorem 1.2], [4]).

Definition 6. [6] Let A be a d-dimensional AS-Gorenstein algebra. We call $\nu \in \mathrm{Aut}_k A$ the generalized Nakayama automorphism of A if $\omega_A \cong {}_{\nu}A(-\ell)$ as graded A-A bimodules. If the generalized Nakayama automorphism $\nu \in \mathrm{Aut}_k A$ is id_A, then A is called symmetric.

A finite dimensional algebra A is called graded Frobenius if $A^* \cong A(-\ell)$ as right and left graded A-modules. Let A be a graded Frobenius algebra. Then $A^* \cong {}_{\nu}A(-\ell)$ as graded A-A bimodules where ν is the usual Nakayama automorphism. Since A is a noetherian AS-Gorenstein algebra of $\mathrm{id}(A) = 0$ and

$$\omega_A = \mathrm{H}^0_{\mathfrak{m}}(A)^* \cong A^* \cong {}_{\nu}A(-\ell),$$

the generalized Nakayama automorphism of A is the usual Nakayama automorphism ([6]).

Let $A = \mathcal{A}(E, \sigma)$ be a geometric AS-Gorenstein algebra of Gorenstein parameter ℓ. If ν is the generalized Nakayama automorphism of A, then it restricts to an automorphism $\nu \in \mathrm{Aut}_k V = \mathrm{Aut}_k A_1$. So its dual induces an automorphism $\nu^* \in \mathrm{Aut}_k \mathbb{P}(V^*)$, and which induces an automorphism $\nu^* \in \mathrm{Aut}_k E$ (see [6]). Therefore we define a new graded algebra $\overline{A}$ by

$$\overline{A} := \mathcal{A}(E, \nu^* \sigma^{\ell}).$$

3. Main results

The following theorem motivates this research.

Theorem 7. [5, Theorem 5.4] *Let $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$ be 3-dimensional Sklyanin algebras. If $\sigma^9, \sigma'^9 \neq \mathrm{id}$, then the following are equivalent:*

(1) $\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A'$.
(2) $\mathcal{A}(E, \sigma^3) \cong \mathcal{A}(E', \sigma'^3)$ *as graded algebras.*

Now, we state our main theorem in this article.

Theorem 8. [7], [11]

(1) *Let A, A' be noetherian geometric AS-regular algebras. Then*

$$\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A' \implies \overline{A} \cong \overline{A'} \text{ as graded algebras.}$$

(2) *In particular, if $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$ are 3-dimensional quadratic AS-regular algebras of finite GK-dimension such that (E, σ) and (E', σ') are of the following Type: $\mathbb{P}^2$, S_1, S_2, S_3, S_1' or S_2', then*

$$\mathrm{GrMod}\, A \cong \mathrm{GrMod}\, A' \iff \overline{A} \cong \overline{A'} \text{ as graded algebras.}$$

The generalized Nakayama automorphism of a 3-dimensional Sklyanin algebra A is id_A (cf. [8, Example 10.1]). Thus Theorem 8 (2) also hold for 3-dimensional Sklyanin algebras $A = \mathcal{A}(E, \sigma), A' = \mathcal{A}(E', \sigma')$ with $\sigma^9, \sigma'^9 \neq \mathrm{id}$.

Theorem 9. [11] *Let $A = \mathcal{A}(E, \sigma)$ be a 3-dimensional quadratic AS-regular algebra of finite GK-dimension such that (E, σ) is of the following Type: $\mathbb{P}^2$, S_1, S_2, S_3, S_1' or S_2', then $\overline{A}$ is a 3-dimensional symmetric AS-regular algebra.*

By Theorem 8 (2) and Theorem 9, graded Morita equivalences of 3-dimensional generic geometric AS-regular algebras are characterized by isomorphisms of 3-dimensional symmetric AS-regular algebras. In general, it is more difficult to check if two graded algebras are graded Morita equivalent than to check if they are isomorphic as graded algebras. In this sence, Theorem 8 is useful.

4. Example

In this last section, we give an example by applying Theorem 8 to 3-dimensional skew polynomial algebras.

Example 10. If

$$A = k\langle x, y, z\rangle/(yz - \alpha zy, zx - \beta xz, xy - \gamma yx)$$

where $\alpha, \beta, \gamma \in k$, $\alpha\beta\gamma \neq 0, 1$, then $A = \mathcal{A}(E, \sigma)$ is a 3-dimensional quadratic AS-regular algebra of GK-dimension 3 and Gorenstein parameter 3 such that

$$E = l_1 \cup l_2 \cup l_3 \subset \mathbb{P}^2 \quad \text{where } l_1 = \mathcal{V}(x), l_2 = \mathcal{V}(y), l_3 = \mathcal{V}(z)$$

is a triangle, and $\sigma \in \operatorname{Aut}_k E$ is given by

$$\begin{aligned} \sigma|_{l_1}(0, b, c) &= (0, b, \alpha c) \\ \sigma|_{l_2}(a, 0, c) &= (\beta a, 0, c) \\ \sigma|_{l_3}(a, b, 0) &= (a, \gamma b, 0), \end{aligned}$$

so (E, σ) is of Type S_1. In this case, the automorphism $\nu^* \in \operatorname{Aut}_k E$ induced by the generalized Nakayama automorphism $\nu \in \operatorname{Aut}_k A$ is given by

$$\nu^*(a, b, c) = ((\beta/\gamma)a, (\gamma/\alpha)b, (\alpha/\beta)c)\,,$$

so $\nu^*\sigma^3 \in \operatorname{Aut}_k E$ is given by

$$\begin{aligned} \nu^*\sigma^3|_{l_1}(0, b, c) &= (0, b, \alpha\beta\gamma c) \\ \nu^*\sigma^3|_{l_2}(a, 0, c) &= (\alpha\beta\gamma a, 0, c) \\ \nu^*\sigma^3|_{l_3}(a, b, 0) &= (a, \alpha\beta\gamma b, 0). \end{aligned}$$

It follows that

$$\overline{A} = \mathcal{A}(E, \nu^*\sigma^3) = k\langle x, y, z\rangle/(yz - \alpha\beta\gamma zy,\ zx - \alpha\beta\gamma xz,\ xy - \alpha\beta\gamma yx).$$

Similarly, if

$$A' = k\langle x, y, z\rangle/(yz - \alpha' zy, zx - \beta' xz, xy - \gamma' yx)$$

where $\alpha', \beta', \gamma' \in k, \alpha'\beta'\gamma' \neq 0, 1$, then

$$\overline{A'} = k\langle x, y, z\rangle/(yz - \alpha'\beta'\gamma' zy,\ zx - \alpha'\beta'\gamma' xz,\ xy - \alpha'\beta'\gamma' yx).$$

By Theorem 8 (2),

$$\operatorname{GrMod} A \cong \operatorname{GrMod} A' \Longleftrightarrow \overline{A} \cong \overline{A'}.$$

Moreover,

$$\overline{A} \cong \overline{A'} \Longleftrightarrow \alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}$$

by [12, Lemma 2.1]. Hence we have

$$\operatorname{GrMod} A \cong \operatorname{GrMod} A' \Longleftrightarrow \alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}.$$

References

[1] M. Artin and W. Schelter, *Graded algebras of global dimension 3*, Adv. Math. **66** (1987), 171–216.

[2] M. Artin, J. Tate and M. Van den Bergh, *Some algebras associated to automorphisms of elliptic curves*, The Grothendieck Festschrift Vol. **1** Birkhauser, (1990), 33–85.

[3] P. Jørgensen, *Local cohomology for non-commutative graded algebras*, Comm. Algebra **25** (1997), 575–591.

[4] H. Minamoto and I. Mori, *The structure of AS-Gorenstein algebras*, Adv. Math., to appear.

[5] I. Mori, *Noncommutative projective schemes and point schemes*, Algebras, Rings and Their Representations, World Sci. Publ. (2006), 215–239.

[6] ———, *Co-point modules over Frobenius Koszul algebras*, Comm. Algebra **36** (2008), 4659–4677.

