ON Ω-PERFECT MODULES AND SEQUENCES OF BETTI NUMBERS

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Abstract. Let R be a selfinjective algebra. In this paper we consider Ω-perfect modules and show how to use them to get information about the shapes of the Auslander-Reiten components containing modules of finite complexity. We also look at the growth of the sequence of Betti numbers for modules belonging to certain types of Auslander-Reiten components.

1. Introduction, background and motivation

The notion of complexity of a module has been around for more than thirty years. In depth studies have started in parallel at around the same time for group representations (see [1, 2, 7, 8, 21] for instance) and also in commutative algebra (see [4, 5, 16, 23] and [24]). In both cases the interest in complexity arose from the desire to understand the growth of minimal projective resolutions.

We will recall now the definition of complexity. For this definition we don't need to restrict ourselves to finite dimensional algebras, so R can be either a finite dimensional algebra over a field $\Bbbk$, or $R = (R, \mathbf{m}, \Bbbk)$ can be a local noetherian ring with maximal ideal $\mathbf{m}$ and residue field $\Bbbk$. Let M be a finitely generated R-module and let

$$P^\bullet : \quad \cdots \to P^2 \xrightarrow{\delta^2} P^1 \xrightarrow{\delta^1} P^0 \xrightarrow{\delta^0} M \to 0$$

be a minimal projective (free in the local case) resolution of M. The *i-th Betti number* of M, denoted $\beta_i(M)$, is the number of indecomposable summands of P^i. Then, the *complexity of M* is defined as

$$\operatorname{cx} M = \inf\{n \in \mathbb{N} | \beta_i(M) \leq ci^{n-1} \text{ for some positive } c \in \mathbb{Q} \text{ and all } i \geq 0\}$$

For instance $\operatorname{cx} M = 0$ is equivalent to M having finite projective dimension, and $\operatorname{cx} M = 1$ means that the Betti numbers of M are all bounded. If no such n exists, then we say that the complexity of M is infinite (at some point in time people also used to say that the complexity does not exist in this case). Let Ω denote the syzygy operator. Then it is clear from the definition that if M is a finitely generated R-module, then $\operatorname{cx} M = \operatorname{cx} \Omega M$, and

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an immediate application of the horseshoe lemma also shows that if $0 \to A \to B \to C \to 0$ is a short exact sequence of R-modules, then $\operatorname{cx} B \leq \max\{\operatorname{cx} A, \operatorname{cx} C\}$.

Note also that every Ω-periodic module (that is a module M with the property that $\Omega^k M \cong M$ for some positive integer k) has complexity 1. In fact, Eisenbud has proved that if $R = \Bbbk G$ is the group algebra of a finite group, or if R is a complete intersection, then every module of complexity 1 is Ω-periodic [10]. The converse need not hold in general, not even in the symmetric local case; we have the following example due to Liu and Schulz, [22]: Consider $R = \Bbbk\langle x, y\rangle/\langle x^2, y^2, xy + qyx\rangle$ where $0 \neq q \in \Bbbk$ is not a root of 1, and let T be the trivial extension of R by $\operatorname{Hom}_{\Bbbk}(R, \Bbbk)$. Then T is a local symmetric algebra. Let M be the T-module $(x+y)T$. For all $i \in \mathbb{Z}$ the modules $\Omega^i M$ have dimension 4, and are pairwise non-isomorphic. Since T is symmetric, $\tau M = \Omega^2 M$ holds, where τ is the Auslander-Reiten translation. Hence the module M has complexity 1 and is neither Ω- nor τ-symmetric. The module M therefore is contained in a $\mathbb{Z}A_\infty$ component. There are also counterexamples in the commutative case (see Gasharov and Peeva [15]).

Throughout this paper, R will denote a finite dimensional selfinjective algebra over an algebraically closed field $\Bbbk$ with Jacobson radical $\mathbf{r}$. Then, by an induction on the Loewy length, it follows readily from the definition and the above remarks, that for every finitely generated R-module M, we have $\operatorname{cx} M \leq \operatorname{cx} R/\mathbf{r}$. D will denote the usual duality$D = \operatorname{Hom}_{\Bbbk}(-, \Bbbk)$, and ν will denote the Nakayama equivalence

$$\nu = D\operatorname{Hom}_R(-, R)$$

Also, since R is selfinjective, then $\nu\Omega = \Omega\nu$. Moreover in this case, the Auslander-Reiten translate τ is given by $\tau = \nu\Omega^2$. Since ν is a dimension preserving equivalence that takes projective modules into projective modules, we have that $\operatorname{cx} M = \operatorname{cx} \nu M$, hence $\operatorname{cx} M = \operatorname{cx} \tau M$ for every finitely generated R-module M.

The paper is organized as follows. In the second section we talk about the shape of the Auslander-Reiten components containing modules of finite complexity as obtained in [20] and about the methods used in approaching this problem. In particular, we talk about a very special class of modules called Ω-perfect. In section three, we study the existence of Ω-perfect modules. Finally, in the last section we look at some special cases where we analyze the growth of the Betti numbers.

2. Auslander-Reiten components containing modules of finite complexity

We start this section with the following easy observation:

Lemma 1. *Let R be a selfinjective algebra and let $\mathcal{C}_s$ be a stable component of its Auslander-Reiten quiver. The complexity is constant on $\mathcal{C}_s$.*

Proof. Let $B \to C \in \mathcal{C}_s$ be an irreducible morphism. Then there exists an Auslander-Reiten sequence $0 \to \tau C \to B \oplus E \to C \to 0$ for some module E. Hence we have $\operatorname{cx} B \leq \operatorname{cx}(B \oplus C) \leq \max\{\operatorname{cx} C, \operatorname{cx} \tau C\} = \operatorname{cx} C$. Since there is an irreducible morphism from τC to B we use the same reasoning to get the reverse inequality. There is also an "extreme" case to prove: the case when the only irreducible morphisms to modules $C \in \mathcal{C}_s$ are from projective modules. But it is not hard to prove that this corresponds to the case when R is a Nakayama algebra of Loewy length two. In that case, the only non projective modules are the simple modules and they are all periodic, hence their complexity is 1. □

