A NOTE ON DIMENSION OF TRIANGULATED CATEGORIES.

HIROYUKI MINAMOTO

ABSTRACT. In this note we study the behavior of the dimension of the perfect derived category Perf(A) of a dg-algebra A over a field k under a base field extension K/k. In particular we show that the dimension of a perfect derived category is invariant under a separable algebraic extension K/k. As an application we prove the following statement: Let A be a self-injective algebra over a perfect field k. If the dimension of the stable category $\underline{\text{mod}}A$ is 0, then A is of finite representation type. This theorem is proved by M. Yoshiwaki in the case when k is an algebraically closed field. Our proof depends on his result.

1. INTRODUCTION

In [3] R. Rouquier introduced the dimension of triangulated categories and showed that it gives an upper bound or a lower bound of other dimensions in algebraic geometry or in representation theory(see also [4]). The dimension of triangulated categories is studied by many researchers.

In this note we study the behavior of the dimension of the perfect derived category Perf(A) of a dg-algebra A over a field k under a base field extension K/k. For a field extension K/k, we denote $A \otimes_k K$ by A_K.

Theorem 1. (1) *For an algebraic extension K/k, we have*

$$\text{tridim Perf}(A) \leq \text{tridim Perf}(A_K).$$

(2) *If moreover K/k is separable, then equality holds.*

As an application we prove the following theorem, which gives evidence that dimension of triangulated categories captures some representation theoretic properties.

The stable category $\underline{\text{mod}}A$ plays an important role in the study of self-injective algebra A (cf. [2, 4]). If a self-injective algebra A is of finite representation type then the dimension of the stable category $\underline{\text{mod}}A$ is zero. Then a natural question arises as to whether the converse should also hold.

Theorem 2. *Let A be a self-injective finite dimensional algebra over a perfect field k. If* tridim $\underline{\text{mod}}A = 0$*, then A is of finite representation type.*

In the case when k is an algebraically closed field, this theorem is proved by M. Yoshiwaki in [5]. Our proof depends on his result.

The final version of this paper has been submitted for publication elsewhere.

2. Dimension of triangulated categories.

We review the definition of dimension of triangulated categories due to R. Rouquier We need to prepare a bit of notations.

Let $\mathcal{T}$ be a triangulated category. For a full subcategory $\mathcal{I}$ of $\mathcal{T}$ we denote by $\langle \mathcal{I} \rangle$ the smallest full subcategory of $\mathcal{T}$ containing $\mathcal{I}$ which is closed under taking shifts, finite direct sums, direct summands and isomorphisms. For full subcategories $\mathcal{I}$ and $\mathcal{J}$ of $\mathcal{T}$ we denote by $\mathcal{I} * \mathcal{J}$ the full subcategory of $\mathcal{T}$ consisting of those object $M \in \mathcal{T}$ such that there exists an exact triangle $I \to M \to J \xrightarrow{[1]}$ with $I \in \mathcal{I}$ and $J \in \mathcal{J}$. Set $\mathcal{I} \diamond \mathcal{J} := \langle \mathcal{I} * \mathcal{J} \rangle$. For $n \geq 1$ we define inductively

$$\langle \mathcal{I} \rangle_n := \begin{cases} \langle \mathcal{I} \rangle & \text{for } n = 1; \\ \langle \mathcal{I} \rangle \diamond \langle \mathcal{I} \rangle_{n-1} & \text{for } n \geq 2. \end{cases}$$

Now we define the dimension of a triangulated category $\mathcal{T}$ to be

$$\operatorname{tridim} \mathcal{T} := \min\{n \mid \langle E \rangle_{n+1} = \mathcal{T} \text{ for some } E \in \mathcal{T}\}.$$

3. Sketch of proof of Theorem 1 and 2

First we consider the case when K/k is a finite extension. Let $\overline{E}$ be an object of $\mathrm{Perf}(A_K)$ such that $\langle \overline{E} \rangle_n = \mathrm{Perf}(A_K)$ for some $n \in \mathbb{N}$. Then we see that $\langle U\overline{E} \rangle_n = \mathrm{Perf}(A)$ where $U : \mathrm{Perf}(A_K) \to \mathrm{Perf}(A)$ is the forgetful functor.

