WEAK GORENSTEIN DIMENSION FOR MODULES AND GORENSTEIN ALGEBRAS

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Abstract. We will generalize the notion of Gorenstein dimension and introduce that of weak Gorenstein dimension. Using this notion, we will characterize Gorenstein algebras.

1. Introduction

1.1. **Notation and definitions.** For a ring A we denote by $\mathrm{rad}(A)$ the Jacobson radical of A. Also, we denote by Mod-A the category of right A-modules, by mod-A the full subcategory of Mod-A consisting of finitely presented modules and by $\mathcal{P}_A$ the full subcategory of mod-A consisting of projective modules. For each $X \in$ Mod-A we denote by $E_A(X)$ its injective envelope. Left A-modules are considered as right A^{op}-modules, where A^{op} denotes the opposite ring of A. In particular, we denote by inj dim A (resp., inj dim A^{op}) the injective dimension of A as a right (resp., left) A-module and by $\mathrm{Hom}_A(-,-)$ (resp., $\mathrm{Hom}_{A^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-A (resp., Mod-A^{op}). Sometimes, we use the notation X_A (resp., ${}_AX$) to stress that the module considered is a right (resp., left) A-module.

In this note, complexes are cochain complexes and modules are considered as complexes concentrated in degree zero. For a complex $X^\bullet$ and an integer $n \in \mathbb{Z}$, we denote by $\mathrm{H}^n(X^\bullet)$ the nth cohomology. We denote by $\mathcal{K}(\text{Mod-}A)$ the homotopy category of complexes over Mod-A, by $\mathcal{K}^-(\mathcal{P}_A)$ (resp., $\mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$) the full triangulated subcategory of $\mathcal{K}(\text{Mod-}A)$ consisting of bounded above (resp., bounded) complexes over $\mathcal{P}_A$ and by $\mathcal{K}^{-,\mathrm{b}}(\mathcal{P}_A)$ the full triangulated subcategory of $\mathcal{K}^-(\mathcal{P}_A)$ consisting of complexes with bounded cohomology. We denote by $\mathcal{D}(\text{Mod-}A)$ the derived category of complexes over Mod-A. Also, we denote by $\mathrm{Hom}^\bullet_A(-,-)$ (resp., $-\otimes^\bullet -$) the associated single complex of the double hom (resp., tensor) complex and by $\mathbf{R}\mathrm{Hom}^\bullet_A(-,A)$ the right derived functor of $\mathrm{Hom}^\bullet_A(-,A)$. We refer to [4], [9] and [15] for basic results in the theory of derived categories.

Definition 1 ([5]). A module $X \in$ Mod-A is said to be coherent if it is finitely generated and every finitely generated submodule of it is finitely presented. A ring A is said to be left (resp., right) coherent if it is coherent as a left (resp., right) A-module.

Throughout the first three sections, A is a left and right coherent ring. Note that mod-A consists of the coherent modules and is a thick abelian subcategory of Mod-A in the sense of [9].

We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ the full triangulated subcategory of $\mathcal{D}(\text{Mod-}A)$ consisting of complexes over mod-A with bounded cohomology.

The detailed version of this paper will be submitted for publication elsewhere.

Definition 2 ([9])**.** A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ is said to have finite projective dimension if $\mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet[i], -)$ vanishes on mod-A for $i \ll 0$. We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the épaisse subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ consisting of complexes of finite projective dimension.

