WEAKLY CLOSED GRAPH

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Abstract. We introduce the notion of weak closedness for connected simple graphs. This notion is a generalization of closedness introduced by Herzog-Hibi-Hreindóttir-Kahle-Rauh. We give a characterization of weakly closed graphs and prove that the binomial edge ideal J_G is F-pure for weakly closed graph G.

Key Words: binomial edge ideal, F-purity, weakly closed graph.

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1. Introduction

This article is based on [6].

Throughout this article, let k be an F-finite field of positive characteristic. Let G be a graph on the vertex set $V(G) = [n]$ with edge set $E(G)$. We assume that a graph G is always connected and simple, that is, G is connected and has no loops and multiple edges. And the term "labeling" means numbering of $V(G)$ from 1 to n.

For each graph G, we call $J_G := ([i, j] = X_iY_j - X_jY_i \mid \{i, j\} \in E(G))$ the *binomial edge ideal* of G (see [4], [8]). J_G is an ideal of $S := k[X_1, \dots, X_n, Y_1, \dots, Y_n]$.

2. Weakly closed graph

In this section, we give the definition of weakly closed graphs and the first main theorem of this chapter, which is a characterization of weakly closed graphs.

Until we define the notion of weak closedness, we fix a graph G and a labeling of $V(G)$.

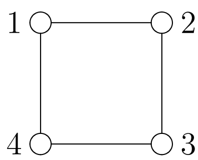
Let $(a_1, \dots, a_n)$ be a sequence such that $1 \le a_i \le n$ and $a_i \ne a_j$ if $i \ne j$.

Definition 1. We say that a_i is *interchangeable with* a_{i+1} if $\{a_i, a_{i+1}\} \in E(G)$. And we call the following operation $\{a_i, a_{i+1}\}$-*interchanging* :

$$(a_1, \dots, a_{i-1}, \underline{a_i}, \underline{a_{i+1}}, a_{i+2}, \dots, a_n) \to (a_1, \dots, a_{i-1}, \underline{a_{i+1}}, \underline{a_i}, a_{i+2}, \dots, a_n)$$

Definition 2. Let $\{i, j\} \in E(G)$. We say that i is *adjacentable with* j if the following assertion holds: for a sequence $(1, 2, \dots, n)$, by repeating interchanging, one can find a sequence $(a_1, \dots, a_n)$ such that $a_k = i$ and $a_{k+1} = j$ for some k.

Example 3. About the following graph G, 1 is adjacentable with 4:



The detailed version of this paper will be submitted for publication elsewhere.

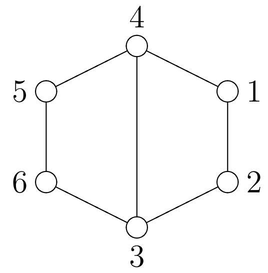
Indeed,

$$(\underline{1},2,3,\underline{4}) \xrightarrow{\{1,2\}} (2,\underline{1},3,\underline{4}) \xrightarrow{\{3,4\}} (2,\underline{1},\underline{4},3).$$

Now, we can define the notion of weakly closed graph.

Definition 4. Let G be a graph. G is said to be *weakly closed* if there exists a labeling which satisfies the following condition: for all i,j such that $\{i,j\} \in E(G)$, i is adjacentable with j.

Example 5. The following graph G is weakly closed:



Indeed,

$$(\underline{1},2,3,\underline{4},5,6) \xrightarrow{\{1,2\}} (2,\underline{1},3,\underline{4},5,6) \xrightarrow{\{3,4\}} (2,\underline{1},\underline{4},3,5,6),$$
$$(1,2,\underline{3},4,5,\underline{6}) \xrightarrow{\{3,4\}} (1,2,4,\underline{3},5,\underline{6}) \xrightarrow{\{5,6\}} (1,2,4,\underline{3},\underline{6},5).$$

Hence 1 is adjacentable with 4 and 3 is adjacentable with 6.

Before stating the first main theorem of this chapter, which is a characterization of weakly closed graphs, we recall that the definition of closed graphs.