[7] I. Mori and K. Ueyama, *Graded Morita equivalences for geometric AS-regular algebras*, in preparation.

[8] S. P. Smith, *Some finite dimensional algebras related to elliptic curves*, in Representation Theory of Algebras and Related Topics (Mexico City, 1994), CMS Conf. Proc. **19**, Amer. Math. Soc., Providence, (1996), 315–348.

[9] D. R. Stephenson, *Artin-Schelter regular algebras of global dimension three*, J. Algebra **183** (1996), 55–73.

[10] ———, *Algebras associated to elliptic curves*, Trans. Amer. Math. Soc. **349** (1997), 2317–2340.

[11] K. Ueyama, *Graded Morita equivalences for generic Artin-Schelter regular algebras*, Kyoto J. Math., to appear.

[12] J. Vitoria, *Equivalences for noncommutative projective spaces*, preprint.

[13] J. J. Zhang, *Non-noetherian regular rings of dimension 2*, Proc. Amer. Math. Soc. **126** (1998), 1546–1653.

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ON SELFINJECTIVE ALGEBRAS OF STABLE DIMENSION ZERO

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Abstract. This paper is based on our lecture giving at the '43rd Symposium on Ring Theory and Representation Theory' held at Naruto University of Education in September 2010. In this paper, we consider the stable dimension of selfinjective algebra, which is the dimension of its stable module category in the sense of Rouquier. We give a proof that a non-semisimple selfinjective algebra A is representation-finite if the stable dimension of A is zero. Moreover, we verify that selfinjective algebras obtained from some hereditary algebra have stable dimension at most one.

Key Words: Representation-finite algebra, Selfinjective algebra, Stable dimension.

2000 *Mathematics Subject Classification*: Primary 16G60; Secondary 18E30.

1. Notation

Throughout this article, k denotes an algebraically closed field, and all algebras are finite-dimensional associative k-algebras with an identity, unless otherwise stated.

For any k-algebra Λ, we denote by $\operatorname{mod}\Lambda$ the abelian category of finite-dimensional (over k) left Λ-modules and by $\Gamma(\Lambda)$ the Auslander-Reiten quiver of Λ. We may identify the vertices of $\Gamma(\Lambda)$ with the indecomposable Λ-modules. Then we have the Auslander-Reiten translation $\tau_\Lambda = D\operatorname{Tr}$ and $\tau_\Lambda^{-1} = \operatorname{Tr} D$, where $D: \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda^{\mathrm{op}}$ is the standard duality $\operatorname{Hom}_k(\text{-}, k)$. Moreover, we denote by $\mathcal{D}^b(\operatorname{mod}\Lambda)$ the bounded derived category of Λ, by $\operatorname{gl.dim}\Lambda$ the global dimension of Λ, by $\mathrm{T}(\Lambda)$ the trivial extension of Λ and by $\hat{\Lambda}$ the repetitive category of Λ.

For any selfinjective k-algebra A, we denote by $\underline{\operatorname{mod}}A$ the stable module category of A. Let $\Omega = \Omega_A : \underline{\operatorname{mod}}A \to \underline{\operatorname{mod}}A$ be a syzygy functor. Note that if X is indecomposable, then $\Omega(X)$ remains indecomposable. And moreover, Ω is an equivalence and $\underline{\operatorname{mod}}A$ is a triangulated category with shift functor Ω^{-1} (see Happel [14]). Similarly, the stable module category $\underline{\operatorname{mod}}\hat{\Lambda}$ of a repetitive category $\hat{\Lambda}$ can be defined and then is a triangulated category.

Furthermore, we denote by ${}_s\Gamma(A)$ the stable Auslander-Reiten quiver of A, which is obtained from $\Gamma(A)$ by removing the projective-injective vertices and the arrows attached to them. Then the set ${}_s\Gamma(A)_0$ of vertices of ${}_s\Gamma(A)$ coincides with the set of isoclasses of non-projective indecomposable A-modules. It is well-known that we can recover $\Gamma(A)$ from ${}_s\Gamma(A)$. Note that the Auslander-Reiten translation τ_A is an automorphism of the quiver ${}_s\Gamma(A)$ with an inverse τ_A^{-1} and that $\tau_A \cong \Omega^2\nu \cong \nu\Omega^2$ since $\Omega\nu \cong \nu\Omega$, where $\nu = D\operatorname{Hom}_A(\text{-}, A)$ is the Nakayama functor.

The detailed version of this paper has been submitted for publication elsewhere.

2. Preliminaries

First, we define the dimension of triangulated category in the sense of Rouquier.

Definition 1 (Rouquier [21])**.** Let $\mathcal{T}$ be a triangulated category with shift functor [1]. Then the *dimension* of $\mathcal{T}$ is defined to be

$$\dim \mathcal{T} \quad := \quad \min\{n \geq 0 \mid \langle M \rangle_{n+1} = \mathcal{T} \text{ for some } M \in \mathcal{T}\}$$

or ∞ when there is no such an object M, where $\langle M \rangle_{n+1}$ is defined inductively:

$$\begin{aligned} &\text{for } n = 0, \quad \langle M \rangle_1 := \operatorname{add}\{M[i] \mid i \in \mathbb{Z}\}, \text{ and} \\ &\text{if } n > 0, \quad \langle M \rangle_{n+1} := \operatorname{add}\{M_{n+1} \mid \text{there is a triangle} : M_n \to M_{n+1} \to M_1 \to M_n[1], \\ &\qquad\qquad\qquad\qquad\qquad \text{where } M_n \in \langle M \rangle_n \text{ and } M_1 \in \langle M \rangle_1\}. \end{aligned}$$

Let $\mathcal{F}$ be another triangulated category, and let $F : \mathcal{T} \to \mathcal{F}$ be a triangle functor. Then we mention some fundamental remarks.

Remark 2.
- (a) If a functor F is dense, then $\dim \mathcal{T} \geq \dim \mathcal{F}$.
- (b) If a functor F is an equivalence, then $\dim \mathcal{T} = \dim \mathcal{F}$.

Second, we define the stable dimension of selfinjective algebra.

Let A be a non-semisimple selfinjective algebra (over a field). Recall that the stable module category $\underline{\operatorname{mod}}A$ of A is a triangulated category. Thus we can define the stable dimension of A.

Definition 3. The *stable dimension* of A is defined to be

$$\operatorname{stab.dim} A \quad := \quad \dim(\underline{\operatorname{mod}}A) \ \ \text{(in the sense of Definition 1).}$$

Recall also that $\underline{\operatorname{mod}}A$ and $\mathcal{D}^b(\operatorname{mod} A)/\operatorname{per} A$ are equivalent as triangulated categories (see Rickard [19]), where $\operatorname{per} A$ is the épaisse subcategory of $\mathcal{D}^b(\operatorname{mod} A)$ consisting of perfect complexes. Then by Remark 2, we have the fundamental properties for the stable dimension.

Remark 4.
- (a) If there exists a dense functor $F : \mathcal{T} \to \underline{\operatorname{mod}}A$, where $\mathcal{T}$ is a suitable triangulated category: *e.g.*, $\mathcal{T} = \mathcal{D}^b(\operatorname{mod} A)$ and $\underline{\operatorname{mod}}\hat{\Lambda}$, then $\dim \mathcal{T} \geq \operatorname{stab.dim} A$ (see Subsection 4.1).
- (b) Let B be another selfinjective algebra. If $\underline{\operatorname{mod}}A$ and $\underline{\operatorname{mod}}B$ are equivalent as triangulated categories, then $\operatorname{stab.dim} A = \operatorname{stab.dim} B$; For instance, A and B are derived equivalent, then $\operatorname{stab.dim} A = \operatorname{stab.dim} B$.

Rouquier introduced a notion of dimension of a triangulated category in [21]. One of his aims was to give a lower bound for Auslander's representation dimension of selfinjective algebras (see Proposition 6), and then he gave the first example of algebras having representation dimension at least four (see Theorem 23).

