APR TILTING MODULES AND QUIVER MUTATIONS

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Abstract. We study the quiver with relations of the endomorphism algebra of an APR tilting module. We give an explicit description of the quiver with relations by graded quivers with potential (QPs) and mutations. Consequently, mutations of QPs provide a rich source of derived equivalence classes of algebras.

1. Introduction

Derived categories have been one of the important tools in the study of many areas of mathematics. In the representation theory of algebras, tilting modules play an essential role to give an equivalence of derived categories. More precisely, the endomorphism algebra of a tilting module is derived equivalent to the original algebra. Therefore the relationship of quivers with relations of these algebras has been investigated for a long time.

The first well-known result of these studies appears in the work of [5]. It is the origin of tilting theory and formulated in terms of an APR tilting module now [4]. Let us recall an important property of APR tilting modules.

Theorem 1. [4] *Let KQ be a path algebra of a finite acyclic quiver Q and T_k be the APR tilting KQ-module associated with a source $k \in Q$. Then we have an algebra isomorphism*

$$\mathrm{End}_{KQ}(T_k) \cong K(\mu_k Q),$$

where μ_k is a mutation at k.

Thus the quiver of the endomorphism algebra is completely determined by combinatorial methods and the *mutation* can be considered as a generalization of BGP reflection. The notion of mutation was introduced by Fomin-Zelevinsky [11], which is an important ingredient of cluster algebras, and many links with other subjects have been discovered and widely investigated. In particular, Derksen-Weyman-Zelevinsky applied mutations to quivers with potential (QPs). It has been found that mutations of QPs have close connections with tilting theory, for example [9, 17].

The main purposes of this paper is to generalize the above result for a more general class of algebras by using mutations of QPs. Since we have gl.dim$KQ \leq 1$, it is natural to consider algebras Λ with gl.dim$\Lambda \leq 2$. In this case, we can describe the quiver and relations by the following steps.

1. Define the associated graded QP $(Q_\Lambda, W_\Lambda, C_\Lambda)$.
2. Apply left mutation μ_k^L to $(Q_\Lambda, W_\Lambda, C_\Lambda)$.
3. Take the truncated Jacobian algebras $\mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda))$.

The detailed version of this paper will be submitted for publication elsewhere.

Then we have the following result.

Theorem 2. (*Theorem 7*) *Let* Λ *be a finite dimensional algebra with* gl.dim$\Lambda \le 2$ *and* T_k *be the APR tilting* Λ*-module associated with a source* k*. Then we have an algebra isomorphism*

$$\mathrm{End}_\Lambda(T_k) \cong \mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda)).$$

We give three remarks about the theorem. First, we can show that $\mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda))$ coincides with $K(\mu_k Q)$ if gl.dim$\Lambda = 1$, so that Theorem 2 gives a generalization of Theorem 1. Second, the condition gl.dim$\Lambda \le 2$ is actually not necessary, and it is enough to assume that the associated projective module has the injective dimension at most 2. Finally, this isomorphic provides a bridge of the two notions which have entirely different origins, and it implies that the contemporary concepts have a profound connection with the classical ones.

Conventions and notations. We always suppose that K is an algebraically closed field for simplicity. All modules are left modules and the composition fg of morphisms means first f, then g. We denote the set of vertices by Q_0 and the set of arrows by Q_1 of a quiver Q. We denote by $a : s(a) \to e(a)$ the start and end vertices of an arrow or path a.

2. Preliminaries

In this section, we give a brief summary of the definitions and results we will use in the next sections. See references for more detailed arguments and precise definitions.

2.1. **Quivers with potentials.** We review the notions initiated in [10].

