THE NOETHERIAN PROPERTIES OF THE RINGS OF DIFFERENTIAL OPERATORS ON CENTRAL 2-ARRANGEMENTS

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Abstract. P. Holm began to study the ring of differential operators of the coordinate ring of a hyperplane arrangement. In this paper, we introduce Noetherian properties of the ring differential operators of the coordinate ring of a central 2-arrangement and its graded ring associated to the order filtration.

Key Words: Ring of differential operators, Noetherian property, Hyperplane arrangement.

2010 *Mathematics Subject Classification*: Primary 13N10; Secondary 32S22.

1. Introduction

For a commutative algebra R over a field K of characteristic zero, define vector spaces inductively by

$$\mathscr{D}^0(R) := \{\theta \in \mathrm{End}_K(R) \mid a \in R, \theta a - a\theta = 0\},$$
$$\mathscr{D}^m(R) := \left\{\theta \in \mathrm{End}_K(R) \mid a \in R, \theta a - a\theta \in \mathscr{D}^{m-1}(R)\right\} \quad (m \geq 1).$$

We define the ring $\mathscr{D}(R) := \bigcup_{m\geq 0} \mathscr{D}^m(R)$ of differential operators of R.

Let $S := K[x_1, \ldots, x_n]$ be the polynomial ring. The ring $\mathscr{D}(S)$ is the n-th Weyl algebra $K[x_1, \ldots, x_n]\langle \partial_1, \ldots, \partial_n\rangle$ where $\partial_i := \frac{\partial}{\partial x_i}$ (see for example [3]). We use the multi-index notetions, for example, $\partial^{\boldsymbol{\alpha}} := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ and $|\boldsymbol{\alpha}| := \alpha_1 + \cdots + \alpha_n$ for $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$. Define $\mathscr{D}^{(m)}(S) := \bigoplus_{|\boldsymbol{\alpha}|=m}$. Then $\mathscr{D}(S) = \bigoplus_{m\geq 0} \mathscr{D}^{(m)}(S)$. It is well known $\mathscr{D}(R)$ that $\mathscr{D}(R)$ is Noetherian, if R is a regular domain (see [3]).

Holm [2] showed that $\mathscr{D}(R)$ is finitely generated as a K-algebra when R is a coordinate ring of a generic hyperplane arrangement. Holm [1] also proved that the ring of differential operators of a central 2-arrangement is a free S-module, and gave a basis of it. We can write any ellement in $\mathscr{D}(R)$ as a linearly combination of this basis ellements.

In this paper, we introduce the Noetherian property of $\mathscr{D}(R)$ when R is the coordinate ring of a central arrangement. In particular, the case $n = 2$, $\mathscr{D}(R)$ is a Noetherian ring . We give an example of a finitely generated ideal in the end of this paper.

The details of this note are in [4].

2. Hyperplane arrangement

In this section, we fix some notation, and we introduce some properties of the ring of differential operators of a central arrangement. Let $\mathscr{A} = \{H_i \mid i = 1, \ldots, r\}$ be a central (hyperplane) arrangement (i.e., every hyperplane in $\mathscr{A}$ contains the origin) in K^n. Fix a

The detailed version of this paper will be submitted for publication elsewhere.

polynomial p_i with $\ker(p_i) = H_i$, and put $Q := p_1 \cdots p_r$. Thus Q is a product of certain homogeneous polynomials of degree 1. Let I denote the principal ideal of S generated by Q. Then S/I is the coordinate ring of the hyperplane arrangement defined by Q.

For any ideal J of S, we define an S-submodule $\mathscr{D}^{(m)}(J)$ of $\mathscr{D}^{(m)}(S)$ and a subring $\mathscr{D}(J)$ of $\mathscr{D}(S)$ by

$$\mathscr{D}^{(m)}(J) := \{\theta \in \mathscr{D}^{(m)}(S) \mid \theta(J) \subseteq J\},$$
$$\mathscr{D}(J) := \{\theta \in \mathscr{D}(S) \mid \theta(J) \subseteq J\}.$$

Holm [2] proved the following proposition.

Proposition 1 (Proposition 4.3 in [2])**.**

$$\mathscr{D}(I) = \bigoplus_{m \geq 0} \mathscr{D}^{(m)}(I).$$

There is a ring isomorphism $\mathscr{D}(S/J) \simeq \mathscr{D}(J)/J\mathscr{D}(S)$ (see [3, Theorem 15.5.13]). Thus we can express $\mathscr{D}(S/J)$ as a subquotient of Weyl algebra.