Definition 6 (See [4])**.** G is *closed with respect to the given labeling* if the following condition is satisfied: for all $\{i,j\},\{k,l\} \in E(G)$ with $i<j$ and $k<l$ one has $\{j,l\} \in E(G)$ if $i=k$ but $j \neq l$, and $\{i,k\} \in E(G)$ if $j=l$ but $i \neq k$.

In particular, G is *closed* if there exists a labeling for which it is closed.

Remark 7. (1) [4, Theorem 1.1] G is closed if and only if J_G has a quadratic Gröbner basis. Hence if G is closed then S/J_G is Koszul algebra.

(2) [2, Theorem 2.2] Let G be a graph. Then the following conditions are equivalent:
   (a) G is closed.
   (b) There exists a labeling of $V(G)$ such that all facets of $\Delta(G)$ are intervals $[a,b] \subset [n]$, where $\Delta(G)$ is the clique complex of G.

The following characterization of closed graphs is a reinterpretation of Crupi and Rinaldo's one. This is relevant to the first main theorem of this chapter deeply.

Proposition 8 (See [1, Proposition 2.6])**.** *Let G be a graph. Then the following conditions are equivalent:*

(1) *G is closed.*
(2) *There exists a labeling which satisfies the following condition: for all i, j such that $\{i, j\} \in E(G)$ and $j > i + 1$, the following assertion holds: for all $i < k < j$, $\{i, k\} \in E(G)$* and *$\{k, j\} \in E(G)$.*

Proof. (1) $\Rightarrow$ (2): Let $\{i, j\} \in E(G)$. Since G is closed, there exists a labeling satisfying $\{i, i+1\}, \{i+1, i+2\}, \ldots, \{j-1, j\} \in E(G)$ by [HeHiHrKR, Proposition 1.4]. Then we have that $\{i, i+2\}, \ldots, \{i, j-2\}, \{i, j-1\} \in E(G)$ by the definition of closedness. Similarly, we also have that $\{k, j\} \in E(G)$ for all $i < k < j$.

(2) $\Rightarrow$ (1): Assume that $i < k < j$. If $\{i, k\}, \{i, j\} \in E(G)$, then $\{k, j\} \in E(G)$ by assumption. Similarly, if $\{i, j\}, \{k, j\} \in E(G)$, then $\{i, k\} \in E(G)$. Therefore G is closed. □

The following theorem characterizes weakly closed graph.

Theorem 9. *Let G be a graph. Then the following conditions are equivalent:*

(1) *G is weakly closed.*
(2) *There exists a labeling which satisfies the following condition: for all i, j such that $\{i, j\} \in E(G)$ and $j > i + 1$, the following assertion holds: for all $i < k < j$, $\{i, k\} \in E(G)$* or *$\{k, j\} \in E(G)$.*

Proof. (1) $\Rightarrow$ (2): Assume that $\{i, j\} \in E(G)$, $\{i, k\} \notin E(G)$ and $\{k, j\} \notin E(G)$ for some $i < k < j$. Then i is not adjacentable with j, which is in contradiction with weak closedness of G.

(2) $\Rightarrow$ (1): Let $\{i, j\}E(G)$. By repeating interchanging along the following algorithm, we can see that i is adjacentable with j:

(a): Let $A := \{k \mid \{k, j\} \in E(G), i < k < j\}$ and $C := \emptyset$.

(b): If $A = \emptyset$ then go to (g), otherwise let $s := \max\{A\}$.

(c): Let $B := \{t \mid \{s, t\} \in E(G), s < t \leq j\} \setminus C = \{t_1, \ldots, t_m = j\}$, where $t_1 < \ldots < t_m = j$.

(d): Take $\{s, t_1\}$-interchanging, $\{s, t_2\}$-interchanging, $\ldots$, $\{s, t_m = j\}$-interchanging in turn.

(e): Let $A := A \setminus \{s\}$ and $C := C \cup \{s\}$.

(f): Go to (b).

(g): Let $U := \{u \mid i < u < j, \{i, u\} \in E(G)$ and $\{u, j\} \notin E(G)\}$ and $W := \emptyset$.

(h): If $U = \emptyset$ then go to (m), otherwise let $u := \min\{U\}$.

(i): Let $V := \{v \mid \{v, u\} \in E(G), i \leq v < u\} \setminus W = \{v_1 = i, \ldots, v_l\}$, where $v_1 = i < \ldots < v_l$.