Definition 5 (Auslander [3])**.** The representation dimension of a non-semisimple artin algebra Λ is defined to be

$$\operatorname{rep.dim} \Lambda := \min\{\operatorname{gl.dim} \operatorname{End}_\Lambda(M) \mid M \text{ is a generator and a cogenerator in} \operatorname{mod} \Lambda\}.$$

For semisimple artin algebra, the dimension is defined to be one.

In [20], Rouquier showed the following result.

Proposition 6 (Rouquier [20] cf. Auslander [3]). *Let A be a non-semisimple selfinjective algebra* (*over a field*). *Then*

$$\mathrm{LL}(A) \geq \mathrm{rep}\,.\dim A \geq \mathrm{stab}\,.\dim A + 2,$$

where the Loewy length $\mathrm{LL}(A)$ *is the smallest integer* r *such that* $\mathrm{rad}(A)^r = 0$.

After Auslander proved in [3] (see Proposition p.55) that $\mathrm{LL}(A) + 1 \geq \mathrm{rep}\,.\dim A$, Rouquier has improved it by indicating that the equality does not occur, and hence the first inequality in Proposition 6.

Remark 7. The stable dimension is always finite by the first inequality in Proposition 6. Recall also that for any artin algebra, the representation dimension is always finite (see Iyama [15]).

Auslander introduced the representation dimension in [3], and hoped that the representation dimension should be a good measure of how far a representation-infinite algebra is from being representation-finite. Actually, he showed the following result.

Theorem 8 (Auslander [3]). *For any artin algebra Λ, Λ is representation-finite if and only if* $\mathrm{rep}\,.\dim \Lambda \leq 2$.

Thus by Proposition 6 and Theorem 8, we observe that any (non-semisimple) representation-finite selfinjective algebra (over a field) has stable dimension zero, which also follows from definition immediately. Then we have a natural question whether the converse should also hold.

3. On selfinjective algebras of stable dimension zero

3.1. **Main results.** In this subsection, we assume that A is a non-semisimple selfinjective algebra over an algebraically closed field k, unless otherwise stated.

Our main result is to prove that if A has stable dimension zero, then A is representation-finite. Namely, we verify that the converse of the observation above indeed holds provided that the base field is algebraically closed. Although this was expected to hold by some experts, it had not been proved before.

Our main theorem is the following.

Theorem 9 (Yoshiwaki [24]). *Let A be a non-semisimple selfinjective finite-dimensional connected algebra over an algebraically closed field k. If the set ${}_s\Gamma(A)_0$ of isoclasses of non-projective indecomposable A-modules admits only finitely many Ω-orbits, then A is representation-finite.*

To prove this theorem, we need two critical results. The first result is a characterization of representation-finite algebras over an algebraically closed field due to Liu.

Proposition 10 (Liu [17] 3.11 Proposition p.52). *Let A be a finite-dimensional algebra over an algebraically closed field. Then A is representation-finite if and only if $\Gamma(A)$ admits only finitely many τ_A-orbits.*

This follows from the 2nd Brauer-Thrall conjecture. So, we require the assumption in Theorem 9 that the base field k is algebraically closed.

Second, we need the following well-known result due to Auslander.

Theorem 11 (Auslander [5])**.** *Let A be a finite-dimensional connected algebra (over a field), and let $\mathcal{C}$ be a connected component of $\Gamma(A)$. If the length of the modules in $\mathcal{C}$ is bounded, then A is representation-finite and $\mathcal{C} = \Gamma(A)$.*

So, we require the assumption in Theorem 9 that A is connected. As a consequence of Theorem 11, the 1st Brauer-Thrall conjecture follows. Namely, it may be to say that Theorem 9 follows from the two Brauer-Thrall conjectures.

Suppose that $\operatorname{stab.dim} A = 0$. Then by definition we have

$$\underline{\mathrm{mod}}A = \operatorname{add}\{\Omega^i M \mid i \in \mathbb{Z}\}$$

for some $M \in \underline{\mathrm{mod}}A$. Hence it is easy to see the following lemma.

Lemma 12. *The following are equivalent*:

(1) $\operatorname{stab.dim} A = 0$,

(2) ${}_s\Gamma(A)_0$ *admits only finitely many Ω-orbits.*

Thus we obtain the desired result by Theorem 9.

Corollary 13. *If* $\operatorname{stab.dim} A = 0$*, then A is representation-finite.*

Proof. Suppose that $\operatorname{stab.dim} A = 0$. Then by Lemma 12, ${}_s\Gamma(A)_0$ admits only finitely many Ω-orbits, and hence A is representation-finite by Theorem 9. $\square$

Moreover, we have the following result by Proposition 6, Theorem 8 and Corollary 13.

Corollary 14. *If* $\operatorname{rep.dim} A = 3$*, then* $\operatorname{stab.dim} A = 1$.

Proof. If $\operatorname{rep.dim} A = 3$, then $\operatorname{stab.dim} A \leq 1$ and A is not representation-finite by Proposition 6 and Theorem 8. Then by Corollary 13, $\operatorname{stab.dim} A = 1$. $\square$

In the last of this subsection, we give an example of selfinjective algebra having stable dimension zero.

Example 15. Let $A = kQ/I$, where Q is the quiver

$$\begin{tikzcd}
& 2 \arrow[d, shift right, "\alpha_2"'] & \\
& 0 \arrow[u, shift right, "\beta_2"'] \arrow[dl, shift right, "\beta_1"'] \arrow[dr, shift left, "\beta_3"] & \\
1 \arrow[ur, shift right, "\alpha_1"'] & & 3 \arrow[ul, shift left, "\alpha_3"]
\end{tikzcd}$$

and the ideal I is generated by

$$\alpha_1\beta_1 - \alpha_2\beta_2,\ \alpha_2\beta_2 - \alpha_3\beta_3,\ \beta_1\alpha_1,\ \beta_2\alpha_1,\ \beta_1\alpha_2,\ \beta_3\alpha_2,\ \beta_2\alpha_3,\ \beta_3\alpha_3.$$

Then A is a selfinjective algebra of type $\mathbb{D}_4$. Indeed, let $\vec{\Delta}$ be the quiver

$$\begin{tikzcd}
1 \arrow[dr] & 2 \arrow[d] & 3 \arrow[dl] \\
& 0 &
\end{tikzcd},$$

and let $B = k\,\vec{\Delta}$ be the path algebra of $\vec{\Delta}$. Clearly, B is a tilted algebra of type $\mathbb{D}_4$. Then we have $\hat{B} = k\hat{\Delta}/\hat{I}$, where $\hat{\Delta}$ is the quiver

$$\begin{tikzcd}
(1,1) \arrow[dr] & (1,2) \arrow[d] & (1,3) \arrow[dl] \\
& (1,0) \arrow[dl, "\beta_{0,1}"'] \arrow[d, "\beta_{0,2}"'] \arrow[dr, "\beta_{0,3}"] & \\
(0,1) \arrow[dr, "\alpha_{0,1}"'] & (0,2) \arrow[d, "\alpha_{0,2}"'] & (0,3) \arrow[dl, "\alpha_{0,3}"] \\
& (0,0) \arrow[dl, "\beta_{-1,1}"'] \arrow[d, "\beta_{-1,2}"'] \arrow[dr, "\beta_{-1,3}"] & \\
(-1,1) \arrow[dr] \arrow[rr, leftrightarrow, bend right, "\rho"'] & (-1,2) \arrow[d] & (-1,3) \arrow[dl] \arrow[uu, mapsto, bend right, "\nu_{\hat{B}}"'] \\
& (-1,0) \arrow[dl] \arrow[d] \arrow[dr] & \\
{} & {} & {}
\end{tikzcd}$$

and the ideal $\hat{I}$ is generated by

$$\alpha_{m,i}\beta_{m,i} - \alpha_{m,j}\beta_{m,j},\ \ \beta_{m-1,i}\alpha_{m,j},$$

with $m \in \mathbb{Z}$, $i, j \in \{1, 2, 3\}$, $i \neq j$. Let $\nu_{\hat{B}}$ be the Nakayama automorphism of $\hat{B}$, and let ρ be the automorphism of $\hat{B}$ given by the permutation $\{((m, 1),\ (m, 3))\}$. Then we obtain $A = \hat{B}/\langle \rho\nu_{\hat{B}} \rangle$, and thus the stable Auslander-Reiten quiver ${}_s\Gamma(A)$ of A is of the form:

$$\begin{tikzcd}[column sep=small]
\bullet \arrow[dr] & & \bullet \arrow[dr] & & \bullet \arrow[dr] & & \bullet \arrow[dr] & & \bullet \arrow[dr] & & \bullet_{M} \\
\bullet \arrow[r] & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \bullet \arrow[r] & \bullet \arrow[r] \arrow[ur] \arrow[dr] & \bullet \arrow[r] & \underset{L}{\bullet} \arrow[r] \arrow[ur] \arrow[dr] & \underset{N}{\bullet} \\
\underset{M}{\bullet} \arrow[ur] & & \bullet \arrow[ur] & & \bullet \arrow[ur] & & \bullet \arrow[ur] & & \underset{\Omega^2 M}{\bullet} \arrow[ur] & & \underset{\nu M = \Omega^5 M}{\bullet}
\end{tikzcd}.$$