We can prove that $\mathscr{D}(J)/J\mathscr{D}(S)$ is right Noetherian if and only if $\mathscr{D}(J)/J\mathscr{D}(S)$ is left Noetherian when $J \neq 0$ is a principal ideal. Therefore we conclude that $\mathscr{D}(S/I)$ is right Noetherian if and only if $\mathscr{D}(S/I)$ is left Noetherian.

Theorem 2. *Let $h \neq 0$ be a polynomial, and let $J = hS$. Then the ring $\mathscr{D}(J)/J\mathscr{D}(S)$ is right Noetherian if and only if $\mathscr{D}(J)/J\mathscr{D}(S)$ is left Noetherian.*

Corollary 3. *Let I be the defining ideal of a central arrangement. Then the ring $\mathscr{D}(S/I)$ is right Noetherian if and only if $\mathscr{D}(S/I)$ is left Noetherian.*

To prove that $\mathscr{D}(S/I)$ is a Noetherian ring, we only need to prove that $\mathscr{D}(S/I)$ is a right Noetherian ring.

The operator

$$\varepsilon_m := \sum_{|\boldsymbol{\alpha}|=m} \frac{m!}{\boldsymbol{\alpha}!} x^{\boldsymbol{\alpha}} \partial^{\boldsymbol{\alpha}}$$

is called the Euler operator of order m where $\boldsymbol{\alpha}! = (\alpha_1!) \cdots (\alpha_n!)$ for $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_n)$. Then $\varepsilon_m = \varepsilon_1(\varepsilon_1 - 1) \cdots (\varepsilon_1 - m + 1)$ [2, Lemma 4.9].

3. $n = 2$

In this section, we assume $n = 2$ and $S = K[x, y]$. We introduce the Noetherian property of the ring $\mathscr{D}(S/I) \simeq \mathscr{D}(I)/I\mathscr{D}(S)$. In contrast, the graded ring $\operatorname{Gr} \mathscr{D}(S/I)$ associated to the order filtration is not Noetherian when $r \geq 2$.

Put $P_i := \frac{Q}{p_i}$ for $i = 1, \dots, r$, and define

$$\delta_i := \begin{cases} \partial_y & \text{if } p_i = ax \quad (a \in K^\times) \\ \partial_x + a_i \partial_y & \text{if } p_i = a(y - a_i x) \quad (a \in K^\times). \end{cases}$$

Then $\delta_i(p_j) = 0$ if and only if $i = j$.

Proposition 4 (Paper III, Proposition 6.7 in [1], Proposition 4.14 in [6]). *For any $m \geq 1$, $\mathscr{D}^{(m)}(I)$ is a free left S-module with a basis*

$$\{\varepsilon_m, P_1\delta_1^m, \ldots, P_m\delta_m^m\} \text{ if } m < r-1,$$
$$\{P_1\delta_1^m, \ldots, P_r\delta_r^m\} \text{ if } m = r-1,$$
$$\{P_1\delta_1^m, \ldots, P_r\delta_r^m, Q\eta_{r+1}^{(m)}, \ldots, Q\eta_{m+1}^{(m)}\} \text{ if } m > r-1,$$

where the set $\{\delta_1^m, \ldots, \delta_r^m, \eta_{r+1}^{(m)}, \ldots, \eta_{m+1}^{(m)}\}$ forms a K-basis for $\sum_{|\boldsymbol{\alpha}|=m} K\partial^{\boldsymbol{\alpha}}$ if $m > r-1$.

By Proposition 1, we have

$$\mathscr{D}(I) = S \oplus \left(\bigoplus_{m=1}^{r-2} \left(S\varepsilon_m \oplus SP_1\delta_1^m \oplus \cdots \oplus SP_m\delta_m^m\right)\right) \oplus \left(\bigoplus_{m \geq r-1} \left(SP_1\delta_1^m \oplus \cdots \oplus SP_r\delta_r^m \oplus SQ\eta_{r+1}^{(m)} \oplus \cdots \oplus SQ\eta_{m+1}^{(m)}\right)\right).$$

For $i = 1, \ldots, r$, we define an additive group

$$L_i := \mathscr{D}(I) \cap (p_1 \cdots p_i)\mathscr{D}(S).$$

Proposition 5. *For $i = 1, \ldots, r$, the additive group L_i is a two-sided ideal of $\mathscr{D}(I)$.*

We consider a sequence

$$I\mathscr{D}(S) = L_r \subseteq L_{r-1} \subseteq \cdots \subseteq L_1 \subseteq L_0 = \mathscr{D}(I) \tag{3.1}$$

of two-sided ideals of $\mathscr{D}(I)$. If L_{i-1}/L_i is a right Noetherian $\mathscr{D}(I)$-module for any i, then $\mathscr{D}(I)/I\mathscr{D}(S)$ is a right Noetherian ring. By proving that L_{i-1}/L_i is right Noetherian for all i, we obtain the following main theorem.