In order to describe the shapes of the stable Auslander-Reiten components containing modules of finite complexity we recall first the notion of Ω-perfect modules introduced in [17, 18]. We observe first that if $g\colon B \to C$ is an irreducible epimorphism between two nonprojective modules, then we have an induced irreducible map $\Omega g\colon \Omega B \to \Omega C$, see [3] for instance These modules have a particularly nice behaviour under the syzygy operator. However, there is no reason why Ωg should be again an epimorphism. Being irreducible, we know though that it must be either an epimorphism or a monomorphism. And one could ask the same question about an irreducible monomorphism f: when can we guarantee that its syzygy Ωf is gain an irreducible monomorphism? We have the following definition:

Definition. *An irreducible map $g\colon B \to C$ is called Ω-perfect if for all $n \geq 0$ the induced maps $\Omega^n g\colon \Omega^n B \to \Omega^n C$ are all monomorphisms or are all epimorphisms. An irreducible map g is eventually Ω-perfect if, for some $i > 0$, the induced map $\Omega^i g\colon \Omega^i B \to \Omega^i C$ is Ω-perfect. An indecomposable non projective R-module C is called Ω-perfect, if each irreducible map into C is Ω-perfect. We say that C is it eventually Ω-perfect if some syzygy of C is an Ω-perfect module.*

It was proved in [17] that if $g\colon B \to C$ is an irreducible epimorphism, then Ωg is again an epimorphism if and only if its kernel is not a simple module. Thus, an irreducible map $g\colon B \to C$ is eventually Ω-perfect, if and only if there exists a positive integer n such that for each $i \geq n$, the induced map $\Omega^i g\colon \Omega^i B \to \Omega^i C$ has a non simple kernel. We have the following consequence, see [18]:

Proposition 2. *Let R be a selfinjective algebra having no periodic simple modules. Then every nonprojective R-module is eventually Ω-perfect.* □

We can specialize to the local finite dimensional case to obtain the following:

Corollary 3. *Let $R = (R, \mathbf{m}, \Bbbk)$ be a local selfinjective algebra, and assume that there are modules of complexity two or higher. Then every indecomposable non projective R-module is eventually Ω-perfect.* □

One very nice feature of Ω-perfect maps is that they behave very nice under the syzygy operator. We have the following (see [17]):

Proposition 4. *Let R be a selfinjective algebra, and let $0 \to A \to B \xrightarrow{g} C \to 0$ be a short exact sequence of R-modules where g is an irreducible Ω-perfect map. Then, for each $i \geq 0$ we have induced exact sequences $0 \to \Omega^i A \to \Omega^i B \xrightarrow{\Omega^i g} \Omega^i C \longrightarrow 0$, and thus $\beta_i(B) = \beta_i(A) + \beta_i(C)$ for each $i \geq 0$.* □

It turns out that every indecomposable not τ-periodic module of complexity one is eventually Ω-perfect ([17]). The proof of this result is somehow involved and it would be interesting to have a more direct and possibly elementary proof. Note also that a recent result of Dugas ([9]), proves that if a *simple* module over a selfinjective algebra has complexity 1, then it must be periodic. As mentioned above in the introduction, this need not hold for all modules with bounded Betti numbers so the assumption that the module is simple, is essential.

We would also like to mention the following facts. Let C be an Ω-perfect module. Then it is easy to show that τC is also Ω-perfect. Let now B be an indecomposable module, and assume that there is an irreducible monomorphism $B \to C$. Then it was shown in [17] that B must also be Ω-perfect. We would like to know the answer to the following question:

Question 5. Let B and C be two indecomposable R-modules, let $B \to C$ be an irreducible epimorphism and assume C is Ω-perfect. Is B also Ω-perfect?

We will first look at Auslander-Reiten components containing modules that are not eventually Ω-perfect since this is the much easier case. We will show that these components must have a very predictable shape. First, we recall the following definition and theorem, see [19].

Definition. Let R be an artin algebra and let $\mathcal{C}_s$ be a stable component of its Auslander-Reiten quiver. A function $d\colon \mathcal{C}_s \to \mathbb{Q}$ is *additive* if it satisfies the following properties:
(a) $d(C) > 0$ for each $C \in \mathcal{C}_s$.
(b) $2d(C) = \sum_i d(E_i)$ for each indecomposable non projective module C, where the sequence $0 \to \tau C \to \bigoplus_i E_i \bigoplus P \to C \to 0$ is an Auslander-Reiten sequence and P is a (possibly 0) projective R-module.
(c) $d(C) = d(\tau C)$ for each $C \in \mathcal{C}_s$.

The following theorem was proved by Happel-Preiser-Ringel in [19]:

Theorem 6. *Let R be an artin algebra over an algebraically closed field and let $\mathcal{C}_s$ be a stable component of its Auslander-Reiten quiver. Assume that there exists an additive function on $\mathcal{C}_s$. Then the tree class of $\mathcal{C}_s$ is either an extended Dynkin diagram of type $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, or an infinite Dynkin tree of type A_∞, D_∞ or A_∞^∞.*

Assume that a non-periodic stable component $\mathcal{C}_s$ contains a module C that is not eventually Ω-perfect. This means that some syzygy of C is a simple periodic module. Let us denote that module by S, and let n be the Ω-period of S. It is clear that S is also ν-periodic since the Nakayama functor preserves lengths, so let m denote the ν-period of S. Let $T = S \oplus \Omega S \oplus \ldots \oplus \Omega^{n-1} S$, and let $W = T \oplus \nu T \oplus \ldots \oplus \nu^{m-1} T$. It is now immediate that $\tau W = W$. Also, it is not hard to show that the function $d\colon \mathcal{C}_s \to \mathbb{Q}$ given by $d(M) = \dim \underline{\mathrm{Hom}}_R(W, M)$ is an additive function, see [13, 20]. Using the Happel-Preiser-Ringel theorem and the above observations we have the following surprising application (see [20]):

Theorem 7. *Let R be a selfinjective algebra and let $\mathcal{C}_s$ be a stable component of the Auslander-Reiten quiver of R containing a module that is not eventually Ω-perfect. Assume in addition that the component is not τ-periodic. Then $\mathcal{C}_s$ is of the form $\mathbb{Z}\Delta$ where Δ is of type $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, or an infinite Dynkin tree of type D_∞ or A_∞^∞.* □

We should make a few remarks here. First, note the excluded case when the component is τ-periodic is also well understood. They are either infinite tubes or they are periodic components whose tree class is a Dynkin diagram (see [19, 26]). Note also that the theorem says that components of type $\mathbb{Z}A_\infty$ cannot occur. In fact, we shall see in the next section that we cannot have components of type $\mathbb{Z}\Delta$ for $\Delta = \widetilde{A}_1, \widetilde{E}_6, \widetilde{E}_7$, or $\widetilde{E}_8$ either. At this point we would like to state a second question that has actually been around in the area for some time.