• Let Q be a finite connected quiver. We denote by KQ_i the K-vector space with basis consisting of paths of length i in Q, and by $KQ_{i,cyc}$ the subspace of KQ_i spanned by all cycles. We denote *complete path algebra* by

$$\widehat{KQ} = \prod_{i \ge 0} KQ_i.$$

A *quiver with potential* (QP) is a pair (Q, W) consisting of a quiver Q and an element $W \in \prod_{i\ge 2} KQ_{i,\mathrm{cyc}}$, called a *potential*. For each arrow a in Q, the *cyclic derivative* $\partial_a : \widehat{KQ}_{cyc} \to \widehat{KQ}$ is defined by the continuous linear map which sends $\partial_a(a_1 \cdots a_d) = \sum_{a_i = a} a_{i+1} \cdots a_d a_1 \cdots a_{i-1}$. For a QP (Q, W), we define the *Jacobian algebra* by

$$\mathcal{P}(Q, W) = \widehat{KQ}/\mathcal{J}(W),$$

where $\mathcal{J}(W) = \overline{\langle \partial_a W \mid a \in Q_1 \rangle}$ is the closure of the ideal generated by $\partial_a W$ with respect to the $J_{\widehat{KQ}}$-adic topology.

• A QP (Q, W) is called *reduced* if $W \in \prod_{i \ge 3} KQ_{i,\mathrm{cyc}}$.

• For two QPs (Q', W') and (Q'', W''), we define a new QP (Q, W) as a direct sum $(Q', W') \oplus (Q'', W'')$, where $Q_0 = Q'_0 (= Q''_0)$, $Q_1 = Q'_1 \coprod Q''_1$ and $W = W' + W''$.

Definition 3. For each vertex k in Q not lying on a loop nor 2-cycle, we define a *mutation* $\mu_k(Q, W)$ as a *reduced part* of $\widetilde{\mu}_k(Q, W) = (Q', W')$, where (Q', W') is given as follows.

(1) Q' is a quiver obtained from Q by the following changes.
 - Replace each arrow $a : k \to v$ in Q by a new arrow $a^* : v \to k$.
 - Replace each arrow $b : u \to k$ in Q by a new arrow $b^* : k \to u$.
 - For each pair of arrows $u \xrightarrow{b} k \xrightarrow{a} v$, add a new arrow $[ba] : u \to v$

(2) $W' = [W] + \Delta$ is defined as follows.
 - $[W]$ is obtained from the potential W by replacing all compositions ba by the new arrows $[ba]$ for each pair of arrows $u \xrightarrow{b} k \xrightarrow{a} v$.
 - $\Delta = \sum_{\substack{a,b \in Q_1 \\ e(b)=k=s(a)}} [ba]a^*b^*$.

2.2. **Truncated Jacobian algebras.** We introduce the notion of cuts and the truncated Jacobian algebras.

Definition 4. [14] Let (Q, W) be a QP. A subset $C \subset Q_1$ is called a *cut* if each cycle appearing W contains exactly one arrow of C. Then we define the *truncated Jacobian algebra* by

$$\mathcal{P}(Q, W, C) := \mathcal{P}(Q, W)/\overline{\langle C \rangle} = \widehat{KQ_C}/\overline{\langle \partial_c W \mid c \in C \rangle},$$

where Q_C is the subquiver of Q with vertex set Q_0 and arrow set $Q_1 \setminus C$.

Then, we can naturally define a QP with a cut from a given algebra as follows.

Definition 5. [16] Let Q be a finite connected quiver and $\Lambda = \widehat{KQ}/\overline{\langle R \rangle}$ be a finite dimensional algebra with a minimal set of relations.

Then we define a QP (Q_Λ, W_Λ) as follows:

(1) $(Q_\Lambda)_0 = Q_0$
(2) $(Q_\Lambda)_1 = Q_1 \coprod C_\Lambda$, where $C_\Lambda := \{\rho_r : e(r) \to s(r) \mid r \in R\}$.
(3) $W_\Lambda = \sum_{r \in R} \rho_r r$.

Then the set C_Λ gives a cut of (Q_Λ, W_Λ).