Let $\{M,\ N,\ L\}$ be a complete set of representatives of τ_A-orbits in ${}_s\Gamma(A)$, and put $X = M \oplus N \oplus L$. Then $\underline{\mathrm{mod}}A = \mathrm{add}\{\Omega^i X \mid i \in \mathbb{Z}\}$ because the τ_A-orbits and the Ω-orbits in ${}_s\Gamma(A)_0$ coincide, and hence we have $\mathrm{stab}.\dim A = 0$.

3.2. **An application.** We now define the derived dimension of finite-dimensional algebra.

Definition 16. Let Λ be a finite-dimensional k-algebra. Then the *derived dimension* of Λ is defined to be

$$\mathrm{der.dim}\,\Lambda \;:=\; \dim(\mathcal{D}^b(\mathrm{mod}\,\Lambda)) \quad \text{(in the sense of Definition 1)}.$$

For the derived dimension, it is known that it has the following property.

Proposition 17 (Rouquier [20], Krause-Kussin [16], Oppermann [18]). *Let Λ be a finite-dimensional k-algebra. Then*

$$\mathrm{der.dim}\,\Lambda \leq \inf(\mathrm{gl.dim}\,\Lambda,\ \mathrm{rep.dim}\,\Lambda).$$

Remark 18. The derived dimension is always finite because the representation dimension is always finite (see Iyama [15]; also see Remark 7).

Furthermore, we introduce the iterated tilted algebra.

Definition 19. Let Q be a finite connected acyclic quiver, and let kQ be the path algebra of Q. Then Λ is an *iterated tilted algebra* of type Q if there exists a triangle equivalence between $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$ and $\mathcal{D}^b(\mathrm{mod}\,kQ)$. If Q is a Dynkin quiver, Λ is called an iterated tilted algebra of Dynkin type.

There is the original definition of iterated tilted algebra (for instance, see Happel [14] Chapter IV 4.4 p.173). We, however, use Definition 19 for simplicity since Happel has shown that it is equivalent to the original definition (see Happel [14] Chapter IV 5.4 Theorem p.176).

As an application of our result above (see Corollary 13), we obtain the following result.

Theorem 20 (Chen-Ye-Zhang [8], Yoshiwaki [24]). *For any finite-dimensional k-algebra Λ, the following are equivalent*:

1. $\mathrm{der.dim}\,\Lambda = 0$*,*
2. $\mathrm{stab.dim}\,\mathrm{T}(\Lambda) = 0$*,*
3. $\mathrm{T}(\Lambda)$ *is representation-finite,*
4. Λ *is an iterated tilted algebra of Dynkin type.*

Sketch of Proof. Chen-Ye-Zhang have actually shown the implication from (1) to (2). It is easy to see that any iterated tilted algebra of Dynkin type has derived dimension zero. Therefore, since Corollary 13 means that the implication from (2) to (3) holds, the assertion follows from Assem-Happel-Roldán's result in [1] (also see Happel [14] Chapter V 2.1 Theorem p.199) that (3) is equivalent to (4). ☐

4. Some selfinjective algebras have stable dimension one

4.1. **A calculation for the stable dimension.** The proof of Theorem 20 gives us an idea for calculation of the stable dimension. In this subsection, we assume that Λ is an iterated tilted algebra of some finite connected acyclic quiver Q. Then we have the following facts due to Happel.

Facts (Happel [14]).

(a) There exists a triangle equivalence between $\underline{\mathrm{mod}}\hat{\Lambda}$ and $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$ since Λ has finite global dimension (see [14] Chapter II 4.9 Theorem p.88).

(b) $\hat{\Lambda}$ is locally support-finite (see [14] Chapter V 3.1 Lemma p.201).
($i.e.$, for all $x \in \hat{\Lambda}$, $\sharp\{y \in \hat{\Lambda} \mid y \in \mathrm{supp}\, M \text{ with } M(x) \neq 0, M \in \mathrm{ind}\,\hat{\Lambda}\} < \infty$.)

Here, we need the critical result due to Dowbor-Skowroński (see [10] 2.5 Proposition p.319; also see [9] Lemma 2 p. 524).

Theorem 21 (Dowbor-Skowroński). *Let $F_\lambda : \mathrm{mod}\,\hat{\Lambda} \to \mathrm{mod}\,\hat{\Lambda}/G$ be the push-down functor, where G is an admissible torsion-free group of k-linear automorphisms of $\hat{\Lambda}$. If $\hat{\Lambda}$ is locally support-finite, then F_λ is dense.*

The push-down functor preserves the projective modules and the injective modules (see Bongartz-Gabriel [7] 3.2 Proposition p.344), so that the induced functor $\underline{F_\lambda} : \underline{\mathrm{mod}}\hat{\Lambda} \to \underline{\mathrm{mod}}\hat{\Lambda}/G$ is well-defined. Then we have the following commutative diagram

$$\begin{tikzcd} \underline{\mathrm{mod}}\hat{\Lambda} \arrow[r, phantom, "\cong"] \arrow[d, "\text{push-down}"'] & \mathcal{D}^b(\mathrm{mod}\,\Lambda) \arrow[dl, dotted, two heads] \\ \underline{\mathrm{mod}}\hat{\Lambda}/G & \end{tikzcd}$$

Since we have the dense functor from $\mathcal{D}^b(\mathrm{mod}\,\Lambda)$ to $\underline{\mathrm{mod}}\hat{\Lambda}/G$, we obtain

$$1 \geq \mathrm{der}\,.\dim\Lambda \geq \mathrm{stab}\,.\dim\hat{\Lambda}/G$$

by Remark 4 and Proposition 17.

We call such an algebra $\hat{\Lambda}/G$ a selfinjective algebra of type Q. Thus we have the following result by the argument above.

Proposition 22. *Any selfinjective algebra of type Q has stable dimension at most one.*

4.2. **Conjectures.** It has been conjectured, or asked by many experts, whether the following holds.

Conjecture 1. *Any artin algebra of tame representation type has representation dimension at most three.*

This is true for some classes of tame algebras, such as special biserial algebras (see Erdmann-Holm-Iyama-Schröer [11]) and domestic selfinjective algebras socle equivalent to a weakly symmetric algebra of Euclidean type (see Bocian-Holm-Skowroński [6]). Note that the latter algebras have stable dimension at most one by Proposition 6.

Also, any hereditary algebra has representation dimension at most three (see Auslander [3] Proposition p.58). Namely, any wild hereditary algebra must have representation dimension at most three. Hence the converse does not hold in general.

By Corollary 14, we pose a new conjecture for the stable dimension.

Conjecture 2. *Any (non-semisimple) selfinjective k-algebra of tame representation type has stable dimension at most one.*

By Proposition 22, any selfinjective algebra of Euclidean type has stable dimension at most one. According to Skowroński [22], a selfinjective algebra is of Euclidean type if and only if it is standard domestic of infinite type. Therefore, any domestic selfinjective standard algebra of infinite type has stable dimension at most one. Namely, we obtain a partial result for Conjecture 2.