(j): Take $\{v_1 = i, u\}$-interchanging, $\{v_2, u\}$-interchanging, $\ldots$, $\{v_l, u\}$-interchanging in turn.

(k): Let $U := U \setminus \{u\}$ and $W := W \cup \{u\}$.

(l): Go to (h).

(m): Finished. □

By comparing this theorem and Proposition 8, we get the following corollary. A graph G is said to be *complete r-partite* if there exists a partition $V(G) = \coprod_{i=1}^{r} V_i$ such that $\{i,j\} \in E(G)$ if and only of $a \neq b$ for all $i \in V_a$ and $j \in V_b$.

Corollary 10. *Closed graphs and complete r-partite graphs are weakly closed.*

Proof. Assume that G is complete r-partite and $V(G) = \coprod_{i=1}^{r} V_i$. Let $\{i,j\} \in E(G)$ with $i \in V_a$ and $j \in V_b$. Then $a \neq b$. Hence for all $i < k < j$, $k \notin V_a$ or $k \notin V_b$. This implies that $\{i,k\} \in E(G)$ or $\{k,j\} \in E(G)$. □

3. F-purity of binomial edge ideals

In this section, we study about F-purity of binomial edge ideals. Firstly, we recall that the definition of F-purity of a ring R.

Definition 11 (See [5])**.** Let R be an F-finite reduced Noetherian ring of characteristic $p > 0$. R is said to be *F-pure* if the Frobenius map $R \to R$, $x \mapsto x^p$ is pure, equivalently, the natural inclusion $\tau : R \hookrightarrow R^{1/p}$, $(x \mapsto (x^p)^{1/p})$ is pure, that is, $M \to M \otimes_R R^{1/p}$, $m \mapsto m \otimes 1$ is injective for every R-module M.

The following proposition, which is called the Fedder's criterion, is useful to determine the F-purity of a ring R.

Proposition 12 (See [3])**.** *Let $(S, \mathfrak{m})$ be a regular local ring of characteristic $p > 0$. Let I be an ideal of S. Put $R = S/I$. Then R is F-pure if and only if $I^{[p]} : I \not\subseteq \mathfrak{m}^{[p]}$, where $J^{[p]} = (x^p \mid x \in J)$ for an ideal J of S.*

In this section, we consider the following question:

Question. When is S/J_G F-pure ?

In [8], Ohtani proved that if G is complete r-partite graph then S/J_G is F-pure. Moreover, it is easy to show that if G is closed then S/J_G is F-pure. However, there are many examples of G such that G is neither complete r-partite nor closed but S/J_G is F-pure. Namely, there is room for improvement about the above studies.

The second main theorem of this chapter is as follows:

Theorem 13. *If G is weakly closed, then S/J_G is F-pure.*

Proof. For a sequence $v_1, v_2, \dots, v_s$, we put

$$Y_{v_1}(v_1, v_2, \dots, v_s)X_{v_s} := (Y_{v_1}[v_1, v_2][v_2, v_3] \cdots [v_{s-1}, v_s]X_{v_s})^{p-1}.$$

Let $\mathfrak{m} = (X_1, \dots, X_n, Y_1, \dots, Y_n)S$. By taking completion and using Proposition 2.2, it is enough to show that $Y_1(1, 2, \dots, n)X_n \in (J_G^{[p]} : J_G) \setminus \mathfrak{m}^{[p]}$. It is easy to show that $Y_1(1, 2, \dots, n)X_n \notin \mathfrak{m}^{[p]}$ by considering its initial monomial.

Next, we use the following lemmas (see [8]):

Lemma 14 ([8, Formula 1]). *If* $\{a,b\} \in E(G)$, *then*

$$Y_{v_1}(v_1,\ldots,c,\underline{a},\underline{b},d,\ldots,v_n)X_{v_n} \equiv Y_{v_1}(v_1,\ldots,c,\underline{b},\underline{a},d,\ldots,v_n)X_{v_n}$$

modulo $J_G^{[p]}$.

