TILTING MODULES ARISING FROM TWO-TERM TILTING COMPLEXES

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Abstract. We see that every two-term tilting complex over an Artin algebra has a tilting module over a certain factor algebra as a homology group. Also, we determine the endomorphism algebra of such a homology group, which is given as a certain factor algebra of the endomorphism algebra of the two-term tilting complex. Thus, every derived equivalence between Artin algebras given by a two-term tilting complex induces a derived equivalence between the corresponding factor algebras.

Let A be an Artin algebra. We denote by mod-A the category of finitely generated right A-modules and by $\mathcal{P}_A$ the full subcategory of mod-A consisting of projective modules.

Definition 1. A pair $(\mathcal{T}, \mathcal{F})$ of full subcategories $\mathcal{T}$, $\mathcal{F}$ in mod-A is said to be *a torsion theory* for mod-A if the following conditions are satisfied:

(1) $\mathcal{T} \cap \mathcal{F} = \{0\}$;
(2) $\mathcal{T}$ is closed under factor modules;
(3) $\mathcal{F}$ is closed under submodules; and
(4) for any $X \in$ mod-A, there exists an exact sequence $0 \to X' \to X \to X'' \to 0$ with $X' \in \mathcal{T}$ and $X'' \in \mathcal{F}$.

If $\mathcal{T}$ is stable under the Nakayama functor ν, then $(\mathcal{T}, \mathcal{F})$ is said to be *a stable torsion theory* for mod-A.

Let $T^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ be a two-term complex:

$$T^\bullet : \cdots \to 0 \to T^{-1} \xrightarrow{\alpha} T^0 \to 0 \to \cdots ,$$

and set the following subcategories in mod-A:

$\mathcal{T}(T^\bullet) = \mathrm{Ker}\ \mathrm{Hom}_{\mathcal{K}(A)}(T^\bullet[-1], -) \cap \text{mod-}A,$
$\mathcal{F}(T^\bullet) = \mathrm{Ker}\ \mathrm{Hom}_{\mathcal{K}(A)}(T^\bullet, -) \cap \text{mod-}A.$

Proposition 2 ([1, Propositions 5.5 and 5.7])**.** *The following are equivalent.*

(1) $T^\bullet$ *is a tilting complex.*
(2) $(\mathcal{T}(T^\bullet), \mathcal{F}(T^\bullet))$ *is a stable torsion theory for* mod-A.

Furthermore, if these equivalent conditions hold, then the following hold.

(1) $\mathcal{T}(T^\bullet) = \mathrm{gen}(\mathrm{H}^0(T^\bullet))$*, the generated class by* $\mathrm{H}^0(T^\bullet)$*, and* $\mathrm{H}^0(T^\bullet)$ *is Ext-projective in* $\mathcal{T}(T^\bullet)$.
(2) $\mathcal{F}(T^\bullet) = \mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet))$*, the cogenerated class by* $\mathrm{H}^{-1}(\nu T^\bullet)$ *and* $\mathrm{H}^{-1}(\nu T^\bullet)$ *is Ext-injective in* $\mathcal{F}(T^\bullet)$.

The detailed version of this note has been submitted for publication elsewhere.

Conversely, let $(\mathcal{T}, \mathcal{F})$ be a stable torsion theory for mod-A.

Proposition 3 ([1, Theorem 5.8]). *Assume that there exist $X \in \mathcal{T}$ and $Y \in \mathcal{F}$ satisfying the following conditions:*

(1) $\mathcal{T} = \mathrm{gen}(X)$ *and X is Ext-projective in $\mathcal{T}$; and*
(2) $\mathcal{F} = \mathrm{cog}(Y)$ *and Y is Ext-injective in $\mathcal{F}$.*

Let $P_X^\bullet$ be a minimal projective presentation of X and $I_Y^\bullet$ be a minimal injective presentation of Y, and set $T_{X,Y}^\bullet = P_X^\bullet \oplus \nu^{-1} I_Y^\bullet[1]$. Then $T_{X,Y}^\bullet \in \mathcal{K}^{\mathrm{b}}(\mathcal{P}_A)$ is a tilting complex such that $\mathcal{T} = \mathcal{T}(T_{X,Y}^\bullet)$ and $\mathcal{F} = \mathcal{F}(T_{X,Y}^\bullet)$.