Even if Q is a wild quiver, then any selfinjective algebra of type Q has stable dimension at most one. Thus the converse does not hold in general, similar to Conjecture 1.

A basic connected algebra A is *standard* (see [23]) if there exists a Galois covering $R \to R/G = A$ (see [12]) such that R is a simply connected locally bounded category (see [2]) and G is an admissible torsion-free group of k-linear automorphism of R. Thus it will be possible to calculate the stable dimension of selfinjective standard algebra in the same way as subsection 4.1, so that we pose a new conjecture.

Conjecture 3. *Any (non-semisimple) selfinjective standard k-algebra of tame representation type has stable dimension at most one.*

This is a weak version of Conjecture 2.

5. A question

In [20], Rouquier gave the first example of algebras having representation dimension at least four. Namely, he showed the following.

Theorem 23 (Rouquier). *Let $A = \bigwedge(k^n)$ be an exterior algebra. Then* $\operatorname{rep.dim} = n+1$, $\operatorname{der.dim} A = n$ *and* $\operatorname{stab.dim} A = n-1$.

Moreover, Han showed the following result in [13].

Theorem 24 (Han). *Any representation-finite artin algebra has derived dimension at most one.*

Since any non-semisimple selfinjective algebra is not derived equivalent to a hereditary algebra, any (non-semisimple) representation-finite selfinjective algebra has derived dimension one by Theorem 20. So, by Theorem 8, we obtain the following result.

Theorem 25. *If a non-semisimple selfinjective k-algebra A is representation-finite, then* $\operatorname{rep.dim} A = 2$, $\operatorname{der.dim} A = 1$ *and* $\operatorname{stab.dim} A = 0$.

Thus we have the following natural question.

Question. What about $\operatorname{rep.dim} A - \operatorname{der.dim} A$ and $\operatorname{der.dim} A - \operatorname{stab.dim} A$?

Theorems 23 and 25 suggest that the difference in the question above may be at least one.

References

[1] I. Assem, D. Happel and O. Roldán, *Representation-finite trivial extension algebras*, J. Pure Appl. Algebra **33** (1984), no. 3, 235–242.

[2] I. Assem and A. Skowroński, *On some classes of simply connected algebras*, Proc. London Math. Soc. (3) **56** (1988), no. 3, 417–450.
[3] M. Auslander, *Representation dimension of Artin algebras*, Queen Mary College Mathematics Notes (1971), republished in [4].
[4] ———, *Selected works of Maurice Auslander. Part 1*, American Mathematical Society, Providence, RI, 1999, Edited and with a foreword by Idun Reiten, Sverre O. Smalø, and Øyvind Solberg.
[5] ———, I. Reiten and S. O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, **36**, Cambridge University Press, Cambridge, 1995.
[6] R. Bocian, T. Holm and A. Skowroński, *The representation dimension of domestic weakly symmetric algebras*, Cent. Eur. J. Math. **2** (2004), no. 1, 67–75 (electronic).
[7] K. Bongartz and P. Gabriel, *Covering spaces in representation-theory*, Invent. Math. **65** (1981/82), no. 3, 331–378.
[8] X.W. Chen, Y. Ye and P. Zhang, *Algebras of derived dimension zero*, Comm. Algebra **36** (2008), no. 1, 1–10.
[9] P. Dowbor and A. Skowroński, *On Galois coverings of tame algebras*, Arch. Math. (Basel) **44** (1985), no. 6, 522–529.
[10] ——— and ———, *Galois coverings of representation-infinite algebras*, Comment. Math. Helv. **62** (1987), no. 2, 311–337.
[11] K. Erdmann, T. Holm, O. Iyama, and J. Schröer, *Radical embeddings and representation dimension*, Adv. Math. **185** (2004), no. 1, 159–177.
[12] P. Gabriel, *The universal cover of a representation-finite algebra*, Representations of algebras (Puebla, 1980), Lecture Notes in Math., **903**, Springer, Berlin-New York, 1981, pp. 68–105.
[13] Y. Han, *Derived dimensions of representation-finite algebras*, arXiv:0909.0330.
[14] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Mathematical Society Lecture Note Series, **119**, Cambridge University Press, Cambridge, 1988.
[15] O. Iyama, *Finiteness of representation dimension*, Proc. Amer. Math. Soc. **131** (2003), no. 4, 1011-1014 (electronic).
[16] H. Krause and D. Kussin, *Rouquier's theorem on representation dimension*, Trends in representation theory of algebras and related topics, Contemp. Math., **406**, Amer. Math. Soc., Providence, RI, 2006, pp. 95–103.
[17] S. Liu, *Degrees of irreducible maps and the shapes of Auslander-Reiten quivers*, J. London Math. Soc. (2) **45** (1992), no. 1, 32–54.
[18] S. Oppermann, *Lower bounds for Auslander's representation dimension*, Duke Math. J. **148** (2009), no. 2, 211–249.
[19] J. Rickard, *Derived categories and stable equivalence*, J. Pure Appl. Algebra **61** (1989), no. 3, 303–317.
[20] R. Rouquier, *Representation dimension of exterior algebras*, Invent. Math. **165** (2006), no. 2, 357–367.
[21] ———, *Dimensions of triangulated categories*, J. K-theory **1** (2008), no. 2, 193-256 and errata, 257–258.
[22] A. Skowroński, *Selfinjective algebras of polynomial growth*, Math. Ann. **285** (1989), no. 2, 177–199.
[23] A. Skowroński, *Selfinjective algebras: finite and tame type*, Trends in representation theory of algebras and related topics, Contemp. Math., **406**, Amer. Math. Soc., Providence, RI, 2006, pp. 169–238.
[24] M. Yoshiwaki, *On selfinjective algebras of stable dimension zero*, arXiv:1004.1723.

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T-STRUCTURES AND LOCAL COHOMOLOGY FUNCTORS

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Abstract. The section functor Γ_W with support in a specialization closed subset W of $\mathrm{Spec}(R)$ is one of the most important radical functors and basic tools not only for the theory of commutative algebra but also for algebraic geometry. The aim of this article is to characterize the section functor Γ_W (resp. the right derived functor $\mathbf{R}\Gamma_W$ of Γ_W) as elements of the set of all functors on the category of all R-modules (resp. the derived category consisting of all left bounded complexes of R-modules).

1. Introduction

This is a joint work with Yuji Yoshino.

Let R be a commutative noetherian ring. We denote the category of all R-modules by R-Mod and also denote the derived category consisting of all left bounded complexes of R-modules by $\mathcal{D}^+(R\text{-Mod})$.

A radical functor, or more generally a preradical functor, has its own long history in the theory of categories and functors. See [2] or [3] for the case of module category. One of the most useful and important facts is that there is a bijective correspondence between the set of all left exact radical functors on R-Mod and the set of all hereditary torsion theories for R-Mod (See [5, Chapter VI, Proposition 3.1]).

In this paper, one of our purpose is to observe some necessary and sufficient conditions for a functor on R-Mod to be left exact radical functor. Furthermore, we give the notion of abstract local cohomology functors, that is, we say a triangle functor δ on $\mathcal{D}^+(R\text{-Mod})$ is an abstract local cohomology functor if it defines a stable t-structure on $D^+(R\text{-Mod})$ which divides indecomposable injective R-modules. (See Definition 6 for the precise meaning.) We note here that the notion of t-structure was introduced and studied first in the paper [1], but what we need in this paper is the notion of stable t-structure introduced by Miyachi in [4]. We shall also prove that an abstract local cohomology functor is of the form $\mathbf{R}\Gamma_W$ with W being a specialization closed subset of $\mathrm{Spec}(R)$ and show that the set of specialization closed subsets of $\mathrm{Spec}(R)$ bijectively corresponds to $\mathbb{A}(R)$ which is the set of all isomorphism classes of abstract local cohomology functors on $\mathcal{D}^+(R\text{-Mod})$.

2. The definition of abstract local cohomology functors

Let us recall some definitions for functors from the category theory.