Note that the canonical functor $\mathcal{K}(\text{Mod-}A) \to \mathcal{D}(\text{Mod-}A)$ gives rise to equivalences of triangulated categories

$$\mathcal{K}^{-,\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A) \quad \text{and} \quad \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A) \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}.$$

We denote by $D(-)$ both $\mathbf{R}\mathrm{Hom}^\bullet_A(-, A)$ and $\mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(-, A)$. There exists a bifunctorial isomorphism

$$\theta_{M^\bullet, X^\bullet} : \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A^{\mathrm{op}})}(M^\bullet, DX^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}(\text{Mod-}A)}(X^\bullet, DM^\bullet)$$

for $X^\bullet \in \mathcal{D}(\text{Mod-}A)$ and $M^\bullet \in \mathcal{D}(\text{Mod-}A^{\mathrm{op}})$. For each $X^\bullet \in \mathcal{D}(\text{Mod-}A)$ we set

$$\eta_{X^\bullet} = \theta_{DX^\bullet, X^\bullet}(\mathrm{id}_{DX^\bullet}) : X^\bullet \to D^2X^\bullet = D(DX^\bullet).$$

Definition 3. A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ is said to have bounded dual cohomology if $DX^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}})$. We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ consisting of complexes with bounded dual cohomology.

Definition 4 ([2] and [12])**.** A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}})_{\mathrm{bdh}}$ is said to have finite Gorenstein dimension if $\eta_{X^\bullet}$ is an isomorphism. We denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ the full triangulated subcategory of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ consisting of complexes of finite Gorenstein dimension.

For a module $X \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$, we set

$$\text{G-dim } X = \sup\{\, i \geq 0 \mid \mathrm{Ext}^i_A(X, A) \neq 0\}$$

if $X \neq 0$, and G-dim $X = 0$ if $X = 0$. Also, we set G-dim $X = \infty$ for a module $X \in \text{mod-}A$ with $X \notin \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$. Then G-dim X is called the Gorenstein dimension of $X \in \text{mod-}A$. We denote by $\mathcal{G}_A$ the full additive subcategory of mod-A consisting of modules of Gorenstein dimension zero.

Remark 5. A module $X \in \text{mod-}A$ has Gorenstein dimension zero if and only if X is reflexive, i.e., the canonical homomorphism

$$X \to \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A), x \mapsto (f \mapsto f(x))$$

is an isomorphism and $\mathrm{Ext}^i_A(X, A) = \mathrm{Ext}^i_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A) = 0$ for $i \neq 0$.

Remark 6. The following hold.

(1) $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}} \subseteq \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}} \subseteq \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ and $\mathcal{P}_A \subseteq \mathcal{G}_A$.
(2) The pair of functors $\mathbf{R}\mathrm{Hom}^\bullet_A(-, A)$ and $\mathbf{R}\mathrm{Hom}^\bullet_{A^{\mathrm{op}}}(-, A)$ defines a duality between $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ and $\mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}})_{\mathrm{fGd}}$ and a duality between $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ and $\mathcal{D}^{\mathrm{b}}(\text{mod-}A^{\mathrm{op}})_{\mathrm{fpd}}$.
(3) The pair of functors $\mathrm{Hom}_A(-, A)$ and $\mathrm{Hom}_{A^{\mathrm{op}}}(-, A)$ defines a duality between $\mathcal{G}_A$ and $\mathcal{G}_{A^{\mathrm{op}}}$ and a duality between $\mathcal{P}_A$ and $\mathcal{P}_{A^{\mathrm{op}}}$.

1.2. **Introduction.** The notion of Gorenstein dimension has played an important role in the study of Gorenstein algebras (see e.g. [2], [10], [11] and so on). In this note, generalizing this, we will introduce the notion of weak Gorenstein dimension and characterize Gorenstein algebras in terms of weak Gorenstein dimension.

A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ with $\sup\{\ i \mid \mathrm{H}^i(X^\bullet) \neq 0\} = d < \infty$ is said to have finite weak Gorenstein dimension if $\mathrm{H}^i(\eta_{X^\bullet})$ is an isomorphism for all $i < d$ and $\mathrm{H}^d(\eta_{X^\bullet})$ is a monomorphism. Obviously, every $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ has finite weak Gorenstein dimension, the converse of which does not hold true in general (see Example 9 and Proposition 15). Extending the fact announced by Avramov [3], we will characterize complexes of finite weak Gorenstein dimension. Denote by $\mathcal{G}_A/\mathcal{P}_A$ the residue category of $\mathcal{G}_A$ over $\mathcal{P}_A$. Also, denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the quotient category of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ over the épaisse subcategory $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$. Avramov [3] announced that the embedding $\mathcal{G}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ gives rise to an equivalence

$$\mathcal{G}_A/\mathcal{P}_A \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}.$$