Question 8. Let R be a selfinjective algebra and assume that its Auslander-Reiten quiver contains a component of type $\mathbb{Z}\Delta$ where $\Delta = \widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8, D_\infty$ or A_∞^∞. Does this imply that R is a tame algebra?

The answer to the above question is affirmative in the group algebra case, see [12]. Therefore it seems that given a selfinjective algebra, almost all the indecomposable modules are eventually Ω-perfect. We will discuss more about this phenomenon in the next section.

Considering Theorem 2.7, it turns out that a similar result holds for components containing modules of finite complexity. The following result was proved in [20]. It is a generalization of Webb's theorem who had proved it first for group algebras [28].

Theorem 9. *Let R be a selfinjective algebra and let $\mathcal{C}_s$ be a stable component of the Auslander-Reiten quiver of R containing a module of finite complexity. Assume in addition that the component is not τ-periodic. Then $\mathcal{C}_s$ is of the form $\mathbb{Z}\Delta$ where Δ is of type $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, or an infinite Dynkin tree of type A_∞, D_∞ or A_∞^∞.* □

3. Ω-perfect modules

In this section we continue the study of Ω-perfect modules over a selfinjective algebra and show that every component of type $\mathbb{Z}\widetilde{E}_i$ for $i = 6, 7, 8$ or $\mathbb{Z}\widetilde{A}_1$ consists of eventually Ω-perfect modules. We also give an example of a component containing only modules that are not Ω-perfect, and discuss possible values for complexities. We also pose some new questions. We will need the notion of τ-perfect irreducible map. It is obviously very similar to the one of Ω-perfect map: we say that an irreducible map $g\colon B \to C$ is called *τ-perfect* if for all $n \geq 0$ the induced maps $\tau^n g\colon \tau^n B \to \tau^n C$ are all monomorphisms or are all epimorphisms.

If $\mathcal{C}$ is a component of the Auslander-Reiten quiver of R, we will denote by $\mathcal{C}_s$ its stable part, and by $\Omega\mathcal{C}$ the component containing all the modules of the form ΩX for $X \in \mathcal{C}$ non projective. We have the following:

Proposition 10. *Let R be a selfinjective artin algebra and let $\mathcal{C}$ be an Auslander-Reiten component. If the module $X \in \mathcal{C}$ does not have any projective or simple predecessors in $\mathcal{C}$, then ΩX does not have either any simple or projective predecessors in $\Omega\mathcal{C}$.*

Proof. Assume that ΩX has a simple predecessor S in the component $\Omega\mathcal{C}$. By applying the inverse syzygy operator we obtain in $\mathcal{C}$ a chain of irreducible maps $\Omega^{-1}S \to \cdots \to X$. Denote by P the indecomposable projective-injective with socle S. We have an Auslander-Reiten sequence $0 \to \mathbf{r}P \to P \oplus \mathbf{r}P/S \to P/S \to 0$, and since $P/S \cong \Omega^{-1}S$, we see that P is a predecessor of X in $\mathcal{C}$. Assume now that ΩX has a projective predecessor in its component, so there exists a chain of irreducible maps $P \to P/S \to \cdots \to \Omega X$ where S is the socle of P. As before, we have that $P/S \cong \Omega^{-1}S$, so there is a chain of irreducible maps in $\Omega^2\mathcal{C}$ from S to $\Omega^2 X$. Applying the Nakayama functor, we obtain that τX, and hence X have a simple predecessor since the Nakayama functor preserves lengths. □

One can prove in a similar fashion that for a selfinjective algebra, a component $\mathcal{C}$ contains a simple (projective) module, if and only if the component $\Omega\mathcal{C}$ contains a projective (respectively simple) module.

Remark 11. Let $\mathcal{C}$ be an Auslander-Reiten component having a boundary, that is, a component containing indecomposable modules whose Auslander-Reiten sequences have indecomposable middle terms. Assume that $\mathcal{C}$ is not a tube, and let C be an indecomposable module lying on the boundary of $\mathcal{C}$. Without loss of generality we may assume that neither C nor ΩC has a simple module in the positive direction of their τ-orbit. This means that if $0 \to \tau C \to B \to C \to 0$ is the Auslander-Reiten sequence ending at C, then both maps $\tau C \to B$ and $B \to C$ are Ω-perfect and so C is an Ω-perfect module. So we see that nonperiodic components with boundaries, always contain Ω-perfect maps and Ω-perfect modules. As we will see soon, this need not happen in components of the type $\mathbb{Z}A_\infty^\infty$.

Lemma 12. *Let $g\colon B \to C$ be an irreducible map that is not eventually Ω-perfect, where neither B nor C has a nonzero projective summand. Then, there exists a positive integer α such that for each $i \geq 0$ we have $|\ell(\Omega^i B) - \ell(\Omega^i C)| \leq \alpha$.*