Definition 1. Let γ be a functor on R-Mod.

(1) A functor γ is called a preradical functor if γ is a subfunctor of identity functor $\mathbf{1}$.

The detailed version of this paper has been submitted for publication elsewhere.

(2) A preradical functor γ is called a radical functor if $\gamma(M/\gamma(M)) = 0$ for every R-module M.

(3) A functor γ is said to preserve injectivity if $\gamma(I)$ is an injective R-module whenever I is an injective R-module.

Example 2. Let W be a subset of $\mathrm{Spec}(R)$. Recall that W is said to be specialization closed if $\mathfrak{p} \in W$ and $\mathfrak{p} \subseteq \mathfrak{q} \in \mathrm{Spec}(R)$ imply $\mathfrak{q} \in W$.

When W is a specialization closed subset, we can define the section functor $\mathit{\Gamma}_W$ with support in W as

$$\mathit{\Gamma}_W(M) = \{x \in M \mid \mathrm{Supp}(Rx) \subseteq W\}$$

for all $M \in R\text{-Mod}$. Then it is easy to see that $\mathit{\Gamma}_W$ is a left exact radical functor that preserves injectivity.

The notion of stable t-structure was introduced by J. Miyachi.

Definition 3. A pair $(\mathcal{U}, \mathcal{V})$ of full subcategories of a triangulated category $\mathcal{T}$ is called a stable t-structure on $\mathcal{T}$ if it satisfies the following conditions:

(1) $\mathrm{Hom}_{\mathcal{T}}(\mathcal{U}, \mathcal{V}) = 0$.

(2) $\mathcal{U} = \mathcal{U}[1]$ and $\mathcal{V} = \mathcal{V}[1]$.

(3) For any $X \in \mathcal{T}$, there is a triangle $U \to X \to V \to U[1]$ with $U \in \mathcal{U}$ and $V \in \mathcal{V}$.

For a triangle functor δ on triangulated category $\mathcal{T}$, we define two full subcategories of $\mathcal{T}$

$$\begin{aligned} \mathrm{Im}(\delta) &= \{X \in \mathcal{T} \mid X \cong \delta(Y) \text{ for some } Y \in \mathcal{T}\}, \\ \mathrm{Ker}(\delta) &= \{X \in \mathcal{T} \mid \delta(X) \cong 0\}. \end{aligned}$$

The following theorem proved by J. Miyachi is a key to our argument. We shall refer to this theorem as Miyachi's Theorem.

Theorem 4. [4, Proposition 2.6] *Let $\mathcal{T}$ be a triangulated category and $\mathcal{U}$ be a full triangulated subcategory of $\mathcal{T}$. Then the following conditions are equivalent for $\mathcal{U}$.*

(1) *There is a full subcategory $\mathcal{V}$ of $\mathcal{T}$ such that $(\mathcal{U}, \mathcal{V})$ is a stable t-structure on $\mathcal{T}$.*

(2) *The natural embedding functor $i : \mathcal{U} \to \mathcal{T}$ has a right adjoint $\rho : \mathcal{T} \to \mathcal{U}$.*

If it is the case, setting $\delta = i \circ \rho : \mathcal{T} \to \mathcal{T}$, we have the equalities

$$\mathcal{U} = \mathrm{Im}(\delta) \quad \textit{and} \quad \mathcal{V} = \mathcal{U}^{\perp} = \mathrm{Ker}(\delta).$$

Remark 5. Let $(\mathcal{U}, \mathcal{V})$ be a stable t-structure on $\mathcal{T}$, ρ be a right adjoint functor of $i : \mathcal{U} \to \mathcal{T}$ and set $\delta = i \circ \rho$ as in the theorem. The functor ρ, hence δ as well, is unique up to isomorphisms by the uniqueness of right adjoint functors.

Now we can define an abstract local cohomology functor.

Definition 6. We denote $\mathcal{T} = \mathcal{D}^{+}(R\text{-Mod})$ in this definition. Let $\delta : \mathcal{T} \to \mathcal{T}$ be a triangle functor. We call that δ is an abstract local cohomology functor if the following conditions are satisfied:

(1) The natural embedding functor $i : \mathrm{Im}(\delta) \to \mathcal{T}$ has a right adjoint $\rho : \mathcal{T} \to \mathrm{Im}(\delta)$ and $\delta \cong i \circ \rho$. (Hence, by Miyachi's Theorem, $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ is a stable t-structure on $\mathcal{T}$.)

(2) The t-structure $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ divides indecomposable injective R-modules, by which we mean that each indecomposable injective R-module belongs to either $\mathrm{Im}(\delta)$ or $\mathrm{Ker}(\delta)$.

Example 7. We denote by $E_R(R/\mathfrak{p})$ the injective hull of an R-module $R/\mathfrak{p}$ for a prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$.

Let W be a specialization closed subset of $\mathrm{Spec}(R)$. Since the section functor Γ_W is a left exact radical functor on R-Mod, we can define the right derived functor $\mathbf{R}\Gamma_W$ on $\mathcal{D}^+(R\text{-Mod})$. We claim that $\mathbf{R}\Gamma_W$ is an abstract local cohomology functor on $\mathcal{D}^+(R\text{-Mod})$.

In fact, it is known that $\mathcal{D}^+(R\text{-Mod})$ is triangle-equivalent to the triangulated category $\mathcal{K}^+(\mathrm{Inj}(R))$, which is the homotopy category consisting of all left-bounded injective complexes over R. Through this equivalence, for any injective complex $I \in \mathcal{K}^+(\mathrm{Inj}(R))$, $\mathbf{R}\Gamma_W(I) = \Gamma_W(I)$ is the subcomplex of I consisting of injective modules supported in W. Hence every object of $\mathrm{Im}(\mathbf{R}\Gamma_W)$ (resp. $\mathrm{Ker}(\mathbf{R}\Gamma_W)$) is an injective complex whose components are direct sums of $E_R(R/\mathfrak{p})$ with $\mathfrak{p} \in W$ (resp. $\mathfrak{p} \in \mathrm{Spec}(R)\backslash W$). In particular, if $\mathfrak{p} \in W$ (resp. $\mathfrak{p} \in \mathrm{Spec}(R)\backslash W$), then $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\mathbf{R}\Gamma_W)$ (resp. $E_R(R/\mathfrak{p}) \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$). Since $\mathrm{Hom}_R(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{q})) = 0$ for $\mathfrak{p} \in W$ and $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$, we can see that

$$\mathrm{Hom}_{\mathcal{K}^+(\mathrm{Inj}(R))}(I, J) = \mathrm{Hom}_{\mathcal{K}^+(\mathrm{Inj}(R))}(I, \Gamma_W(J))$$

for any $I \in \mathrm{Im}(\mathbf{R}\Gamma_W)$ and $J \in \mathcal{K}^+(\mathrm{Inj}(R))$. Hence it follows from the above equivalence that $\mathbf{R}\Gamma_W$ is a right adjoint of the natural embedding $i : \mathrm{Im}(\mathbf{R}\Gamma_W) \to \mathcal{D}^+(R\text{-Mod})$.

3. Main result

Let W be a specialization closed subset of $\mathrm{Spec}(R)$ and Γ_W be a section functor with support in W. We have pointed out in Example 7 that the right derived functor $\mathbf{R}\Gamma_W$ is an abstract local cohomology functor. In this section we shall prove that every abstract local cohomology functor is of this form. The main result of this paper is the following.

Theorem 8. (1) *The following conditions are equivalent for a left exact preradical functor γ on* R-Mod.

(a) *γ is a radical functor.*
(b) *γ preserves injectivity.*
(c) *γ is a section functor with support in a specialization closed subset of* $\mathrm{Spec}(R)$.
(d) *$\mathbf{R}\gamma$ is an abstract local cohomology functor.*

(2) *Given an abstract local cohomology functor δ on $\mathcal{D}^+(R\text{-Mod})$, there exists a specialization closed subset $W \subseteq \mathrm{Spec}(R)$ such that δ is isomorphic to the right derived functor $\mathbf{R}\Gamma_W$ of the section functor Γ_W.*

The equivalences among the conditions (a), (b) and (c) of the statment (1) in Theorem 8 already appear in several literatures, but they are not explicitly written. A new and significant feature of the statement (1) is that they are equivalent as well to the condition (d) and we have already seen that it holds the implication (c) $\Rightarrow$ (d) in Example 7.