2.3. **APR tilting modules.** We call a Λ-module T *tilting module* if $\mathrm{proj.dim}_\Lambda T \leq 1$, $\mathrm{Ext}^1_\Lambda(T, T) = 0$, and there exists a short exact sequence $0 \to \Lambda \to T_0 \to T_1 \to 0$ with T_0, T_1 in $\mathrm{add}T$.

Definition 6. Let Λ be a basic finite dimensional algebra and P_k be a simple projective non-injective Λ-module associated with a source k of the quiver Λ. Then Λ-module $T := \tau^- P_k \oplus \Lambda/P_k$ is called an *APR tilting module*, where τ^- denotes the inverse of the Auslander-Reiten translation.

3. Main theorem

3.1. **Main result.** Let Q be a finite connected quiver and $\Lambda = \widehat{KQ}/\overline{\langle R \rangle}$ be a finite dimensional algebra with a minimal set of relations. Assume that P_k is the simple projective non-injective Λ-module associated with a source $k \in Q$. Our aim is to determine the quiver and the set of relations giving $\mathrm{End}_\Lambda(T_k)$.

Consider the associated QP $(Q_\Lambda, W_\Lambda, C_\Lambda)$ of Λ and we put $\widetilde{\mu}_k(Q_\Lambda, W_\Lambda) = (Q', W')$. Then W' is given by

$$W' = [\sum_{r\in R}\rho_r r] + \sum_{\substack{a\in Q_1, r\in R\\ s(a)=k=s(r)}} [\rho_r a]a^*\rho_r^*,$$

and it is easy to check that subset

$$C' = \{\ \rho_r \mid r\in R, s(r)\neq k\}\coprod\{\ [\rho_r a] \mid a\in Q_1, r\in R, s(a)=k=s(r)\}$$

of Q' is a cut of (Q', W').

Then we have the following.

Theorem 7. *Let $\Lambda = \widehat{KQ}/\overline{\langle R\rangle}$ be a finite dimensional algebra with a minimal set of relations. Let $T_k := \tau^- P_k \oplus \Lambda/P_k$ be the APR tilting module. Then if* inj.dim$P_k \leq 2$, *we have an algebra isomorphism*

$$\mathrm{End}_\Lambda(T_k) \cong \mathcal{P}(\widetilde{\mu}_k(Q_\Lambda, W_\Lambda), C').$$

Notice that the assumption inj.dim$P_k \leq 2$ is automatic if gl.dim$\Lambda = 2$. Thus our theorem give a generalization from gl.dim$\Lambda = 1$ to gl.dim$\Lambda = 2$.

Here we will explain the choice of C'. In fact C' is naturally obtained by using graded mutations. For this purpose, we recall graded QPs, as introduced by [1].

Graded quivers with potentials. Let (Q, W) be a QP and we define a map $d : Q_1 \to \mathbb{Z}$. We call a QP (Q, W, d) *$\mathbb{Z}$-graded QP* if each arrow $a \in Q_1$ has a degree $d(a) \in \mathbb{Z}$, and *homogeneous of degree l* if each term in W is a degree l.

Definition 8. Let QP (Q, W, d) be a $\mathbb{Z}$-graded QP of degree l. For each vertex k in Q not lying on a loop nor 2-cycle, we define a *left mutation* $\mu_k^L(Q, W, d)$ as a reduced part of $\widetilde{\mu}_k^L(Q, W, d) = (Q', W', d')$, where (Q', W', d') is given as follows.

(1) $(Q', W') = \widetilde{\mu}_k(Q, W)$

(2) The new degree d' is defined as follows:

- $d'(a) = d(a)$ for each arrow $a \in Q \cap Q'$.
- $d'(a^*) = -d(a)$ for each arrow $a : k \to v$ in Q.
- $d'(b^*) = -d(b) + l$ for each arrow $b : u \to k$ in Q.
- $d'([ba]) = d(a) + d(b)$ for each pair of arrows $u \xrightarrow{b} k \xrightarrow{a} v$ in Q.