Proof. By taking enough powers of the Auslander-Reiten translate τ, we may assume without loss of generality that g is onto, and that its kernel S is a simple periodic module. Note that by applying Ω we obtain an induced exact sequence $0 \to \Omega B \xrightarrow{\Omega g} \Omega C \to S \to 0$. If the induced map $\Omega^2 g$ is again a monomorphism, then we get the commutative exact diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^2 B & \xrightarrow{\Omega^2 g} & \Omega^2 C & \longrightarrow & L & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_{\Omega B} & \longrightarrow & P_{\Omega C} & \longrightarrow & Q & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega B & \xrightarrow{\Omega g} & \Omega C & \longrightarrow & S & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

hence the two modules L and ΩS are isomorphic, and we have a short exact sequence $0 \to \Omega^2 B \to \Omega^2 C \to \Omega S \to 0$. If on the other hand, the map $\Omega^2 g$ is an epimorphism, then we obtain a commutative diagram

$$\begin{array}{ccccccccccc}
 & & & & 0 & & 0 & & & & \\
 & & & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & \Omega^2 B & \xrightarrow{\Omega^2 g} & \Omega^2 C & \longrightarrow & 0 & & \\
 & & \| & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & P_{\Omega B} & \longrightarrow & P_{\Omega C} & \longrightarrow & S & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \| & & \\
 & & 0 & \longrightarrow & \Omega B & \xrightarrow{\Omega g} & \Omega C & \longrightarrow & S & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & & & \\
 & & & & 0 & & 0 & & & &
\end{array}$$

and therefore we obtain a short exact sequence $0 \to \Omega^2 S \to \Omega^2 B \to \Omega^2 C \to 0$. Continuing in this fashion we see that for each integer $i \geq 0$ we get either short exact sequences $0 \to \Omega^i S \to \Omega^i B \to \Omega^i C \to 0$, or of the form $0 \to \Omega^{i+1} B \to \Omega^{i+1} C \to \Omega^i S \to 0$. By letting $\alpha = \max_{i \in \mathbb{N}} \{\ell(\Omega^i S)\}$, our result follows, since the simple module S is periodic. □

The following (most probably) well-known lemma will be used to characterize Ω-perfect maps in terms of the "τ-perfect" property. As usual, if M is an indecomposable non-projective module, $\alpha(M)$ denotes the number of non-projective indecomposable direct summands of the middle term of the Auslander-Reiten sequence ending in M.

Lemma 13. *Let Λ be a selfinjective artin algebra and let M be an indecomposable non projective and non simple Λ-module with $\alpha(M) = 2$ with $n = \ell(M) = \ell(\tau M)$. Assume also that there exists an irreducible map $E \to M$ where E is indecomposable and that $\ell(E) = \ell(M) - 1$.*

(a) *The middle term of the Auslander-Reiten sequence ending at M has no nonzero projective summand.*

(b) *If E, M and τM are uniserial, then the remaining summand F of the Auslander-Reiten sequence ending at M is uniserial too and its length is $\ell(F) = \ell(M) + 1$.*

Proof. Let $0 \to \tau M \to E \oplus F \oplus P \to M \to 0$ be the Auslander-Reiten sequence ending at M, where F is indecomposable non projective, and P is a nonzero projective module. Note first that P must be indecomposable since the algebra is selfinjective. Using now the fact that $\tau M = \mathbf{r}P$, a length argument shows that

$$\ell(F) = 2n - \ell(E) - \ell(P) = 2n - (n-1) - (n+1) = 0$$

contradicting our assumption. This proves the first part of the lemma.
For part (b), note first that the Auslander-Reiten sequence ending at M has the form $0 \to \tau M \to E \oplus F \to M \to 0$ where E and F are both indecomposable and $\ell(F) = n+1$. Hence τM is a maximal submodule of F. To prove the uniseriality of F, it suffices to show that $\tau M = \mathbf{r}F$. It is folklore (see also [17], Proposition 2.5.) that, since τM is not simple, we have an induced exact sequence

$$0 \to \mathbf{r}\tau M \to \mathbf{r}E \oplus \mathbf{r}F \to \mathbf{r}M \to 0.$$

Counting lengths, we get $\ell(\mathbf{r}F) = 2(n-1) - (n-2) = n$. Since the image of τM in F contains the radical of F, it follows that $\tau M \cong \mathbf{r}F$, and F is also an uniserial module. □

We are now ready to prove the promised characterization of Ω-perfect maps.

Proposition 14. *Let R be a selfinjective algebra of infinite representation type, and let C be an indecomposable module and let $g\colon B \to C$ be an irreducible map. Then g is eventually Ω-perfect if and only if both g and Ωg are eventually τ-perfect.*

Proof. Obviously, if g is eventually Ω-perfect then both maps g and Ωg are eventually τ-perfect. For the reverse direction, assume that both maps g and Ωg are eventually τ-perfect, but that g is not eventually Ω-perfect. By applying enough powers of Ω, we may assume that $g, \Omega g$ are both τ-perfect, and that for each $i \geq 0$, the maps $\tau^i g$ are onto and $\Omega^i \tau g$ are one-to-one. Thus, for each $i \geq 0$, there exist simple modules S_i and exact sequences $0 \to S_i \to \Omega^{2i} B \xrightarrow{\Omega^{2i} g} \Omega^{2i} C \to 0$ and $0 \to \Omega^{2i+1} B \xrightarrow{\Omega^{2i+1} g} \Omega^{2i+1} C \to S_i \to 0$. But $\Omega^{2i+2} g$ is again surjective so we infer from the proof of lemma 13 that for each $i \geq 0$, $\Omega^2 S_i \cong S_{i+1}$. Since there are only finitely many nonisomorphic simple modules, the sequence $\{S_1, \nu S_2 = \tau S_1, \nu^2 S_3 = \tau^2 S_1, \cdots\}$ is eventually periodic. Therefore without loss of generality we may assume that there is a periodic simple module S, say of period n, whose τ-powers are all simple.

We claim first that the simple modules $S, \tau S, \cdots, \tau^{n-1} S$ lie on the boundary of a regular tube $\mathcal{C}$. To see this, observe first that we can deduce that the middle term E of the Auslander-Reiten sequence $0 \to \tau S \to E \to S \to 0$ is indecomposable. Moreover, E cannot be projective, since otherwise the middle term of each Auslander-Reiten sequence ending at a $\tau^i S$ would be an indecomposable projective-injective module of length two. This would imply that our algebra is selfinjective Nakayama of Loewy length two, contradicting our assumption on the representation type of R. By construction, all the modules in the same τ-orbit of $\mathcal{C}$ have the same length, and these lengths increase by one from a τ-orbit to the next one. We may apply now the previous lemma and infer that the component is a regular component. By the second part of the lemma we get that all the modules in $\mathcal{C}$ are uniserial, a contradiction since we cannot have uniserial module of arbitrary large length. □