We will extend this fact. Denote by $\hat{\mathcal{G}}_A$ the full additive subcategory of mod-A consisting of modules $X \in \text{mod-}A$ with $\mathrm{Ext}^i_A(X, A) = 0$ for $i \neq 0$, by $\hat{\mathcal{G}}_A/\mathcal{P}_A$ the residue category of $\hat{\mathcal{G}}_A$ over $\mathcal{P}_A$ and by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the quotient category of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ over the épaisse subcategory $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$. We will show that the embedding $\hat{\mathcal{G}}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ gives rise to a full embedding

$$F : \hat{\mathcal{G}}_A/\mathcal{P}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$$

(see Proposition 8), that a complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ has finite weak Gorenstein dimension if and only if there exists a homomorphism $Z[m] \to X^\bullet$ in $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ inducing an isomorphism in $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ for some $Z \in \hat{\mathcal{G}}_A$ and $m \in \mathbb{Z}$ (see Lemma 12) and that F is an equivalence if and only if $\hat{\mathcal{G}}_A = \mathcal{G}_A$ (see Proposition 15).

Using the notion of weak Gorenstein dimension, we will characterize Gorenstein algebras. Let R be a commutative noetherian local ring and A a noetherian R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and A is finitely generated as an R-module. We say that A satisfies the condition (G) if the following equivalent conditions are satisfied: (1) Every simple $X \in \text{mod-}A$ has finite weak Gorenstein dimension; and (2) $A/\mathrm{rad}(A)$ has finite weak Gorenstein dimension (see Definition 18). Our main theorem states that the following are equivalent: (1) $\mathrm{inj\ dim}\, A = \mathrm{inj\ dim}\, A^{\mathrm{op}} < \infty$; and (2) $A_{\mathfrak{p}}$ satisfies the condition (G) for all $\mathfrak{p} \in \mathrm{Supp}_R(A)$ (see Theorem 19). Furthermore, in case A is a local ring, we will show that for any $d \geq 0$ the following are equivalent: (1) $\mathrm{inj\ dim}\, A = \mathrm{inj\ dim}\, A^{\mathrm{op}} = d$; (2) $\mathrm{inj\ dim}\, A = \mathrm{depth}\, A = d$; and (3) $A/\mathrm{rad}(A)$ has weak Gorenstein dimension d (see Theorem 20). Note that if $\mathrm{inj\ dim}\, A = \mathrm{depth}\, A < \infty$ then A is a Gorenstein R-algebra in the sense of Goto and Nishida [8].

This note is organized as follows. In Section 2, we will extend the fact announced by Avramov [3] quoted above. Also, we will include an example of A with $\hat{\mathcal{G}}_A \neq \mathcal{G}_A$ which is due to J.-I. Miyachi. In Section 3, we will introduce the notion of weak Gorenstein dimension and study finitely presented modules of finite weak Gorenstein dimension. In Section 4, we will study noetherian algebras of finite selfinjective dimension and prove the main theorem. In Section 5, we will characterize local noetherian algebras of finite selfinjective dimension. Also, we will provide several examples showing what rich properties local noetherian algebras of finite selfinjective dimension enjoy.

2. Full embedding

Let $\mathcal{G}_A/\mathcal{P}_A$ be the residue category of $\mathcal{G}_A$ over the full additive subcategory $\mathcal{P}_A$ and $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the quotient category of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ over the épaisse subcategory $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$. Then, as Avramov [3] announced, the embedding $\mathcal{G}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$ gives rise to an equivalence

$$\mathcal{G}_A/\mathcal{P}_A \xrightarrow{\sim} \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}.$$

In this section, we will extend this fact.

