THE NOETHERIAN PROPERTIES OF THE RINGS OF DIFFERENTIAL OPERATORS ON CENTRAL 2-ARRANGEMENTS

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Abstract. P. Holm began to study the ring of differential operators of the coordinate ring of a hyperplane arrangement. In this paper, we introduce Noetherian properties of the ring differential operators of the coordinate ring of a central 2-arrangement and its graded ring associated to the order filtration.

Key Words: Ring of differential operators, Noetherian property, Hyperplane arrangement.

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1. Introduction

For a commutative algebra $R$ over a field $K$ of characteristic zero, define vector spaces inductively by

$$
\mathcal{D}^0(R) := \{ \theta \in \text{End}_K(R) \mid a \in R, \theta a - a \theta = 0 \},
$$

$$
\mathcal{D}^m(R) := \{ \theta \in \text{End}_K(R) \mid a \in R, \theta a - a \theta \in \mathcal{D}^{m-1}(R) \} \quad (m \geq 1).
$$

We define the ring $\mathcal{D}(R) := \bigcup_{m \geq 0} \mathcal{D}^m(R)$ of differential operators of $R$.

Let $S := K[x_1, \ldots, x_n]$ be the polynomial ring. The ring $\mathcal{D}(S)$ is the $n$-th Weyl algebra $K[x_1, \ldots, x_n][\partial_1, \ldots, \partial_n]$ where $\partial_i := \frac{\partial}{\partial x_i}$ (see for example [3]). We use the multi-index notations, for example, $\partial^\alpha := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Define $\mathcal{D}^{(m)}(S) := \bigoplus_{|\alpha|=m} \mathcal{D}^m(S)$. Then $\mathcal{D}(S) = \bigoplus_{m \geq 0} \mathcal{D}^{(m)}(S)$. It is well known $\mathcal{D}(R)$ that $\mathcal{D}(R)$ is Noetherian, if $R$ is a regular domain (see [3]).

Holm [2] showed that $\mathcal{D}(R)$ is finitely generated as a $K$-algebra when $R$ is a coordinate ring of a generic hyperplane arrangement. Holm [1] also proved that the ring of differential operators of a central 2-arrangement is a free $S$-module, and gave a basis of it. We can write any element in $\mathcal{D}(R)$ as a linearly combination of this basis elements.

In this paper, we introduce the Noetherian property of $\mathcal{D}(R)$ when $R$ is the coordinate ring of a central arrangement. In particular, the case $n = 2$, $\mathcal{D}(R)$ is a Noetherian ring.

We give an example of a finitely generated ideal in the end of this paper.

The details of this note are in [4].

2. Hyperplane arrangement

In this section, we fix some notation, and we introduce some properties of the ring of differential operators of a central arrangement. Let $\mathcal{A} = \{ H_i \mid i = 1, \ldots, r \}$ be a central (hyperplane) arrangement (i.e., every hyperplane in $\mathcal{A}$ contains the origin) in $K^n$. Fix a

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The detailed version of this paper will be submitted for publication elsewhere.
polynomial \( p_i \) with \( \ker(p_i) = H_i \), and put \( Q := p_1 \cdots p_r \). Thus \( Q \) is a product of certain homogeneous polynomials of degree 1. Let \( I \) denote the principal ideal of \( S \) generated by \( Q \). Then \( S/I \) is the coordinate ring of the hyperplane arrangement defined by \( Q \).

For any ideal \( J \) of \( S \), we define an \( S \)-submodule \( \mathcal{D}^{(m)}(J) \) of \( \mathcal{D}^{(m)}(S) \) and a subring \( \mathcal{D}(J) \) of \( \mathcal{D}(S) \) by

\[
\mathcal{D}^{(m)}(J) := \{ \theta \in \mathcal{D}^{(m)}(S) \mid \theta(J) \subseteq J \},
\]

\[
\mathcal{D}(J) := \{ \theta \in \mathcal{D}(S) \mid \theta(J) \subseteq J \}.
\]

Holm [2] proved the following proposition.

**Proposition 1** (Proposition 4.3 in [2]).

\[
\mathcal{D}(I) = \bigoplus_{m \geq 0} \mathcal{D}^{(m)}(I).
\]

There is a ring isomorphism \( \mathcal{D}(S/J) \simeq \mathcal{D}(J)/J\mathcal{D}(S) \) (see [3, Theorem 15.5.13]). Thus we can express \( \mathcal{D}(S/J) \) as a subquotient of Weyl algebra.