Therefore, we shall prove that it holds the implication (d) $\Rightarrow$ (a) of the statement (1) and the statement (2) in Theorem 8. (For details, see the paper [6].)

To prove the statement (2), we introduce several lemmas.

Lemma 9. *Let $X \in \mathcal{D}^+(R\text{-Mod})$ and let W be a specialization closed subset of* $\mathrm{Spec}(R)$*.*

(1) $X \cong 0 \iff \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{p}, X)_\mathfrak{p} = 0$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)$.

(2) $X \in \mathrm{Im}(\mathbf{R}\Gamma_W) \iff \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{q}, X)_\mathfrak{q} = 0$ *for all* $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$.

(3) $X \in \mathrm{Ker}(\mathbf{R}\Gamma_W) \iff \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{p}, X)_\mathfrak{p} = 0$ *for all* $\mathfrak{p} \in W$.

Corollary 10. *Let $(R, \mathfrak{m}, k)$ be a noetherian local ring and let $X \not\cong 0 \in \mathcal{D}^+(R\text{-Mod})$. If $X \in \mathrm{Im}(\mathbf{R}\Gamma_\mathfrak{m})$, then $\mathbf{R}\mathrm{Hom}_R(E_R(k), X) \not\cong 0$.*

It follows from above results that we can show the following lemma.

Lemma 11. *Let $X \in \mathcal{D}^+(R\text{-Mod})$ and let W be a specialization closed subset of* $\mathrm{Spec}(R)$*.*

(1) *If $X \in \mathrm{Ker}(\mathbf{R}\Gamma_W)$ and $\mathbf{R}\mathrm{Hom}_R(X, E_R(R/\mathfrak{q})) = 0$ for all $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$, then $X \cong 0$.*

(2) *If $X \in \mathrm{Im}(\mathbf{R}\Gamma_W)$ and $\mathbf{R}\mathrm{Hom}_R(E_R(R/\mathfrak{p}), X) = 0$ for all $\mathfrak{p} \in W$, then $X \cong 0$.*

Now we can prove that it holds the implication (d) $\Rightarrow$ (a) of the statement (1) and the statement (2) in Theorem 8.

Proof. In this proof we denote $\mathcal{T} = \mathcal{D}^+(R\text{-Mod})$.

(1) (d) $\Rightarrow$ (a) Assume that $\mathbf{R}\gamma$ is an abstract local cohomology functor. We have to show that $\gamma(M/\gamma(M)) = 0$ for any R-module M. It is enough to show that $\gamma(E/\gamma(E)) = 0$ for any injective R-module E. In fact, for any R-module M, taking the injective hull $E(M)$ of M, we have $\gamma(M/\gamma(M)) \subseteq \gamma(E(M)/\gamma(E(M)))$.

We note that the natural inclusion $\gamma \subset \mathbf{1}$ of functors on R-Mod induces a natural morphism $\phi : \mathbf{R}\gamma \to \mathbf{1}$ of functors on $\mathcal{T}$. Since $(\mathrm{Im}(\mathbf{R}\gamma), \mathrm{Ker}(\mathbf{R}\gamma))$ is a stable t-structure on $\mathcal{T}$, it follows from Miyachi's Theorem and the proof of it that every injective R-module E is embedded in a triangle

$$\mathbf{R}\gamma(E) \xrightarrow{\phi(E)} E \longrightarrow V \longrightarrow \mathbf{R}\gamma(E)[1]$$

with $\mathbf{R}\gamma(E) \in \mathrm{Im}(\mathbf{R}\gamma)$ and $V \in \mathrm{Ker}(\mathbf{R}\gamma)$. Since E is an injective R-module and since $\mathbf{R}\gamma$ is the right derived functor of a left exact functor, $\mathbf{R}\gamma(E) = \gamma(E)$ is a submodule of E via the morphism $\phi(E)$. Therefore we have $V \cong E/\gamma(E)$ in $\mathcal{T}$. In particular, $H^0(\mathbf{R}\gamma(E/\gamma(E))) \cong H^0(\mathbf{R}\gamma(V)) = 0$. Since γ is left exact functor, it is concluded that $\gamma(E/\gamma(E)) = 0$ as desired.

(2) Suppose that $\delta : \mathcal{T} \to \mathcal{T}$ is an abstract local cohomology functor. We divides the proof into several steps.

(1st step) : Consider the subset $W = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta)\}$ of $\mathrm{Spec}(R)$. Then W is a specialization closed subset. To see this, we have only to show that $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta)$ implies $E_R(R/\mathfrak{q}) \in \mathrm{Im}(\delta)$ for prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$. Assume contrarily that there are prime ideals $\mathfrak{p} \subseteq \mathfrak{q}$ so that $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta)$ but $E_R(R/\mathfrak{q}) \notin \mathrm{Im}(\delta)$. Since the t-structure $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ divides indecomposable injective modules, we must have $E_R(R/\mathfrak{q}) \in \mathrm{Ker}(\delta)$. Then, from the definition of t-structures, we have

$\mathrm{Hom}_{\mathcal{T}}(E_R(R/\mathfrak{p}), E_R(R/\mathfrak{q})) = 0$, which says that there are no nontrivial R-module homomorphisms from $E_R(R/\mathfrak{p})$ to $E_R(R/\mathfrak{q})$. However, a natural nontrivial map $R/\mathfrak{p} \to R/\mathfrak{q} \hookrightarrow E_R(R/\mathfrak{q})$ extends to a non-zero map $E_R(R/\mathfrak{p}) \to E_R(R/\mathfrak{q})$. This is a contradiction, hence it is proved that W is specialization closed. ■

Our final goal is, of course, to show the isomorphism $\delta \cong \mathbf{R}\varGamma_W$. Notice that, since the both functors δ and $\mathbf{R}\varGamma_W$ are abstract local cohomology functors, we have two stable t-structures $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ and $(\mathrm{Im}(\mathbf{R}\varGamma_W), \mathrm{Ker}(\mathbf{R}\varGamma_W))$ on $\mathcal{T}$.

(2nd step) : Note that if $\mathfrak{p} \in W$, then $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta) \cap \mathrm{Im}(\mathbf{R}\varGamma_W)$. On the other hand, if $\mathfrak{q} \in \mathrm{Spec}(R)\backslash W$, then $E_R(R/\mathfrak{q}) \in \mathrm{Ker}(\delta) \cap \mathrm{Ker}(\mathbf{R}\varGamma_W)$. ■

(3rd step) : To prove the theorem, it is enough to show that $\mathrm{Im}(\delta) = \mathrm{Im}(\mathbf{R}\varGamma_W)$ by Miyachi's Theorem. (See also Remark 5.) ■

(4th step) : Now we prove the inclusion $\mathrm{Im}(\delta) \subseteq \mathrm{Im}(\mathbf{R}\varGamma_W)$.