Definition 7. We denote by $\hat{\mathcal{G}}_A$ the full additive subcategory of mod-A consisting of modules $X \in \text{mod-}A$ with $\mathrm{Ext}^i_A(X, A) = 0$ for $i \neq 0$.

We denote by $\hat{\mathcal{G}}_A/\mathcal{P}_A$ the residue category of $\hat{\mathcal{G}}_A$ over the full additive subcategory $\mathcal{P}_A$ and by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the quotient category of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ over the épaisse subcategory $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$. Also, we denote by $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ the quotient category of $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ over the épaisse subcategory $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$.

Proposition 8. *The embedding $\hat{\mathcal{G}}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ gives rise to a full embedding*

$$F : \hat{\mathcal{G}}_A/\mathcal{P}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}.$$

In the next section, we will characterize a complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ which admits a homomorphism $Z[m] \to X^\bullet$ in $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ inducing an isomorphism $Z[m] \xrightarrow{\sim} X^\bullet$ in $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ for some $Z \in \hat{\mathcal{G}}_A$ and $m \in \mathbb{Z}$. Such a complex does not necessarily belong to $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}}$. Namely, $\hat{\mathcal{G}}_A \neq \mathcal{G}_A$ in general (see Proposition 15 below), which has been pointed out by J.-I. Miyachi in oral communication.

Example 9 (Miyachi)**.** Let k be a field and fix a nonzero element $c \in k$ which is not a root of unity. Let $S = k < x, y >$ be a non-commutative polynomial ring and $I = (x^2, y^2, cxy + yx)$ a two-sided ideal generated by x^2, y^2 and $cxy + yx$. Set $R = S/I$, $z_n = x + c^n y + I \in R$ for $n \in \mathbb{Z}$ and $w = xy + I \in R$. Then R is a selfinjective algebra and for each $n \in \mathbb{Z}$ there exist exact sequences $R \xrightarrow{z_{n+1}} R \xrightarrow{z_n} R$ in mod-R and $R \xrightarrow{z_n} R \xrightarrow{z_{n+1}} R$ in mod-R^{op}. Since c is not a root of unity, $z_nR \not\cong z_mR$ and $\mathrm{Hom}_R(z_nR, z_mR) \cong k$ unless $n = m$. Thus, since we have a projective resolution $\cdots \to R \xrightarrow{z_3} R \xrightarrow{z_2} R \xrightarrow{z_1} z_1R \to 0$ in mod-R, applying $\mathrm{Hom}_R(-, z_0R)$ we have $\mathrm{Ext}^i_R(z_1R, z_0R) = 0$ for all $i \geq 1$ (see [14]).

Now, we set

$$A = \begin{pmatrix} k & z_0R \\ 0 & R \end{pmatrix} \text{ and } e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in A.$$

Then a module $X \in \text{mod-}A$ is given by a triple $(X_1, X_2; \phi)$ of $X_1 \in \text{mod-}k$, $X_2 \in \text{mod-}R$ and $\phi \in \mathrm{Hom}_R(X_1 \otimes_k z_0R, X_2)$, and a module $M \in \text{mod-}A^{\mathrm{op}}$ is given by a triple $(M_1, M_2; \psi)$ of $M_1 \in \text{mod-}k$, $M_2 \in \text{mod-}R^{\mathrm{op}}$ and $\psi \in \mathrm{Hom}_k(z_0R \otimes_R M_2, M_1)$ (see [7]). Set $X = (0, z_1R; 0) \in \text{mod-}A$. Since we have a projective resolution $\cdots \xrightarrow{z_3} e_2A \xrightarrow{z_2}$