Let $T^\bullet$ be a two-term tilting complex. We set $\mathfrak{a} = \mathrm{ann}_A(\mathrm{H}^0(T^\bullet))$, the annihilator of $\mathrm{H}^0(T^\bullet)$. Note that $\mathrm{H}^0(T^\bullet)$ is faithful in mod-$A/\mathfrak{a}$ and the canonical full embedding mod-$A/\mathfrak{a} \hookrightarrow$ mod-A induces $\mathrm{gen}(\mathrm{H}^0(T^\bullet)_{A/\mathfrak{a}}) = \mathrm{gen}(\mathrm{H}^0(T^\bullet)_A)$ which is closed under extensions. Thus, the next lemma follows from Proposition 2.

Lemma 4. *The following hold.*

(1) $\mathrm{proj\ dim}\ \mathrm{H}^0(T^\bullet)_{A/\mathfrak{a}} \leq 1$.
(2) $\mathrm{Ext}^1_{A/\mathfrak{a}}(\mathrm{H}^0(T^\bullet), \mathrm{H}^0(T^\bullet)) = 0$.
(3) *There exists an exact sequence $0 \to A/\mathfrak{a} \to X^0 \to X^1 \to 0$ in* mod-$A/\mathfrak{a}$ *such that $X^0 \in \mathrm{add}(\mathrm{H}^0(T^\bullet)_{A/\mathfrak{a}})$ and $X^1 \in \mathrm{gen}(\mathrm{H}^0(T^\bullet)_{A/\mathfrak{a}})$ which is Ext-projective in $\mathrm{gen}(\mathrm{H}^0(T^\bullet)_{A/\mathfrak{a}})$.*

We set $\mathfrak{a}' = \mathrm{ann}_A(\mathrm{H}^{-1}(\nu T^\bullet))$, the annihilator of $\mathrm{H}^{-1}(\nu T^\bullet)$. The next lemma follows by the dual arguments of Lemma 4

Lemma 5. *The following hold.*

(1) $\mathrm{inj\ dim}\ \mathrm{H}^{-1}(\nu T^\bullet)_{A/\mathfrak{a}'} \leq 1$.
(2) $\mathrm{Ext}^1_{A/\mathfrak{a}'}(\mathrm{H}^{-1}(\nu T^\bullet), \mathrm{H}^{-1}(\nu T^\bullet)) = 0$.
(3) *There exists an exact sequence $0 \to Y^1 \to Y^0 \to A/\mathfrak{a}' \to 0$ in* mod-$A/\mathfrak{a}'$ *such that $Y^0 \in \mathrm{add}(\mathrm{H}^{-1}(\nu T^\bullet)_{A/\mathfrak{a}'})$ and $Y^1 \in \mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet)_{A/\mathfrak{a}'})$ which is Ext-injective in $\mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet)_{A/\mathfrak{a}'})$.*

Let X be the direct sum of all indecomposable non-projective Ext-projective modules in $\mathrm{gen}(\mathrm{H}^0(T^\bullet))$ which are not contained in $\mathrm{add}(\mathrm{H}^0(T^\bullet))$. Then $\mathrm{add}(\mathrm{H}^0(T^\bullet) \oplus X)$ coincides with the class of all Ext-projective modules in $\mathrm{gen}(\mathrm{H}^0(T^\bullet))$. Also, since $\mathrm{gen}(\mathrm{H}^0(T^\bullet)) = \mathrm{gen}(\mathrm{H}^0(T^\bullet) \oplus X)$, the pair $(\mathrm{gen}(\mathrm{H}^0(T^\bullet) \oplus X), \mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet))$ is a stable torsion theory in mod-A. Let $P^\bullet$ be the minimal projective presentation of $\mathrm{H}^0(T^\bullet) \oplus X$ and $I^\bullet$ be the minimal injective presentation of $\mathrm{H}^{-1}(\nu T^\bullet)$, and set $U^\bullet = P^\bullet \oplus \nu^{-1} I^\bullet[1]$. Then $U^\bullet$ is a tilting complex such that $\mathcal{T}(U^\bullet) = \mathrm{gen}(\mathrm{H}^0(T^\bullet) \oplus X)$ and $\mathcal{F}(U^\bullet) = \mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet))$ by Proposition 3. Note that the stable torsion theory induced by $U^\bullet$ coincides with that of $T^\bullet$. From this fact, we can prove that $\mathrm{add}(\mathrm{H}^0(U^\bullet)) = \mathrm{add}(\mathrm{H}^0(T^\bullet))$. Since there exist the inclusions $\mathrm{add}(\mathrm{H}^0(T^\bullet)) \subset \mathrm{add}(\mathrm{H}^0(T^\bullet) \oplus X) \subset \mathrm{add}(\mathrm{H}^0(U^\bullet))$, we conclude that $\mathrm{add}(\mathrm{H}^0(T^\bullet)) = \mathrm{add}(\mathrm{H}^0(T^\bullet) \oplus X)$. Thus, we have the next lemma.