Let Δ be a quiver. A vertex x of Δ is called a *tip*, if only one arrow of Δ starts or ends at x. If $\mathcal{C}$ is a component whose stable part is of type $\mathbb{Z}\Delta$, then a module M corresponds to a tip of Δ if and only if the Auslander-Reiten sequence ending at M is of the form:

$$0 \to \tau M \to Y \oplus P \to M \to 0$$

for some projective (possibly zero) module P and indecomposable non projective module Y. Assume that $\mathcal{C}$ is a connected component of the Auslander-Reiten quiver, and that we have $\mathcal{C}_s \cong \mathbb{Z}\Delta$ for some quiver Δ. Since an Auslander-Reiten component contains at most finitely many indecomposable projective or simple modules, for each indecomposable module $M \in \mathcal{C}_s$ there exists a positive integer r such that $\tau^r M$ has no projective or simple predecessors in $\mathcal{C}$. We have the following immediate consequence:

Corollary 15. *Let $\mathcal{C}_s$ be a stable component of type $\mathbb{Z}\Delta$ and let M be an indecomposable module in $\mathcal{C}_s$. Assume that M corresponds to a tip of Δ. Then M is eventually Ω-perfect.* □

We have the following:

Proposition 16. *Let $\mathcal{C}_s$ be a stable component of type $Z\Delta$ where Δ is one of $\tilde{E}_6, \tilde{E}_7, \tilde{E}_8$, $\tilde{A}_1$, A_∞. Then every module in $\mathcal{C}_s$ is eventually Ω-perfect.*

Proof. The case where $\Delta = A_\infty$ was treated in [20], (Theorem 2.11. and Lemma 2.6.). Consider now the only case when the connected component $\mathbb{Z}\Delta$ has no tip, that is the case when $\Delta = \tilde{A}_1$, that is, the Kronecker quiver. Let $M \in \mathcal{C}_s$ with no projective or simple predecessors. The Auslander-Reiten sequence ending at M is of the form

$$0 \to \tau M \stackrel{[f_1,f_2]^t}{\longrightarrow} E \oplus E \stackrel{[g_1,g_2]}{\longrightarrow} M \to 0$$

and it is obvious that all of the irreducible maps f_1, f_2, g_1, g_2 are epimorphisms, or all are monomorphisms. We claim that they are all epimorphisms. If they are monomorphisms, then in the Auslander-Reiten sequence

$$0 \to \tau E \stackrel{[\tau g_1,\tau g_2]^t}{\longrightarrow} \tau M \oplus \tau M \stackrel{[f_1,f_2]}{\longrightarrow} E \to 0$$

the maps $\tau g_1, \tau g_2$ are also monomorphisms. Continuing in the positive τ direction we obtain an arbitrary long chain or irreducible monomorphisms

$$\cdots \tau^i E \hookrightarrow \tau^i M \hookrightarrow \tau^{i-1} E \hookrightarrow \cdots \hookrightarrow E \hookrightarrow M$$

which is absurd. We can make the same argument for ΩM, and it follows that M is Ω-perfect.
Assume now that Δ is one of the remaining finite quiver $\tilde{E}_i$, take M in $\mathcal{C}_s$ with $\alpha(M) = 3$, such that M has no projective or simple predecessor in $\mathcal{C}$ and let $\mathcal{C}_M$ be the full subquiver of $\mathcal{C}$, defined by the vertices which are predecessors of M. If $X \in \mathcal{C}_M$ with $\alpha(X) = 1$ (hence X corresponds to one of the 3 tips of Δ, then X is Ω-perfect by Remark 3.2, the irreducible map $Y \to \tau X$ is an Ω-perfect epimorphism, and $\tau X \to Y$ is injective and Ω-perfect. Consequently all irreducible maps between indecomposable modules in $\mathcal{C}_M$ are Ω-perfect, hence all indecomposable modules $N \in \mathcal{C}_M$ with $\alpha(N) \leq 2$ are Ω-perfect. Take finally $V \in \mathcal{C}_M$ with $\alpha(V) = 3$ and let

$$0 \to \tau V \stackrel{[f_1,f_2,f_3]^t}{\longrightarrow} \oplus_{i=1}^3 X_i \stackrel{[g_1,g_2,g_3]}{\longrightarrow} V \to 0$$

the Auslander-Reiten sequence ending in V. The irreducible maps f_i all are surjective, while the g_i are injective. Choose $j \leq 3$, let $\oplus_i X_i = Y \oplus X_j$ and let $[g, g_j] : Y \oplus X_j \to V$ be the sink map. Since f_j is an Ω-perfect epimorphism, the same holds for the "parallel" morphism g, hence V is Ω-perfect, too. □

As we will see very soon, it turns out that if $\mathcal{C}$ is a component in which no irreducible map is Ω-perfect, then every non projective module in $\mathcal{C}$ has complexity at most 2. In fact, we have a slightly more general result. We start with the following:

Proposition 17. *Let C be an indecomposable non projective, and non τ-periodic module and assume that there exist irreducible morphisms $B \to C$ and $\tau C \to B$ that are not eventually Ω-perfect. Then, there exists a positive integer α such that for each $n \geq 0$, $\ell(\Omega^{2n} C) \leq \ell(C) + n\alpha$. In particular, C and every nonprojective module in the same Auslander-Reiten component has complexity 2.*

Proof. Observe first that for each indecomposable non projective R-module M, we have $\ell(\Omega^2 M) = \ell(\tau M)$. Now, applying the previous Lemma 3.3, we obtain for each $i \geq 0$ that $\ell(\Omega^i B) \leq \ell(\Omega^i C) + \alpha/2$, and $\ell(\Omega^{i+2} C) \leq \ell(\Omega^i B) + \alpha/2$ for some positive number α. Hence, for each $i \geq 0$ we have $\ell(\Omega^{i+2} C) - \ell(\Omega^i C) \leq \alpha$. In particular, for each $n \geq 0$ we have $\ell(\Omega^{2n} C) - \ell(C) \leq n\alpha$. This means that the complexity of C is bounded by 2. If $\operatorname{cx} C = 1$, then, since it is not τ periodic, the module C must lie in a $\mathbb{Z}A_\infty$-component by [17], but for these components every irreducible map is eventually Ω-perfect. Hence $\operatorname{cx} C = 2$. □

We obtain the following immediate consequence: assume that we have a component $\mathcal{C}$, whose stable part $\mathcal{C}_s$ is of the form $\mathbb{Z}A_\infty^\infty$, and assume also that there exists an Auslander-Reiten sequence $0 \to \tau C \to E \oplus F \to C \to 0$ where E and F are indecomposable, and neither $E \to C$ nor $F \to C$ is eventually Ω-perfect. Observe also that in this case, no irreducible map in $\mathcal{C}_s$ between indecomposable modules is eventually Ω-perfect. It follows immediately from the previous proposition that every non projective module in $\mathcal{C}$ has complexity 2. This situation can actually occur. The following example is due to Ringel.