In particular, $\widetilde{\mu}_k^L(Q, W, d)$ also has a potential of degree l. Similarly, we can define $\widetilde{\mu}_k^R$ at k. In this case, we define $d'(b^*) = -d(b)$ for each arrow $b : u \to k$ in Q and $d'(a^*) = -d(a) + l$ for each arrow $a : k \to v$ in Q.

If (Q, W) has a cut C, we can identify the QP with a $\mathbb{Z}$-graded QP of degree 1 associating a grading on Q by

$$d_C(a) = \begin{cases} 1 & a \in C \\ 0 & a \notin C. \end{cases}$$

We denote by (Q, W, C) the graded QP of degree 1 with this grading. If any arrow of $\widetilde{\mu}_k^L(Q, W, C)$ has degree 0 or 1, degree 1 arrows give a cut of $\widetilde{\mu}_k(Q, W)$ since $\widetilde{\mu}_k^L(Q, W, C)$ is homogeneous of degree 1. Therefore a cut of $\widetilde{\mu}_k(Q_\Lambda, W_\Lambda)$ is naturally induced as degree

1 arrows of $\widetilde{\mu}_k^L(Q_\Lambda, W_\Lambda, C_\Lambda)$ and the above C' is obtained in this way. Thus we identify degree 1 arrows as a cut.

Because we have $\mathcal{P}(\widetilde{\mu}_k^L(Q_\Lambda, W_\Lambda, C_\Lambda)) \cong \mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda))$, we can rewrite Theorem 7 that we have an algebra isomorphism

$$\mathrm{End}_\Lambda(T_k) \cong \mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda)).$$

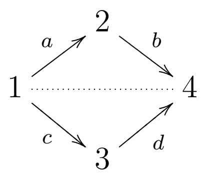
3.2. **Examples.** We explain the theorem with some examples.

Example 9. We keep the assumption of Theorem 7. If gl.dim$\Lambda = 1$, then we have $\Lambda = KQ$ and

$$\mathcal{P}(\mu_k^L(Q_\Lambda, W_\Lambda, C_\Lambda)) = \mathcal{P}(\mu_k^L(Q, 0, 0)) = K(\mu_k Q),$$

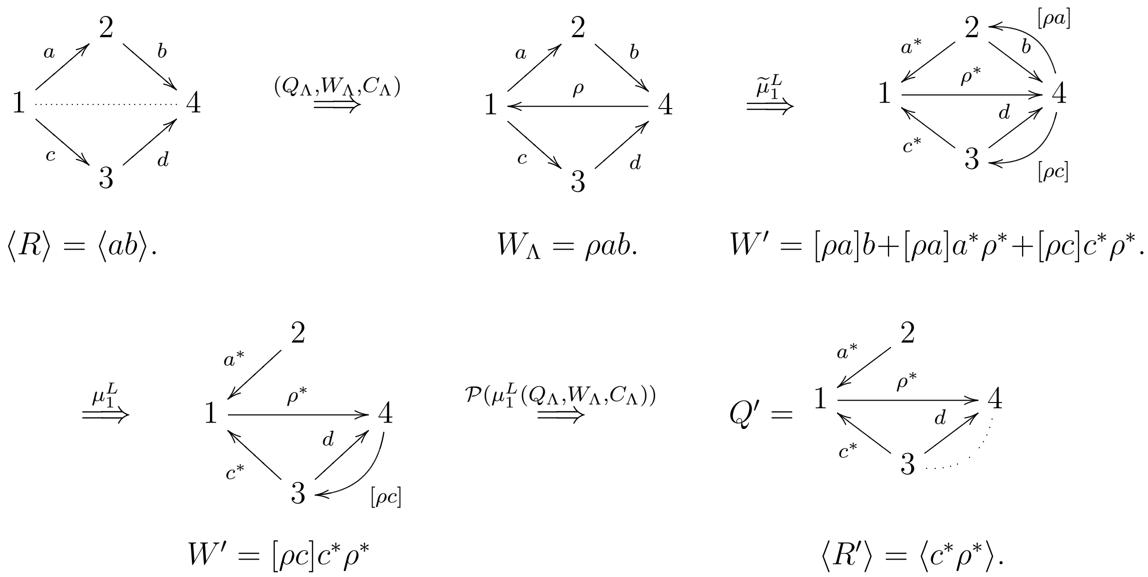
so that the mutation procedure is just reversing arrows having k. Thus the above theorem coincides with the classical result (Theorem 1).