Lemma 15 ([8, Formula 2]). *If* $\{a,b\} \in E(G)$, *then*

$$Y_a(\underline{a},\underline{b},c,\ldots,v_n)X_{v_n} \equiv Y_b(\underline{b},\underline{a},c,\ldots,v_n)X_{v_n},$$

$$Y_{v_1}(v_1,\ldots,c,\underline{a},\underline{b})X_b \equiv Y_{v_1}(v_1,\ldots,c,\underline{b},\underline{a})X_a$$

modulo $J_G^{[p]}$.

Let $\{i,j\} \in E(G)$. Since G is weakly closed, i is adjacentable with j. Hence there exists a polynomial $g \in S$ such that

$$Y_1(1,2,\ldots,n)X_n \equiv g \cdot [i,j]^{p-1}$$

modulo $J_G^{[p]}$ from the above lemmas. This implies $Y_1(1,2,\ldots,n)X_n \in (J_G^{[p]} : J_G)$. □

4. Difference between closedness and weak closedness and some examples

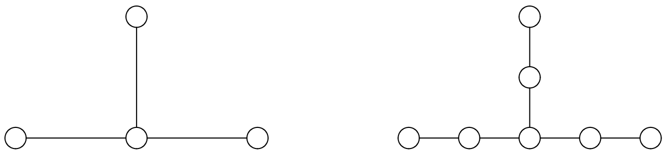
In this section, we state the difference between closedness and weak closedness and give some examples.

Proposition 16. *Let* G *be a graph.*

(1) *[4, Proposition 1.2] If* G *is closed, then* G *is chordal, that is, every cycle of* G *with length* $t > 3$ *has a chord.*
(2) *If* G *is weakly closed, then every cycle of* G *with length* $t > 4$ *has a chord.*

Proof. (2) It is enough to show that the pentagon graph G with edges $\{a,b\}$, $\{b,c\}$, $\{c,d\}$, $\{d,e\}$ and $\{a,e\}$ is not weakly closed. Suppose that G is weakly closed. We may assume that $a = \min\{a,b,c,d,e\}$ without loss of generality. Then $b \neq \max\{a,b,c,d,e\}$. Indeed, if $b = \max\{a,b,c,d,e\}$, then c,d,e are connected with a or b by the definition of weak closedness, but this is a contradiction. Similarly, $e \neq \max\{a,b,c,d,e\}$. Hence we may assume that $c = \max\{a,b,c,d,e\}$ by symmetry. If $b = \min\{b,c,d\}$, then d,e are connected with b or c, a contradiction. Therefore, $b \neq \min\{b,c,d\}$. Similarly, $b \neq \max\{b,c,d\}$. Hence we may assume that $d = \min\{b,c,d\}$ and $e = \max\{b,c,d\}$ by symmetry. Then $\{a,b\}$ and $a < d < b$, but $\{a,d\},\{d,b\} \notin E(G)$. This is a contradiction. □

Next, we give a characterization of closed (resp. weakly closed) tree graphs in terms of claw (resp. bigclaw). A graph G is said to be *tree* if G has no cycles. We consider the following graphs (a) and (b). We call the graph (a) a *claw* and the graph (b) a *bigclaw*.



(a) (b)

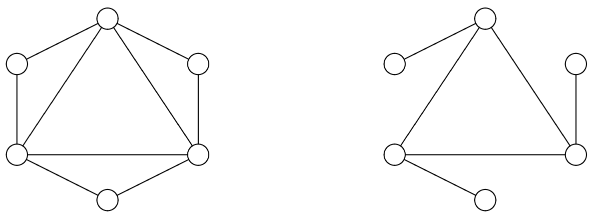
Proposition 17. *Let G be a tree.*

(1) *[4, Corollary 1.3] The following conditions are equivalent:*
  (a) *G is closed.*
  (b) *G is a path.*
  (c) *G is a claw-free graph.*

(2) *The following conditions are equivalent:*
  (a) *G is weakly closed.*
  (b) *G is a caterpillar, that is, a tree for which removing the leaves and incident edges produces a path graph.*
  (c) *G is a bigclaw-free graph.*

Proof. (2) One can see that a bigclaw graph is not weakly closed. □

Remark 18. From Proposition 17(2), we have that chordal graphs are not always weakly closed. As other examples, the following graphs are chordal, but not weakly closed:



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