Example 18. Let R be the finite dimensional selfinjective string algebra given by the quiver

$$1 \underset{\gamma}{\overset{\alpha}{\rightleftarrows}} 2 \underset{\delta}{\overset{\beta}{\rightleftarrows}} 3$$

modulo the relations $\alpha\beta = 0$, $\delta\gamma = 0$, $\gamma\alpha\gamma\alpha = \beta\delta\beta\delta$ and $\alpha\gamma\alpha\gamma\alpha = \delta\beta\delta\beta\delta = 0$. There exists a $\mathbb{Z}A_\infty^\infty$ component where none of the irreducible maps between the indecomposable modules is eventually Ω-perfect, (or even τ-perfect). For instance, consider the string module $M = \mathbf{r}^3 P_2$. It is easy to see that M is not eventually Ω-perfect, that $\alpha(M) = 2$, and that no irreducible map from an indecomposable module to M is eventually Ω-perfect. Moreover, by [6], M lies in a component consisting entirely of string modules. But the only string modules lying on the boundary of an Auslander-Reiten component can lie on tubes (see [12], II.6.4), so this module belongs to a $\mathbb{Z}A_\infty^\infty$ component. Note also that the simple modules S_1 and S_3 are Ω-periodic of period 6, and that they both lie on tubes of rank 3.

Example 19. Following Erdmann [12], for each positive integer m, we denote by Λ_m the local symmetric string algebra over a field K,

$$\Lambda_m = K\langle x, y\rangle / \langle x^2, (xy)^{m+1} - (yx)^{m+1}, x^2 - (yx)^m y, x^3\rangle$$

If the characteristic of K is 2, and $m+1 = 2^n \geq 4$, then the algebra Λ_m modulo its socle is isomorphic to the group algebra of the semidihedral group of order 2^{n+2} modulo its socle. Motivated by this fact, Erdmann calls this algebra *semidihedral.* She proves that Λ_m has infinitely many stable components of type $\mathbb{Z}A_\infty^\infty$ and $\mathbb{Z}D_\infty$ ([12], Propositions II,10.1 and II,10.2), and that the other stable components are tubes of rank 1 and 2. Moreover, she shows that the unique simple module lies in a component of type $\mathbb{Z}D_\infty$ so it is not periodic. Therefore, every indecomposable non projective Λ_m-module is eventually Ω-perfect by [18]. Note that in the same book, Erdmann generalizes the notion of semidihedral algebra to that of algebras of *semidihedral type* and one also obtains interesting examples for the non local case ([12], Lemma VIII. 2.1.).

3.1. $\mathbb{Z}D_\infty$**-components.** We assume for the remainder of this section that $\mathcal{C}$ is a connected Auslander-Reiten component whose stable part is of the form $\mathbb{Z}D_\infty$. Let C be an indecomposable module lying on the boundary of $\mathcal{C}$. Then, without loss of generality we may assume that C is Ω-perfect, by Remark 11. In this context we have the following:

Lemma 20. *Let A and B be two indecomposable modules lying on the boundary of $\mathcal{C}$ with Auslander-Reiten sequences $0 \to \tau A \xrightarrow{f_1} M \xrightarrow{g_1} A \to 0$ and $0 \to \tau B \xrightarrow{f_2} M \xrightarrow{g_2} B \to 0$. Then the irreducible map $[g_1, g_2]^t \colon M \to A \oplus B$ is an epimorphism if and only if the map $[f_1, f_2] \colon \tau A \oplus \tau B \to M$ is also an epimorphism.*

Proof. Counting lengths, we have $\ell(\tau A) + \ell(A) + \ell(\tau B) + \ell(B) = 2\ell(M)$. This means that $\ell(A) + \ell(B) < \ell(M)$ if and only if $\ell(\tau A) + \ell(\tau B) > \ell(M)$. The result follows, since an irreducible map is either a monomorphism, or an epimorphism. □

Keeping the notation from the lemma, we may clearly assume that the modules A and B lying on the boundary of the component are Ω-perfect, and that the Auslander-Reiten sequence ending at M is $0 \to \tau M \to \tau A \oplus \tau B \oplus \tau X \to M \to 0$ for some indecomposable module X. Since the irreducible epimorphisms $M \to A$ and $M \to B$ are Ω-perfect, then the irreducible epimorphisms $\tau A \oplus \tau X \to M$ and $\tau B \oplus \tau X \to M$ are also Ω-perfect being "parallel" to Ω-perfect epimorphisms. Similarly, the irreducible monomorphisms $\tau M \to \tau X \oplus \tau A$ and $\tau M \to \tau X \oplus \tau B$ are also Ω-perfect. Putting together our remarks, we have:

Proposition 21. *Let $\mathcal{C}$ be an Auslander-Reiten component whose stable part is of type $\mathbb{Z}D_\infty$. Assume that there is an irreducible map between indecomposable non projective modules $X \to Y$ that is not eventually Ω-perfect. Then each non projective module in $\mathcal{C}$ has complexity 2.*

Proof. From the shape of our component, it follows by looking at "parallel" maps one at a time, that we may assume that there exists an irreducible map of the form $\tau M \to \tau A \oplus \tau B$ or $\tau A \oplus \tau B \to M$ that is not eventually Ω-perfect, where A and B are indecomposable modules lying on the boundary of $\mathcal{C}$, and M is an indecomposable module. Observe that, neither $\tau M \to \tau A \oplus \tau B$ nor $\tau A \oplus \tau B \to M$ can be eventually Ω-perfect by Lemma 20. Being of type $\mathbb{Z}D_\infty$ means also that $\mathcal{C}$ cannot contain modules of complexity 1 by [17]. We apply now 17. and the result follows. □

We would like to propose the following questions summarizing the discussion in the first three sections. The first one has been around for some time and is due to Rickard [25].