Example 10. Let $\Lambda = \widehat{KQ}/\overline{\langle R\rangle}$ be a finite dimensional algebra given by the following quiver with a relation.



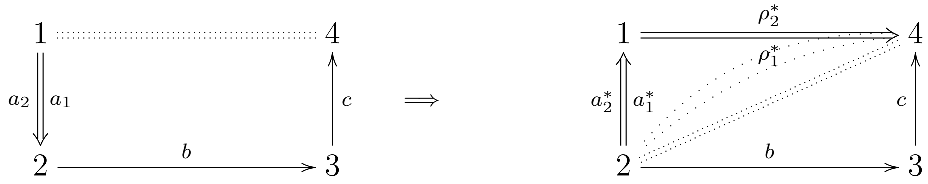
$\langle R\rangle = \langle ab\rangle$.

Then we consider the APR tilting module $T_1 := \tau^- P_1 \oplus \Lambda/P_1$ and calculate Q' and R' satisfying $\widehat{KQ'}/\overline{\langle R'\rangle} \cong \mathrm{End}_\Lambda(T_1)$ by the following steps.



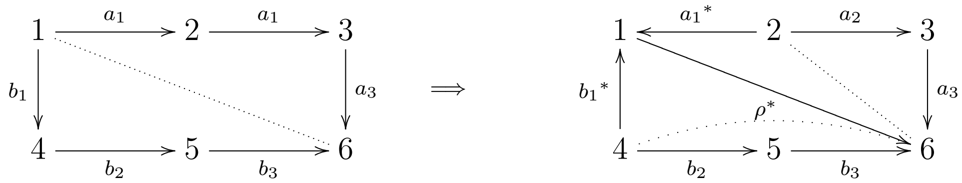
Similarly from the left-hand side algebra Λ, we obtain the quiver and the set of relations giving $\mathrm{End}_\Lambda(T_1)$, which is given by right-hand side picture.

(1)



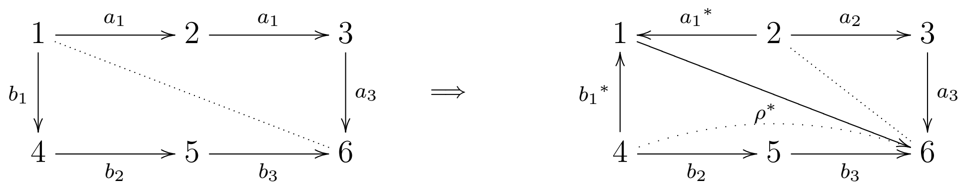
$\langle R\rangle = \langle a_1bc, a_2bc\rangle$ $\qquad$ $\langle R'\rangle = \langle a_1^*\rho_1^*+bc, a_2^*\rho_2^*+bc, a_1^*\rho_2^*, a_2^*\rho_1^*\rangle.$

(2) (i)



$\langle R\rangle = \langle a_1a_2a_3\rangle$ $\qquad$ $\langle R'\rangle = \langle a_2a_3 + {a_1}^*\rho^*, {b_1}^*\rho^*\rangle.$

(ii)



$\langle R\rangle = \langle a_1a_2a_3 = b_1b_2b_3\rangle$ $\qquad$ $\langle R'\rangle = \langle a_2a_3 + {a_1}^*\rho^*, b_2b_3 + {b_1}^*\rho^*\rangle.$

As examples show, we interpret the degree 1 arrows as relations.

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