$e_2A \xrightarrow{z_1} X \to 0$ in mod-A, it follows that $\mathrm{Ext}^i_A(X, e_1A) \cong \mathrm{Ext}^i_R(z_1R, z_0R) = 0$ and $\mathrm{Ext}^i_A(X, e_2A) \cong \mathrm{Ext}^i_R(z_1R, R) = 0$ for $i > 0$. Thus $X \in \hat{\mathcal{G}}_A$. On the other hand, we have

$$\begin{aligned}\mathrm{Hom}_A(X, A) &\cong \mathrm{Ker}(Ae_2 \xrightarrow{z_2} Ae_2)\\ &\cong (wR, Rz_1; 0)\\ &\cong (wR, 0; 0) \oplus (0, Rz_1; 0)\end{aligned}$$

in mod-A^{op} and hence $\mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A)$ is decomposable, so that we have $X \ncong \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A)$ and $X \notin \mathcal{G}_A$.

3. Weak Gorenstein dimension

In this section, we will introduce the notion of weak Gorenstein dimension for finitely presented modules and study finitely presented modules of finite weak Gorenstein dimension.

Definition 10. A complex $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ with $\sup\{\, i \mid \mathrm{H}^i(X^\bullet) \neq 0\} = d < \infty$ is said to have finite weak Gorenstein dimension if $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$, $\mathrm{H}^i(\eta_{X^\bullet})$ is an isomorphism for $i < d$ and $\mathrm{H}^d(\eta_{X^\bullet})$ is a monomorphism.

For a module $X \in \text{mod-}A$ of finite weak Gorenstein dimension we set

$$\hat{\mathrm{G}}\text{-dim } X = \sup\{\, i \mid \mathrm{Ext}^i_A(X, A) \neq 0\}$$

if $X \neq 0$ and $\hat{\mathrm{G}}\text{-dim } X = 0$ if $X = 0$. Also, we set $\hat{\mathrm{G}}\text{-dim } X = \infty$ if $X \in \text{mod-}A$ does not have finite weak Gorenstein dimension. Then $\hat{\mathrm{G}}\text{-dim } X$ is called the weak Gorenstein dimension of $X \in \text{mod-}A$.

Remark 11. For any $X \in \text{mod-}A$ the following hold.

(1) $\hat{\mathrm{G}}\text{-dim } X = 0$ if and only if X is embedded in some $P \in \mathcal{P}_A$, i.e., the canonical homomorphism

$$X \to \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, A), A), x \mapsto (f \mapsto f(x))$$

is a monomorphism and $X \in \hat{\mathcal{G}}_A$.

(2) If $\mathrm{G}\text{-dim } X = d < \infty$ then $\hat{\mathrm{G}}\text{-dim } X = d$.

(3) If $\hat{\mathrm{G}}\text{-dim } X = d < \infty$ then $\hat{\mathrm{G}}\text{-dim } X' \leq d$ for all $X' \in \mathrm{add}(X)$, the full additive subcategory of mod-A consisting of direct summands of finite direct sums of copies of X.

Lemma 12. *A complex* $X^\bullet \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)$ *with* $\sup\{\, i \mid \mathrm{H}^i(X^\bullet) \neq 0\} = d < \infty$ *has finite weak Gorenstein dimension if and only if there exists a distinguished triangle in* $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)$

$$X^\bullet \to Y^\bullet \to Z[-d] \to$$

with $Y^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$, $Y^i = 0$ *for* $i > d$, *and* $Z \in \hat{\mathcal{G}}_A$.

Corollary 13 (cf. [6, Lemma 2.17])**.** *For any* $X \in \text{mod-}A$ *with* $\hat{\mathrm{G}}\text{-dim } X < \infty$ *there exists an exact sequence* $0 \to X \to Y \to Z \to 0$ *in* mod-A *with* $\hat{\mathrm{G}}\text{-dim } X = \text{proj dim } Y$ *and* $Z \in \hat{\mathcal{G}}_A$.