Questions 22. Let R be a selfinjective algebra.

(1) Assume that there exists an indecomposable R-module of complexity greater than 2. Is R is of wild representation type?
(2) Assume that R has stable components of type $\mathbb{Z}D_\infty$ or $\mathbb{Z}A_\infty^\infty$. Is R of tame representation type? Must these components have complexity 2?
(3) Assume that R has a stable component of type $\mathbb{Z}A_\infty$. Is R necessarily of wild representation type?

The answer to the first question is known to be yes if R admits a theory of support varieties, for instance in the group algebras case. See also [14]. The answer to the second and third question is also known to be affirmative in the group algebra case [12] but almost nothing is known outside this case.

4. Growth of Betti numbers. The local case

Let us return to the situation where $R = (R, \mathbf{m}, \Bbbk)$ is a local noetherian $\Bbbk$-algebra. The following questions were among questions posed in the late 1970s and the early 1980s. They are still open even in the commutative artinian case, and even if we also add the selfinjective assumption.

Questions 23. Let $R = (R, \mathbf{m}, \Bbbk)$ be a local noetherian $\Bbbk$-algebra. Let M be an indecomposable finitely generated R-module of infinite projective dimension.

(1) Assume that M has complexity 1. Is the sequence of Betti numbers $\{\beta_i(M)\}_i$ eventually constant?
(2) Is the sequence of Betti numbers $\{\beta_i(M)\}_i$ eventually nondecreasing?

The first question has an affirmative answer if R is a complete intersection ([10]). In the radical square zero case the answer is also affirmative. We sketch the proof below (see also [15])

Proposition 24. *Let $R = (R, \mathbf{m}, \Bbbk)$ is a local artinian ring with $\mathbf{m}^2 = 0$ and let M be a finitely generated R-module with $\operatorname{cx} M = 1$. Then the Betti numbers of M are eventually constant.*

Proof. Let F be a finitely generated free R-module. We observe first that since $\mathbf{m}^2 = 0$, every submodule of $\mathbf{m}F$ is semisimple [3], so all the syzygies of M must be semisimple. Let k denote the largest possible value of a Betti number of M and assume that it corresponds to the i-th Betti number, that is $\beta_i(M) = k$. This means that the i-th syzygy of M is a direct sum of k simple modules, hence $\beta_{i+1}(M) \geq k$. Our choice of k implies now that $\beta_{i+j}(M) = k$ for all $j \geq 0$ and the result follows. □

Question 1 also has an affirmative answer in the case where $R = (R, \mathbf{m}, \Bbbk)$ is a commutative Gorenstein artinian ring with $\mathbf{m}^3 = 0$, see [15]. Question 2 is also pretty much unresolved. In the local commutative artinian case, Gasharov and Peeva have shown ([15]) that for a finitely generated module M, we have the following:

$$\beta_{i+1}(M) \geq (2e - \ell(R) + h - 1)\beta_i(M)$$

for large enough i. Here $e = \dim_\Bbbk \mathbf{m}/\mathbf{m}^2$, h is the Loewy length of R, and $\ell(R)$ is the length of R. They have also shown that if the constant $2e - \ell(R) + h - 1 \geq 2$, then the sequence of Betti numbers has exponential growth. However it is not hard to produce examples of local commutative artinian rings where the constant $2e - \ell(R) + h - 1$ is a negative number. We also want to mention the following two results due to Ramras [23, 24]:

Theorem 25. *Let $(R, \mathbf{m}, \Bbbk)$ be a regular local ring of dimension at least two, and let $S = R/\mathbf{m}^k$ for some $k \geq 2$. Let M be a finitely generated non free S-module. Then, for each $i \geq 1$ we have $\beta^S_{i+2}(M) > \beta^S_i(M)$.* □

and

Theorem 26. *Let R be a local artinian ring, and let M be a finitely generated non free R-module. Then, for each $i \geq 1$ we have*

$$\ell(R)\beta_i(M) > \beta_{i+1}(M) > \frac{\ell(socR)}{\ell(R)}\beta_i(M)$$

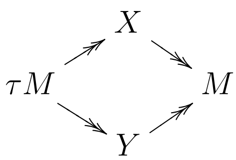
Observe that if we assume in the last theorem that R is also selfinjective, then its socle has length equal to 1, so we don't get any extremely useful information about the growth of the sequence of Betti numbers.

It turns out that in certain cases we can prove a similar theorem to Ramras' first theorem. For this type of result we might restrict ourselves only to the local selfinjective case $R = (R, \mathbf{m}, \Bbbk)$ but this is not necessary. Recall that since R is selfinjective, then for each integer $n \geq 0$ we have that $\beta_i(\tau M) = \beta_{i+2}(M)$ if M is an indecomposable non projective R-module. We will assume that $\operatorname{cx} M > 1$. Next we want to make sure that the stable component of M consists of modules that are eventually Ω-perfect. As mentioned in the introduction, this can be easily achieved if we assume that every simple R-module is non periodic ($\operatorname{cx} \Bbbk > 1$ for the local case) by [18].

We have the following:

Lemma 27. *Let R be a selfinjective algebra and let M be a finitely generated non projective indecomposable R-module. Assume that the stable component of the Auslander-Reiten quiver containing M is of the form $\mathbb{Z}A_\infty^\infty$ and that it consists entirely of eventually Ω-perfect modules. Then the sequences $\{\beta_{2n}(M)\}_n$ and $\{\beta_{2n+1}(M)\}_n$ are eventually strictly increasing.*

Proof. Let M be a module in this component. We may assume that M is Ω-perfect by taking enough powers of theAuslander-Reiten translate. The Auslander-Reiten sequence ending at M must have the following form [18, 20]



so we have an epimorphism $\tau M \to M$ that is the composition of two Ω-perfect epimorphisms. But we can infer from 4 that whenever we have an Ω-perfect epimorphism $f\colon B \to C$, then for each i we have $\beta_i(B) > \beta_i(C)$ since $\beta_i(\mathrm{Ker} f) > 0$. This implies that $\beta_{i+2}(M) = \beta_i(\tau M) > \beta_i(X) > \beta_i(M)$ for all $i \geq 0$ and the result follows. □

We now treat the D_∞ case.