Lemma 6. *For any $M, N \in$ mod-A, the following hold.*

(1) $M \in \mathrm{add}(\mathrm{H}^0(T^\bullet))$ *if and only if M is* Ext-*projective in* $\mathrm{gen}(\mathrm{H}^0(T^\bullet))$.
(2) $N \in \mathrm{add}(\mathrm{H}^{-1}(\nu T^\bullet))$ *if and only if N is* Ext-*injective in* $\mathrm{cog}(\mathrm{H}^{-1}(\nu T^\bullet))$.

The next theorem is a direct consequence of the previous three lemmas.

Theorem 7. *The following hold.*

(1) $\mathrm{H}^0(T^\bullet)$ *is a tilting module in* mod-$A/\mathfrak{a}$.
(2) $\mathrm{H}^{-1}(\nu T^\bullet)$ *is a cotilting module in* mod-$A/\mathfrak{a}'$, *i.e.,* $D(\mathrm{H}^{-1}(\nu T^\bullet))$ *is a tilting module in* mod-$(A/\mathfrak{a}')^{\mathrm{op}}$.

We determine the endomorphism algebras of $\mathrm{H}^0(T^\bullet)$. Set $B = \mathrm{End}_{\mathcal{K}(A)}(T^\bullet)$. Since there exists a surjective algebra homomorphism

$$\theta : B \to \mathrm{End}_{A/\mathfrak{a}}(\mathrm{H}^0(T^\bullet)),$$

which is induced by the functor $\mathrm{H}^0(-)$, we have an algebra isomorphism

$$\mathrm{End}_{A/\mathfrak{a}}(\mathrm{H}^0(T^\bullet)) \cong B/\mathrm{Ker}\ \theta.$$

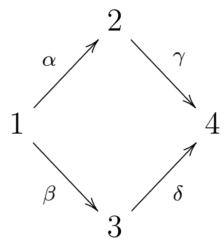
Also, we can prove that $\mathrm{Ker}\ \theta = \mathrm{ann}_B(\mathrm{Hom}_{\mathcal{K}(A)}(A, T^\bullet)) = \mathrm{ann}_B(\mathrm{H}^0(T^\bullet))$. Thus, we have the next theorem.

Theorem 8. *We have the following algebra isomorphisms.*

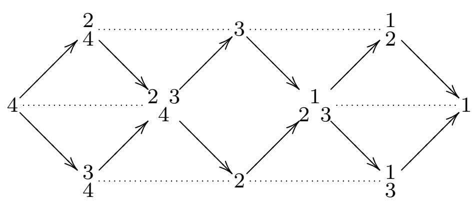
(1) $\mathrm{End}_{A/\mathfrak{a}}(\mathrm{H}^0(T^\bullet)) \cong B/\mathfrak{b}$, *where* $\mathfrak{b} = \mathrm{ann}_B(\mathrm{H}^0(T^\bullet))$.
(2) $\mathrm{End}_{A/\mathfrak{a}'}(\mathrm{H}^{-1}(\nu T^\bullet)) \cong B/\mathfrak{b}'$, *where* $\mathfrak{b}' = \mathrm{ann}_B(\mathrm{H}^{-1}(\nu T^\bullet))$.