Lemma 14. *For any exact sequence* $0 \to X \to Y \to Z \to 0$ *in* mod-A *the following hold.*

(1) *If* $\hat{\mathrm{G}}$-dim $Z < \infty$, *then* $\hat{\mathrm{G}}$-dim $X < \infty$ *if and only if* $\hat{\mathrm{G}}$-dim $Y < \infty$.
(2) *If* $\hat{\mathrm{G}}$-dim $Y < \infty$, *then* $\hat{\mathrm{G}}$-dim $X < \infty$ *if and only if* $Z \in \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$ *and* $\mathrm{H}^i(D^2 Z) = 0$ *for* $i < -1$.
(3) *If* $\hat{\mathrm{G}}$-dim $X < \infty$ *and* $\mathrm{H}^0(\eta_X)$ *is an isomorphism, then* $\hat{\mathrm{G}}$-dim $Y < \infty$ *if and only if* $\hat{\mathrm{G}}$-dim $Z < \infty$.

Proposition 15. *The following are equivalent.*

(1) $\mathcal{G}_A = \hat{\mathcal{G}}_A$.
(2) $\hat{\mathrm{G}}$-dim $X = 0$ *for all* $X \in \hat{\mathcal{G}}_A$.
(3) $\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fGd}} = \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}$.
(4) *The embedding* $\hat{\mathcal{G}}_A/\mathcal{P}_A \to \mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{bdh}}/\mathcal{D}^{\mathrm{b}}(\text{mod-}A)_{\mathrm{fpd}}$ *is dense.*

4. Finiteness of selfinjective dimension

Throughout the rest of this note, A is a left and right noetherian ring.

In this section, using the notion of weak Gorenstein dimension, we will characterize noetherian algebras of finite selfinjective dimension.

Lemma 16. *For any injective* $I \in \text{Mod-}A$ *the following hold.*

(1) flat dim $I \leq$ inj dim A^{op} *and the equality holds if* I *is an injective cogenerator.*
(2) *Let* $d \geq 0$ *and assume that there exists a direct system* $(\{X_\lambda\}, \{f_\mu^\lambda\})$ *in* mod-A *over a directed set* Λ *such that* $\varinjlim X_\lambda \cong I$ *and* $\hat{\mathrm{G}}$-dim $X_\lambda \leq d$ *for all* $\lambda \in \Lambda$. *Then* flat dim $I \leq d$.

Corollary 17. *For any* $d \geq 0$ *the following are equivalent.*

(1) inj dim $A =$ inj dim $A^{\mathrm{op}} \leq d$.
(2) $\hat{\mathrm{G}}$-dim $X \leq d$ *for all* $X \in$ mod-A.

Throughout the rest of this note, R is a commutative noetherian local ring with the maximal ideal $\mathfrak{m}$ and A is a noetherian R-algebra, i.e., A is a ring endowed with a ring homomorphism $R \to A$ whose image is contained in the center of A and A is finitely generated as an R-module. It should be noted that $A/\mathfrak{m}A$ is a finite dimensional algebra over a field $R/\mathfrak{m}$.

We denote by $\mathrm{Spec}(R)$ the set of prime ideals of R. For each $\mathfrak{p} \in \mathrm{Spec}(R)$ we denote by $(-)_\mathfrak{p}$ the localization at $\mathfrak{p}$ and for each $X \in$ Mod-R we denote by $\mathrm{Supp}_R(X)$ the set of $\mathfrak{p} \in \mathrm{Spec}(R)$ with $X_\mathfrak{p} \neq 0$. Also, we denote by dim X the Krull dimension of $X \in$ mod-R. We refer to [13] for basic commutative ring theory.

Definition 18. We say that A satisfies the condition (G) if the following equivalent conditions are satisfied:

(1) $\hat{\mathrm{G}}$-dim $X < \infty$ for all simple $X \in$ mod-A.
(2) $\hat{\mathrm{G}}$-dim $A/\mathrm{rad}(A) < \infty$.

Theorem 19. *The following are equivalent.*

(1) inj dim $A =$ inj dim $A^{\mathrm{op}} < \infty$.
(2) $A_\mathfrak{p}$ *satisfies the condition (G) for all* $\mathfrak{p} \in \mathrm{Supp}_R(A)$.