Lemma 28. *Let R be a selfinjective algebra. Let $\mathcal{C}_s$ be a stable component of the Auslander-Reiten quiver of the form $\mathbb{Z}D_\infty$ consisting entirely of eventually Ω-perfect modules.*

1. *Let M be a module in $\mathcal{C}_s$ not lying on the border of the component. Then the sequences $\{\beta_{2n}(M)\}_n$ and $\{\beta_{2n+1}(M)\}_n$ are eventually strictly increasing.*
2. *Let Y and Z be two indecomposable modules in $\mathcal{C}_s$ lying in the two different τ-orbits that form the border of the component. Then the sequences $\{\beta_{2n}(Y \oplus Z)\}_n$ and $\{\beta_{2n+1}(Y \oplus Z)\}_n$ are eventually strictly increasing.*

Proof. Let M be an indecomposable module in this component. We may assume that M is Ω-perfect by taking enough powers of the Auslander-Reiten translate. If the Auslander-Reiten sequence ending at M has three indecomposable terms in the middle, the Auslander-Reiten sequence ending at M must have the following form [18, 20]

$$\begin{array}{ccccc} & & X & & \\ & \nearrow & & \searrow & \\ \tau M & \twoheadrightarrow & Y & \hookrightarrow & M \\ & \searrow & & \nearrow & \\ & & Z & & \end{array}$$

so as in the previous lemma we have an epimorphism $\tau M \to M$ that is the composition of two Ω-perfect epimorphisms. and $\beta_{i+2}(M) = \beta_i(\tau M) > \beta_i(M)$ for all $i \geq 0$. So the sequences of odd ,and of even Betti numbers for M are strictly increasing. Next we look at the module X. It is clear that we may assume that X is also Ω-perfect. The Auslander-Reiten sequence ending at X is of the form

$$0 \to \tau X \to \tau M \oplus X_1 \to X \to 0$$

where the irreducible map $X_1 \to X$ is an epimorphism. We proceed as in the proof of the previous lemma and obtain that for large enough n, the sequences $\{\beta_{2n}(X)\}_n$ and $\{\beta_{2n+1}(X)\}_n$ are strictly increasing. We proceed by induction along the sectional path of irreducible epimorphisms

$$\cdots X_n \to \cdots \to X_2 \to X_1 \to X$$

and we conclude that for each module X_j the two sequences $\{\beta_{2n}(X_j)\}_n$ and $\{\beta_{2n+1}(X_j)\}_n$ are eventually strictly increasing. This implies that the result holds for every module in the component, whose Auslander-Reiten sequence has the middle term decomposing into two indecomposable summands. This proves the first part of the lemma. By 20 we see that we have a composition of two irreducible epimorphisms fro $\tau Y \oplus \tau Z \to Y \oplus Z$ and we may also assume that both Y and Z are Ω-perfect. This shows that $\{\beta_{2n}(Y \oplus Z)\}_n$ and $\{\beta_{2n+1}(Y \oplus Z)\}_n$ are eventually strictly increasing. □

For the case when the stable component is of type $\mathbb{Z}\widetilde{A}_n$ or $\mathbb{Z}\widetilde{D}_n$ we proceed as above. We have the following similar proposition:

Proposition 29. *Let R be a selfinjective algebra and let M be a finitely generated non projective indecomposable R-module. Assume that the stable component of the Auslander-Reiten quiver containing M is of the form $\mathbb{Z}\widetilde{A}_n$ or $\mathbb{Z}\widetilde{D}_n$ and consists entirely of eventually Ω-perfect modules. Assume that the Auslander-Reiten sequence ending at M has a decomposable middle term. Then the sequences $\{\beta_{2n}(M)\}_n$ and $\{\beta_{2n+1}(M)\}_n$ are eventually increasing.*

Proof. Note first that M has complexity 2, by [20]. We use now the fact that a component of type $\mathbb{Z}\widetilde{A}_n$ is of tree type A_∞^∞ and use the same argument as above. For the case when the component is of type $\mathbb{Z}\widetilde{D}_n$ with $n > 4$, we can use the same proof as in the $\mathbb{Z}\widetilde{D}_\infty$ case, so it remains to look at the case when $n = 4$. In that case, if M is an Ω-perfect module, by [18] the Auslander-Reiten sequence ending at M has the form

$$0 \to \tau M \xrightarrow{[f_1,f_2,f_3,f_4]^T} E_1 \oplus E_2 \oplus E_3 \oplus E_4 \xrightarrow{[g_1,g_2,g_3,g_4]} M \to 0$$

where each f_i is an irreducible epimorphism and each g_i is an irreducible monomorphism. We show first that at least one of the two induced irreducible maps $E_1 \oplus E_2 \to M$ or $E_3 \oplus E_4 \to M$ is an irreducible epimorphism. Assume they are both monomorphisms. Then both $\ell(E_1 \oplus E_2) < \ell(M)$ and $\ell(E_3 \oplus E_4) < \ell(M)$ hence

$$\ell(M) + \ell(\tau M) = \ell(E_1) + \ell(E_2) + \ell(E_3) + \ell(E_4) < 2\ell(M)$$

implying $\ell(\tau M) < \ell(M)$. Since M is Ω-perfect we can repeat this argument and we obtain that the sequence $\{\ell(\tau^n M)\}$ is strictly decreasing; clearly a contradiction. Therefore we may assume that $E_3 \oplus E_4 \to M$ is an irreducible epimorphism. This means that we can look at our sequence as being

$$0 \to \tau M \xrightarrow{[f_1, f_2, f_3']^T} E_1 \oplus E_2 \oplus E_3' \xrightarrow{[g_1, g_2, g_3']} M \to 0$$

where $E_3' = E_3 \oplus E_4$. Now we obtain again from [18] that the induced map f_3' is an irreducible epimorphism and since M is Ω-perfect, $\beta_i(\tau M) > \beta_i(M)$ for all $i \geq 0$. The result follows now immediately. □

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