In the case K/k is an infinite algebraic extension, the key of the proof is the following lemma.

Lemma 3. *Let K/k be an algebraic extension and E an object of $\mathcal{D}(A)$.*

If an object $\overline{G}$ of $\mathcal{D}(A_K)$ belongs to $\langle E \otimes_k K \rangle_n$, then there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over k such that there exists an object G' of $\langle E \otimes_k K_0 \rangle_n$, such that $G' \otimes_{K_0} K \cong \overline{G}$ in $\mathcal{D}(A_K)$.

Let $\overline{E}$ be an object of $\mathrm{Perf}(A_K)$ such that $\langle \overline{E} \rangle_n = \mathrm{Perf}(A_K)$ for some $n \in \mathbb{N}$. Since $\mathrm{Perf}(A_K) = \cup_{i \in \mathbb{N}} \langle A_K \rangle_i$, by the above lemma there exists an intermediate field $k \subset K_0 \subset K$ which is finite dimensional over k such that there exists an object E' of $\mathrm{Perf}(A_{K_0})$ such that $E' \otimes_{K_0} K \simeq \overline{E}$. Then we see that $\langle U_0(E') \rangle_n = \mathrm{Perf}(A)$ where $U_0 : \mathrm{Perf}(A_{K_0}) \to \mathrm{Perf}(A)$ is the forgetful functor.

To prove the second statement, we use the fact that when K/k is a finite separable field extension, the canonical morphism $K \otimes_k K \to K$ splits as $K - K$ bimodules. In the case when K/k is an infinite separable field extension, we reduce to the finite separable extension case by the above lemma.

Theorem 2 is reduced to the case when the base field k is an algebraically closed field by Theorem 1 and the following lemma.

Lemma 4. *Let A be a finite dimensional k-algebra. If $A_{\overline{k}}$ is of finite representation type, then A is of finite representation type.*

4. Examples which show that we need to impose conditions on Theorem 1

To conclude this note we give examples which show that we need to impose conditions on Theorem 1.

Example 5. If an algebraic extension K/k is not separable, then the dimension tridim $\mathrm{Perf}(A_K)$ is possibly larger than the dimension tridim $\mathrm{Perf}(A)$.

Here is an example. Let F be a field of characteristic $p > 0$. Let $K := F(t)$ be a rational function field in one variable and define $k := F(t^p) \subset K = F(t)$. Set $A := K$. Then it is easy to see $A_K \cong K[x]/(x^p)$. Since $\operatorname{gldim} A_K = \infty$, we see that tridim $\mathrm{Perf}(A_K) = \infty$ by [3, Proposition 7.26]. However since $A = K$ is a field, we have tridim $\mathrm{Perf}(A) = 0$.

Example 6. In the case when the extension K/k is not algebraic, the dimension tridim $\mathrm{Perf}(A_K)$ is possibly larger than tridim $\mathrm{Perf}(A)$ even if an extension K/k is separable.

Here is an example. Assume that for simplicity k is algebraically closed. Let $K = k(y)$ and $A = k(x)$ be rational function fields in one variable over k. Then we can easily see that tridim $\mathrm{Perf}(A_K) = 1$ by the method of the proof of [3, Theorem 7.17]. However since $A = k(x)$ is a field, we see that tridim $\mathrm{Perf}(A) = 0$.

References

[1] M. Bökstedt and A. Neeman, Homotopy limits in triangulated categories, Compositio Math., 86(2) (1993), 209-234.

[2] D. Happel, Triangulated categories in the representation theory of finite dimensional algebras, London Mathematical Society Lecture Note Series 119, Cambridge University Press, Cambridge, 1988.

[3] R. Rouquier. Dimensions of triangulated categories, Journal of K-theory 1 (2008), 193-256

[4] R. Rouquier, Representation dimension of exterior algebras, Invent. Math. 165 (2006), no. 2, 357–367.

[5] M. Yoshiwaki. On selfinjective algebras of stable dimension zero, Nagoya Mathematical Journal (accepted for publication).

Research Institute for Mathematical Sciences,
Kyoto University,
Kyoto 606-8502, JAPAN

E-mail address: minamoto@kurims.kyoto-u.ac.jp