Theorem 6. *The ring $\mathscr{D}(S/I) \simeq \mathscr{D}(I)/I\mathscr{D}(S)$ of differential operators of the coordinate ring of a central 2-arrangement is Noetherian (i.e., $\mathscr{D}(S/I)$ is right Noetherian and left Noetherian).*

In contrast, $\operatorname{Gr}\mathscr{D}(S/I)$ is not Noetherian when $r \geq 2$.

Remark 7. The graded ring $\operatorname{Gr}\mathscr{D}(S/I)$ associated to the order filtration is a commutative ring. We consider the ideal $M := \langle \overline{P_1\delta_1^m} \mid m \geq 1\rangle$ of $\operatorname{Gr}\mathscr{D}(S/I)$.

Assume that M is finitely generated with generators $\eta_1, \ldots, \eta_\ell$. Then there exists a positive integer m such that

$$M = \langle \eta_1, \ldots, \eta_\ell \rangle \subseteq \langle \overline{P_1\delta_1}, \ldots, \overline{P_1\delta_1^{m-1}} \rangle.$$

Since $\overline{P_1\delta_1^m} \in M$, we can write

$$\overline{P_1\delta_1^m} = \overline{P_1\delta_1} \cdot \overline{\theta_1} + \cdots + \overline{P_1\delta_1^{m-1}} \cdot \overline{\theta_{m-1}} \tag{3.2}$$

for some $\theta_1, \ldots, \theta_{m-1} \in \mathscr{D}(I)$.

If $\theta \in \mathscr{D}(I)$ with $\operatorname{ord}(\theta) \leq 1$, then the polynomial degree of θ is greater than or equal to 1 by Proposition 4. Since the order of the LHS of (3.2) is m, there exists at least one θ_j such that the order of θ_j is greater than or equal to 1. Thus the polynomial degree of the RHS of (3.2) is greater than $r-1$. However, the polynomial degree of the LHS of (3.2) is exactly $r-1$. This is a contradiction.

Hence M is not finitely generated, and thus we have proved that $\mathrm{Gr}\,\mathscr{D}(S/I)$ is not Noetherian.

4. Exaple

Let $n=2$ and $S=K[x,y]$. Let $Q=xy(x-y)$ and $I=QS$. Put $p_1=x, p_2=y, p_3=x-y$. Then $P_1=y(x-y)$ and $\delta_1=\partial_y$. We consider the right ideal $\langle y(x-y)\partial_y^m \mid m\geq 1\rangle$ of $\mathscr{D}(I)$.

For $\ell\geq 4$, we have

$$y(x-y)\partial_y\cdot y(x-y)\partial_y^{\ell+1}=y^2(x-y)^2\partial_y^{\ell+2}+y(x-2y)\partial_y^{\ell+1},$$
$$y(x-y)\partial_y^2\cdot y(x-y)\partial_y^{\ell}=y^2(x-y)^2\partial_y^{\ell+2}+2y(x-2y)\partial_y^{\ell+1}-2y(x-y)\partial_y^{\ell},$$
$$y(x-y)\partial_y^3\cdot y(x-y)\partial_y^{\ell-1}=y^2(x-y)^2\partial_y^{\ell+2}+3y(x-2y)\partial_y^{\ell+1}-6y(x-y)\partial_y^{\ell}.$$

Thus we obtain

$$y(x-y)\partial_y\cdot y(x-y)\partial_y^{\ell+1}-2y(x-y)\partial_y^2\cdot y(x-y)\partial_y^{\ell}+y(x-y)\partial_y^3\cdot y(x-y)\partial_y^{\ell-1}=-2y(x-y)\partial_y^{\ell}.$$

This leads to

$$y(x-y)\partial_y^{\ell}\in\langle y(x-y)\partial_y^m \mid m=1,2,3\rangle$$

since $y(x-y)\partial_y^m\in\mathscr{D}(I)$ for any $m\geq 1$. We have the identity

$$\langle y(x-y)\partial_y^m \mid m\geq 1\rangle=\langle y(x-y)\partial_y^m \mid m=1,2,3\rangle$$

as right ideals. Hence the right ideal $\langle y(x-y)\partial_y^m \mid m\geq 1\rangle$ is finitely generated.

In contrast, the right ideal $\langle \overline{y(x-y)\partial_y^m} \mid m\geq 1\rangle$ of $\mathrm{Gr}\,\mathscr{D}(S/I)$ is not finitely generated by Remark 7.

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