To do this, assume $X \in \mathrm{Im}(\delta)$. Then there is a triangle in $\mathcal{T}$: $\mathbf{R}\varGamma_W(X) \to X \to V \to \mathbf{R}\varGamma_W(X)[1]$, where $V \in \mathrm{Ker}(\mathbf{R}\varGamma_W)$. Let $\mathfrak{q}$ be an arbitrary element of $\mathrm{Spec}(R)\backslash W$. Since $(\mathrm{Im}(\delta), \mathrm{Ker}(\delta))$ and $(\mathrm{Im}(\mathbf{R}\varGamma_W), \mathrm{Ker}(\mathbf{R}\varGamma_W))$ are stable t-structures and since $E_R(R/\mathfrak{q})$ belongs to $\mathrm{Ker}(\delta) \cap \mathrm{Ker}(\mathbf{R}\varGamma_W)$, it follows that

$$\mathrm{Hom}_{\mathcal{T}}(X, E_R(R/\mathfrak{q})[n]) = \mathrm{Hom}_{\mathcal{T}}(\mathbf{R}\varGamma_W(X), E_R(R/\mathfrak{q})[n]) = 0$$

for any integer n. Then by the above triangle we have

$$\mathrm{Hom}_{\mathcal{T}}(V, E_R(R/\mathfrak{q})[n]) = 0$$

for any integer n. This is equivalent to that $\mathbf{R}\mathrm{Hom}_R(V, E_R(R/\mathfrak{q})) \cong 0$. In fact, the n-th cohomology module of $\mathbf{R}\mathrm{Hom}_R(V, E_R(R/\mathfrak{q}))$ is just $\mathrm{Hom}_{\mathcal{T}}(V, E_R(R/\mathfrak{q})[n]) = 0$. Since $V \in \mathrm{Ker}(\mathbf{R}\varGamma_W)$, Lemma 11(1) forces $V \cong 0$, therefore $X \cong \mathbf{R}\varGamma_W(X)$. Hence we have $X \in \mathrm{Im}(\mathbf{R}\varGamma_W)$ as desired. ■

(5th step) : For the final step of the proof, we show the inclusion $\mathrm{Im}(\delta) \supseteq \mathrm{Im}(\mathbf{R}\varGamma_W)$.

Let $X \in \mathrm{Im}(\mathbf{R}\varGamma_W)$. Then there are triangles $\delta(X) \to X \to Y \to \delta(X)[1]$ with $Y \in \mathrm{Ker}(\delta)$, and $\mathbf{R}\varGamma_W(Y) \to Y \to V \to \mathbf{R}\varGamma_W(Y)[1]$ with $V \in \mathrm{Ker}(\mathbf{R}\varGamma_W)$. Let $\mathfrak{p}$ be an arbitrary prime ideal belonging to W. Similarly to the 4th step, since $E_R(R/\mathfrak{p}) \in \mathrm{Im}(\delta) \cap \mathrm{Im}(\mathbf{R}\varGamma_W)$, we see that $\mathrm{Hom}_{\mathcal{T}}(E_R(R/\mathfrak{p})[n], Y) = \mathrm{Hom}_{\mathcal{T}}(E_R(R/\mathfrak{p})[n], V) = 0$ for any integer n, hence we have $\mathrm{Hom}_{\mathcal{T}}(E_R(R/\mathfrak{p})[n], \mathbf{R}\varGamma_W(Y)) = 0$ for any n. This shows $\mathbf{R}\mathrm{Hom}_R(E_R(R/\mathfrak{p}), \mathbf{R}\varGamma_W(Y)) = 0$, then by Lemma 11(2) we have $\mathbf{R}\varGamma_W(Y) = 0$. Thus $Y \in \mathrm{Ker}(\mathbf{R}\varGamma_W)$. Then, since $(\mathrm{Im}(\mathbf{R}\varGamma_W), \mathrm{Ker}(\mathbf{R}\varGamma_W))$ is a stable t-structure, the morphism $X \to Y$ in the triangle $\delta(X) \to X \to Y \to \delta(X)[1]$ is zero. It then follows that $\delta(X) \cong X \oplus Y[-1]$. Since there is no nontrivial morphisms $\delta(X) \to Y[-1]$ in $\mathcal{T}$, it is concluded that $\delta(X) \cong X$, hence $X \in \mathrm{Im}(\delta)$ as desired, and the proof is completed. □

4. Lattice structure of the set of abstract local cohomology functors

We consider the following sets.

Definition 12. (1) We denote by $\mathbb{S}(R)$ the set of all left exact radical functors on R-Mod.
(2) We denote by $\mathbb{A}(R)$ the set of the isomorphism classes $[\delta]$ where δ ranges over all abstract local cohomology functors on $\mathcal{D}^+(R\text{-Mod})$.
(3) We denote by $\mathrm{sp}(R)$ the set of all specialization closed subsets of $\mathrm{Spec}(R)$.

All these sets are bijectively corresponding to one another. Actually we can define mappings among these sets. First of all, we are able to give a mapping

$$\mathbb{S}(R) \longrightarrow \mathrm{sp}(R) \quad : \quad \gamma \mapsto W_\gamma,$$

which has the inverse mapping

$$\mathrm{sp}(R) \longrightarrow \mathbb{S}(R) \quad : \quad W \mapsto \mathit{\Gamma}_W,$$

where $W_\gamma = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \gamma(R/\mathfrak{p}) = R/\mathfrak{p}\}$. We also have a mapping

$$\mathbb{S}(R) \longrightarrow \mathbb{A}(R) \quad : \quad \gamma \mapsto [\mathbf{R}\gamma],$$

which is surjective by Theorem 8. It is injective as well. In fact, since $\gamma(M) = H^0(\mathbf{R}\gamma(M))$ for $\gamma \in \mathbb{S}(R)$ and $M \in R$-Mod, γ is uniquely determined by $\mathbf{R}\gamma$.

Furthermore, we can see that these sets have complete lattice structure as follows. If $\{W_\lambda \mid \lambda \in \Lambda\}$ is a set of specialization closed subsets of Spec(R), then $\bigcap_\lambda W_\lambda$ and $\bigcup_\lambda W_\lambda$ are also specialization closed subset. By this reason sp(R) is a complete lattice.

By above correspondences, we can define $\bigcap$ and $\bigcup$ for any subsets of $\mathbb{S}(R)$. Actually, if $\{\gamma_\lambda \mid \lambda \in \Lambda\}$ is a set of elements in $\mathbb{S}(R)$, then $\gamma := \bigcap_\lambda \gamma_\lambda$ (resp. $\delta := \bigcup_\lambda \gamma_\lambda$) is well-defined as an element of $\mathbb{S}(R)$ so that $W_\gamma = \bigcap_\lambda W_{\gamma_\lambda}$ (resp. $W_\delta = \bigcup_\lambda W_{\gamma_\lambda}$). In this way we have shown that $\mathbb{S}(R)$ has a structure of complete lattice and the bijective mapping $\mathrm{sp}(R) \to \mathbb{S}(R)$ gives an isomorphism as lattices.

We can define a lattice structure as well on the set $\mathbb{A}(R)$ so that the bijection $\mathbb{A}(R) \cong \mathbb{S}(R)$ is an isomorphism as complete lattices. More precisely, we define the order on $\mathbb{A}(R)$ by

$$[\mathbf{R}\gamma_1] \subseteq [\mathbf{R}\gamma_2] \iff \gamma_1 \subseteq \gamma_2$$

for $\gamma_1, \gamma_2 \in \mathbb{S}(R)$. Notice that $\bigcap_\lambda [\mathbf{R}\gamma_\lambda] = [\mathbf{R}(\bigcap_\lambda \gamma_\lambda)]$, and $\bigcup_\lambda [\mathbf{R}\gamma_\lambda] = [\mathbf{R}(\bigcup_\lambda \gamma_\lambda)]$.

Summing all up we have the following result.

Theorem 13. *The mapping* $\mathbb{S}(R) \to \mathbb{A}(R)$ *which maps* γ *to* $[\mathbf{R}\gamma]$ *(resp.* $\mathrm{sp}(R) \to \mathbb{A}(R)$ *which sends* W *to* $[\mathbf{R}\mathit{\Gamma}_W]$*) gives an isomorphism of complete lattices.*

References

[1] A. A. Beilinson, J. Bernstein and P. Deligne, *Faisceaux Pervers*, Astérisque **100** (1982).
[2] O. Goldman, *Rings and modules of quotients*, J. Algebra **13** (1969), 10–47.
[3] J. M. Maranda, *Injective structures*, Trans. Amer. Math. Soc. **110** (1964), 98–135.
[4] J. Miyachi, *Localization of Triangulated Categories and Derived Categories*, J. Algebra **141** (1991), 463–483.
[5] B. Stenström, *Rings of Quotients*, Springer (1975).
[6] Y. Yoshino and T. Yoshizawa, *Abstract local cohomology functors*, To appear in Math. J. Okayama Univ.

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