We can prove that \( \mathcal{D}(J)/J\mathcal{D}(S) \) is right Noetherian if and only if \( \mathcal{D}(J)/J\mathcal{D}(S) \) is left Noetherian when \( J \neq 0 \) is a principal ideal. Therefore we conclude that \( \mathcal{D}(S/I) \) is right Noetherian if and only if \( \mathcal{D}(S/I) \) is left Noetherian.

**Theorem 2.** Let \( h \neq 0 \) be a polynomial, and let \( J = hS \). Then the ring \( \mathcal{D}(J)/J\mathcal{D}(S) \) is right Noetherian if and only if \( \mathcal{D}(J)/J\mathcal{D}(S) \) is left Noetherian.

**Corollary 3.** Let \( I \) be the defining ideal of a central arrangement. Then the ring \( \mathcal{D}(S/I) \) is right Noetherian if and only if \( \mathcal{D}(S/I) \) is left Noetherian.

To prove that \( \mathcal{D}(S/I) \) is a Noetherian ring, we only need to prove that \( \mathcal{D}(S/I) \) is a right Noetherian ring.

The operator

\[
\varepsilon_m := \sum_{|\alpha|=m} \frac{m!}{\alpha!} x^\alpha \partial^\alpha
\]

is called the Euler operator of order \( m \) where \( \alpha! = (\alpha_1!)(\alpha_n!) \) for \( \alpha = (\alpha_1, \ldots, \alpha_n) \). Then \( \varepsilon_m = \varepsilon_1(\varepsilon_1 - 1) \cdots (\varepsilon_1 - m + 1) \) [2, Lemma 4.9].

3. \( n = 2 \)

In this section, we assume \( n = 2 \) and \( S = K[x, y] \). We introduce the Noetherian property of the ring \( \mathcal{D}(S/I) \simeq \mathcal{D}(I)/I\mathcal{D}(S) \). In contrast, the graded ring \( \text{Gr} \mathcal{D}(S/I) \) associated to the order filtration is not Noetherian when \( r \geq 2 \).

Put \( P_i := \frac{Q}{p_i} \) for \( i = 1, \ldots, r \), and define

\[
\delta_i := \begin{cases} 
\partial_y & \text{if } p_i = ax \ (a \in K^\times) \\
\partial_x + a_i \partial_y & \text{if } p_i = a(y - a_i x) \ (a \in K^\times).
\end{cases}
\]

Then \( \delta_i(p_j) = 0 \) if and only if \( i = j \).
Proposition 4 (Paper III, Proposition 6.7 in [1], Proposition 4.14 in [6]). For any $m \geq 1$, $\mathcal{D}^{(m)}(I)$ is a free left $S$-module with a basis
\[
\{\varepsilon_m, P_1^m \delta_1^m, \ldots, P_m^m \delta_m^m\} \text{ if } m < r - 1,
\{P_1^m \delta_1^m, \ldots, P_r^m \delta_r^m\} \text{ if } m = r - 1,
\{P_1^m \delta_1^m, \ldots, P_r^m \delta_r^m, Q\eta_{r+1}^{(m)}, \ldots, Q\eta_{m+1}^{(m)}\} \text{ if } m > r - 1,
\]
where the set $\{\delta_1^m, \ldots, \delta_r^m, \eta_{r+1}^{(m)}, \ldots, \eta_{m+1}^{(m)}\}$ forms a $K$-basis for $\sum_{|\alpha|=m} K\partial^\alpha$ if $m > r - 1$.

By Proposition 1, we have
\[
\mathcal{D}(I) = S \oplus \left( \bigoplus_{m=1}^{r-2} (S\varepsilon_m \oplus SP_1^m \delta_1^m \oplus \cdots \oplus SP_m^m \delta_m^m) \right)
\oplus \left( \bigoplus_{m \geq r-1} (SP_1^m \delta_1^m \oplus \cdots \oplus SP_r^m \delta_r^m \oplus SQ\eta_{r+1}^{(m)} \oplus \cdots \oplus SQ\eta_{m+1}^{(m)}) \right).
\]

For $i = 1, \ldots, r$, we define an additive group
\[
L_i := \mathcal{D}(I) \cap (p_1 \cdots p_i) \mathcal{D}(S).
\]

Proposition 5. For $i = 1, \ldots, r$, the additive group $L_i$ is a two-sided ideal of $\mathcal{D}(I)$.

We consider a sequence
\[
I \mathcal{D}(S) = L_r \subseteq L_{r-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = \mathcal{D}(I)
\]
of two-sided ideals of $\mathcal{D}(I)$. If $L_{i-1}/L_i$ is a right Noetherian $\mathcal{D}(I)$-module for any $i$, then $\mathcal{D}(I)/I \mathcal{D}(S)$ is a right Noetherian ring. By proving that $L_{i-1}/L_i$ is right Noetherian for all $i$, we obtain the following main theorem.