As the final of this note, we demonstrate our results through an example.

Example 9. Let A be the path algebra defined by the quiver



with relations $\alpha\gamma = \beta\delta = 0$. We denote by e_i the empty path corresponding to the vertex $i = 1, \cdots, 4$. The Auslander–Reiten quiver of A is given by the following:



where each indecomposable module is represented by its composition factors. It is not difficult to see that the following pair gives a stable torsion theory for mod-A:

$$\mathcal{T} = \left\{ \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}, \begin{smallmatrix} 1 \\ 3 \end{smallmatrix}, 1 \right\} \text{ and } \mathcal{F} = \left\{ 4, \begin{smallmatrix} 2 \\ 4 \end{smallmatrix}, \begin{smallmatrix} 3 \\ 4 \end{smallmatrix}, \begin{smallmatrix} 2 & & 3 \\ & 4 & \end{smallmatrix}, 3, 2 \right\},$$

where $\mathcal{T}$ is a torsion class and $\mathcal{F}$ is a torsion-free class. We set

$$X = \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix}, \quad Y = \begin{smallmatrix} 2 & & 3 \\ & 4 & \end{smallmatrix} \oplus 3 \oplus 2.$$

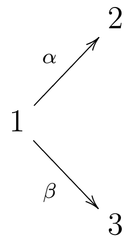
Then $\mathcal{T} = \mathrm{gen}(X)$ and X is Ext-projective in $\mathcal{T}$, and $\mathcal{F} = \mathrm{cog}(Y)$ and Y is Ext-injective in $\mathcal{F}$. According to Proposition 3, we have a two-term tilting complex $T^\bullet = T_1^\bullet \oplus T_2^\bullet \oplus T_3^\bullet \oplus T_4^\bullet$, where

$$T_1^\bullet = 0 \to \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix}, \quad T_2^\bullet = \begin{smallmatrix} 2 \\ 4 \end{smallmatrix} \to \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix}, \quad T_3^\bullet = \begin{smallmatrix} 3 \\ 4 \end{smallmatrix} \to \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix}, \quad T_4^\bullet = \begin{smallmatrix} 4 \end{smallmatrix} \to 0.$$

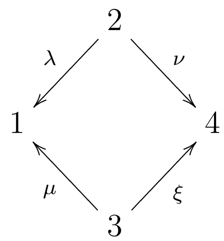
Thus, we have

$$\mathrm{H}^0(T^\bullet) = \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$$

as a right A-module. Since $\mathfrak{a} = \mathrm{ann}_A(\mathrm{H}^0(T^\bullet))$ is a two-sided ideal generated by e_4, γ, δ, the factor algebra $A/\mathfrak{a}$ is defined by the quiver



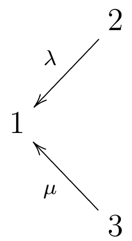
without relations. Next, it is not difficult to see that $B = \mathrm{End}_{\mathcal{K}(A)}(T^\bullet)$ is defined by the quiver



without relations. Then we have

$$\begin{aligned} \mathrm{Hom}_{\mathcal{K}(A)}(A, T^\bullet) &= \bigoplus_{i=1}^{4} \mathrm{Hom}_{\mathcal{K}(A)}(e_i A, T^\bullet) \\ &= \begin{smallmatrix} & 1 & \\ 2 & & 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 3 \end{smallmatrix} \oplus \begin{smallmatrix} 1 \\ 2 \end{smallmatrix} \oplus 0 \end{aligned}$$

as a left B-module. Thus, $\mathfrak{b} = \mathrm{ann}_B(\mathrm{Hom}_{\mathcal{K}(A)}(A, T^\bullet))$ is a two-sided ideal generated by ν, ξ and the empty path corresponding to the vertex 4. Therefore, the factor algebra $B/\mathfrak{b}$ is defined by the quiver



without relations. It follows by Theorems 7 and 8 that $A/\mathfrak{a}$ and $B/\mathfrak{b}$ are derived equivalent to each other.

References

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