5. Gorenstein algebras

In this section, we will deal with the case where inj dim A = inj dim A^{op} = depth A. In that case, A is a Gorenstein R-algebra in the sense of Goto and Nishida (see [8]). Also, we will characterize local Gorenstein algebras in terms of weak Gorenstein dimension.

We set $S = A/\mathrm{rad}(A)$ and denote by depth X the depth of $X \in$ mod-R. Throughout the rest of this note, we assume that A is a local ring, i.e., S is a division ring. Note that $S \in$ mod-A is a unique simple module up to isomorphism and that every $X \in$ mod-A admits a minimal projective resolution.

Theorem 20. *For any $d \geq 0$ the following are equivalent.*

(1) inj dim A = inj dim $A^{\mathrm{op}} = d$.
(2) inj dim A = depth $A = d$.
(3) $\hat{\mathrm{G}}$-dim $S_A = d$.

Corollary 21. *Assume that* inj dim A = inj dim $A^{\mathrm{op}} = d < \infty$. *Then A is Cohen-Macaulay as an R-module and $I^d \cong \mathrm{Hom}_R(A, E_R(R/\mathfrak{m}))$ for a minimal injective resolution $A \to I^\bullet$ in* Mod-A.

Example 22. Even if inj dim A = inj dim $A^{\mathrm{op}} < \infty$, it may happen that A is not Cohen-Macaulay as an R-module. For instance, let R be a Gorenstein local ring with dim $R \geq 1$ and set

$$A = \begin{pmatrix} R & R/xR \\ 0 & R/xR \end{pmatrix}$$

with $x \in \mathfrak{m}$ a regular element. Then A is not Cohen-Macaulay as an R-module but inj dim A = inj dim $A^{\mathrm{op}} < \infty$ (see [1, Example 4.7]).

Example 23. Even if A is a Cohen-Macaulay R-module and inj dim A = inj dim $A^{\mathrm{op}} < \infty$, it may happen that inj dim $A \neq$ depth A. For instance, let R be a Gorenstein local ring with dim $R = d$ and set

$$A = \begin{pmatrix} R & R \\ 0 & R \end{pmatrix}.$$

Then A is a Cohen-Macaulay R-module with depth $A = d$ but inj dim A = inj dim $A^{\mathrm{op}} = d + 1$.

Example 24. Even if A is a Cohen-Macaulay R-module and inj dim A = inj dim A^{op} = depth $A = d < \infty$, it may happen that $I^d \ncong \mathrm{Hom}_R(A, E_R(R/\mathfrak{m}))$ for a minimal injective resolution $A \to I^\bullet$ in Mod-A. For instance, let R be a Gorenstein local ring with dim $R = d$ and A a free R-module with a basis $\{e_{ij}\}_{1 \leq i,j \leq 3}$. Define a multiplication on A subject to the following axioms: (A1) $e_{ij}e_{kl} = 0$ unless $j = k$; (A2) $e_{ii}e_{ij} = e_{ij} = e_{ij}e_{jj}$ for all i, j; (A3) $e_{12}e_{21} = e_{11}$ and $e_{21}e_{12} = e_{22}$; and (A4) $e_{i3}e_{3j} = e_{3j}e_{i3} = 0$ for all $i, j \neq 3$. Set $e_i = e_{ii}$ for all i. Then A is an R-algebra with $1 = e_1 + e_2 + e_3$ and Cohen-Macaulay as an R-module. Also, setting $\Omega = \mathrm{Hom}_R(A, R)$, we have $e_1A \cong e_2A \cong e_3\Omega$ and $e_1\Omega \cong e_2\Omega \cong e_3A$. It follows that inj dim A = inj dim $A^{\mathrm{op}} = d$ but $I^d \cong \mathrm{Hom}_R(\Omega, E_R(R/\mathfrak{m})) \ncong \mathrm{Hom}_R(A, E_R(R/\mathfrak{m}))$ in Mod-A.

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