Theorem 6. The ring $\mathcal{D}(S)/I \simeq \mathcal{D}(I)/I \mathcal{D}(S)$ of differential operators of the coordinate ring of a central 2-arrangement is Noetherian (i.e., $\mathcal{D}(S)/I$ is right Noetherian and left Noetherian).

In contrast, $\text{Gr} \mathcal{D}(S)/I$ is not Noetherian when $r \geq 2$.

Remark 7. The graded ring $\text{Gr} \mathcal{D}(S)/I$ associated to the order filtration is a commutative ring. We consider the ideal $M := \langle P_i^m \delta_i^m \mid m \geq 1 \rangle$ of $\text{Gr} \mathcal{D}(S)/I$.

Assume that $M$ is finitely generated with generators $\eta_1, \ldots, \eta_k$. Then there exists a positive integer $m$ such that
\[
M = \langle \eta_1, \ldots, \eta_k \rangle \subseteq \langle P_1^m \delta_1^m, \ldots, P_m^m \delta_m^m \rangle.
\]

Since $P_1^m \delta_1^m \in M$, we can write
\[
P_1^m \delta_1^m = P_1^m \delta_1 \cdot \overline{\theta_1} + \cdots + P_1^{m-1} \delta_1 \cdot \overline{\theta_{m-1}}
\]
for some $\theta_1, \ldots, \theta_{m-1} \in \mathcal{D}(I)$.

If $\theta \in \mathcal{D}(I)$ with $\text{ord}(\theta) \leq 1$, then the polynomial degree of $\theta$ is greater than or equal to 1 by Proposition 4. Since the order of the LHS of (3.2) is $m$, there exists at least one $\theta_j$ such that the order of $\theta_j$ is greater than or equal to 1. Thus the polynomial degree of
the RHS of (3.2) is greater than \( r - 1 \). However, the polynomial degree of the LHS of (3.2) is exactly \( r - 1 \). This is a contradiction.

Hence \( M \) is not finitely generated, and thus we have proved that \( \text{Gr} \mathcal{D}(S/I) \) is not Noetherian.

4. Example

Let \( n = 2 \) and \( S = K[x, y] \). Let \( Q = xy(x - y) \) and \( I = QS \). Put \( p_1 = x, p_2 = y, p_3 = x - y \). Then \( P_1 = y(x - y) \) and \( \delta_1 = \partial_y \). We consider the right ideal \( \langle y(x - y)\partial_y^m \mid m \geq 1 \rangle \) of \( \mathcal{D}(I) \).

For \( \ell \geq 4 \), we have

\[
\begin{align*}
y(x-y)\partial_y \cdot y(x-y)\partial_y^{\ell+1} &= y^2(x-y)^2\partial_y^{\ell+2} + y(x-2y)\partial_y^{\ell+1}, \\
y(x-y)\partial_y^2 \cdot y(x-y)\partial_y^{\ell} &= y^2(x-y)^2\partial_y^{\ell+2} + 2y(x-2y)\partial_y^{\ell+1} - 2y(x-y)\partial_y^{\ell}, \\
y(x-y)\partial_y^3 \cdot y(x-y)\partial_y^{\ell-1} &= y^2(x-y)^2\partial_y^{\ell+2} + 3y(x-2y)\partial_y^{\ell+1} - 6y(x-y)\partial_y^{\ell}.
\end{align*}
\]

Thus we obtain

\[
y(x-y)\partial_y \cdot y(x-y)\partial_y^{\ell+1} - 2y(x-y)\partial_y^2 \cdot y(x-y)\partial_y^{\ell} + y(x-y)\partial_y^3 \cdot y(x-y)\partial_y^{\ell-1} = -2y(x-y)\partial_y^{\ell}.
\]

This leads to

\[
y(x-y)\partial_y^\ell \in \langle y(x-y)\partial_y^m \mid m = 1, 2, 3 \rangle
\]

since \( y(x-y)\partial_y^m \in \mathcal{D}(I) \) for any \( m \geq 1 \). We have the identity

\[
\langle y(x-y)\partial_y^m \mid m \geq 1 \rangle = \langle y(x-y)\partial_y^m \mid m = 1, 2, 3 \rangle
\]

as right ideals. Hence the right ideal \( \langle y(x-y)\partial_y^m \mid m \geq 1 \rangle \) is finitely generated.

In contrast, the right ideal \( \langle y(x-y)\partial_y^m \mid m \geq 1 \rangle \) of \( \text{Gr} \mathcal{D}(S/I) \) is not finitely generated by Remark 7.

References