HOCHSCHILD COHOMOLOGY OF QUIVER ALGEBRAS DEFINED BY TWO CYCLES AND A QUANTUM-LIKE RELATION

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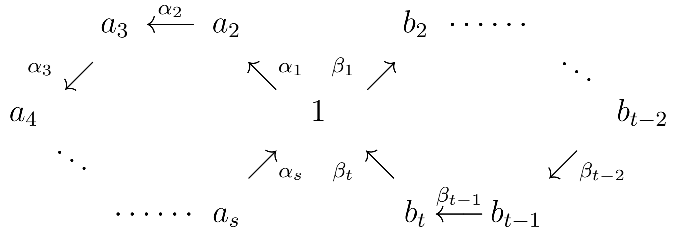
Abstract. This paper is based on my talk given at the Symposium on Ring Theory and Representation Theory held at Okayama University, Japan, 25–27 September 2011. In this paper, we consider quiver algebras A_q over a field k defined by two cycles and a quantum-like relation depending on a non-zero element q in k. We determine the ring structure of the Hochschild cohomology ring of A_q modulo nilpotence and give a necessary and sufficient condition for A_q to satisfy the finiteness condition given in [19].

1. Introduction

Let A be an indecomposable finite dimensional algebra over a field k. We denote by A^e the enveloping algebra $A \otimes_k A^{op}$ of A, so that left A^e-modules correspond to A-bimodules. The Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n \geq 0} \mathrm{Ext}^n_{A^e}(A, A)$ with Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative ring, that is, for homogeneous elements $\eta \in \mathrm{HH}^m(A)$ and $\theta \in \mathrm{HH}^n(A)$, we have $\eta\theta = (-1)^{mn}\theta\eta$. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ which is generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $\mathrm{HH}^*(A)$, so that the maximal ideals of $\mathrm{HH}^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

Let q be a non-zero element in k and s, t integers with $s, t \geq 1$. We consider the quiver algebra $A_q = kQ/I_q$ defined by the two cycles Q with $s+t-1$ vertices and $s+t$ arrows as follows:



and the ideal I_q of kQ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}$$

for $a, b \geq 2$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$. We denote the trivial path at the vertex $a(i)$ and at the vertex $b(j)$ by $e_{a(i)}$ and by $e_{b(j)}$ respectively. We regard the numbers i in the subscripts of $e_{a(i)}$ modulo s and j in the subscripts of $e_{b(j)}$ modulo t. In this paper, we describe the ring structure of $\mathrm{HH}^*(A_q)/\mathcal{N}$.

In [17], Snashall and Solberg used the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$ to define a support variety for any finitely generated module over A. This led us to consider the structure of $\mathrm{HH}^*(A)/\mathcal{N}$. In [17], Snashall and Solberg conjectured that $\mathrm{HH}^*(A)/\mathcal{N}$ is always finitely generated as a k-algebra. But a counterexample to this conjecture was given by Snashall [16] and Xu [21]. This example makes us consider whether we can give necessary and sufficient conditions on a finite dimensional algebra A for $\mathrm{HH}^*(A)/\mathcal{N}$ to be finitely generated as a k-algebra.

On the other hand, in the theory of support varieties, it is interesting to know when the variety of a module is trivial. In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the necessary and sufficient conditions on a module for it to have trivial variety under some finiteness conditions on A. In [19], Solberg gave a condition which is equivalent to the finiteness conditions. In the paper, we show that A_q satisfies the finiteness condition given in [19] if and only if q is a root of unity.

The content of the paper is organized as follows. In Section 1 we deal with the definition of the support variety given in [17] and precedent results about the Hochschild cohomology ring modulo nilpotence. In Section 2, we describe the finiteness condition given in [19] and introduce precedent results about this condition. In Section 3, we determine the Hochschild cohomology ring of A_q modulo nilpotence and show that A_q satisfies the finiteness condition if and only if q is a root of unity.

2. Support variety

In [17], Snasall and Solberg defined the support variety of a finitely generated A-module M over a noetherian commutative graded subalgebra H of $\mathrm{HH}^*(A)$ with $H^0 = \mathrm{HH}^0(A)$. In this paper, we consider the case $H = \mathrm{HH}^*(A)$.

Definition 1 ([17])**.** The support variety of M is given by

$$V(M) = \{m \in \operatorname{MaxSpec} \mathrm{HH}^*(A)/\mathcal{N} |\, \operatorname{AnnExt}^*_A(M, M) \subseteq m'\}$$

where $\operatorname{AnnExt}^*_A(M, M)$ is the annihilator of $\operatorname{Ext}^*_A(M, M)$, m' is the pre-image of m for the natural epimorphism and the $\mathrm{HH}^*(A)$-action on $\operatorname{Ext}^*_A(A, A)$ is given by the graded algebra homomorphism $\mathrm{HH}^*(A) \xrightarrow{-\otimes M} \operatorname{Ext}^*_A(M, M)$.

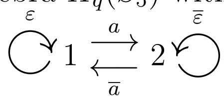
Since A is indecomposable, we have that $\mathrm{HH}^0(A)$ is a local ring. Thus $\mathrm{HH}^*(A)/\mathcal{N}$ has a unique maximal graded ideal $m_{gr} = \langle \operatorname{rad} \mathrm{HH}^*(A), \mathrm{HH}^{\geq 1}(A)\rangle/\mathcal{N}$. We say that the variety of M is trivial if $V(M) = \{m_{gr}\}$.

In [16], Snashall gave the following question.
Question ([16]). Whether we can give necessary and sufficient conditions on a finite dimensional algebra for the Hochschild cohomology ring modulo nilpotence to be finitely generated as a k-algebra.

With respect to sufficient condition, it is shown that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as a k-algebra for various classes of algebras by many authors as follows:

(1) In [6], [20], Evens and Venkov showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for any block of a group ring of a finite group.
(2) In [7], Friedlander and Suslin showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for any block of a finite dimensional cocommutative Hopf algebra.
(3) In [9], Green, Snashall and Solberg showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional self-injective algebras of finite representation type over an algebraically closed field.

(4) In [10], Green, Snashall and Solberg showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional monomial algebras.
(5) In [11], Happel showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for finite dimensional algebras of finite global dimension.
(6) In [15], Schroll and Snashall showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for the principal block of the Heche algebra $H_q(S_5)$ with $q = -1$ defined by the quiver

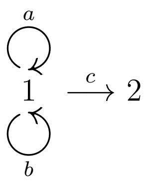


and the ideal I of kQ generated by

$$\alpha\bar{\varepsilon}, \bar{\alpha}\varepsilon, \bar{\varepsilon}\bar{\alpha}, \varepsilon^2 - \alpha\bar{\alpha}, \bar{\varepsilon}^2 - \bar{\alpha}\alpha.$$

(7) In [18], Snashall and Taillefer showed that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated for a class of special biserial algebras.
(8) In [12], Koenig and Nagase produced many examples of finite dimensional algebras with a stratifying ideal for which $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as a k-algebra.
(9) In [16] and [21], Snashall and Xu gave the example of a finite dimensional algebra for which $\mathrm{HH}^*(A)/\mathcal{N}$ is not a finitely generated k-algebra.

Example 2. ([16, Example 4.1]) Let $A = kQ/I$ where Q is the quiver



and $I = \langle a^2, b^2, ab - ba, ac\rangle$. Then $\mathrm{HH}^*(A)/\mathcal{N}$ is not finitely generated as a k-algebra.

Xu showed this in the case $\operatorname{char} k = 2$ in [21].

3. Finiteness condition

In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer gave the following two conditions **(Fg1)** and **(Fg2)** for an algebra A and a graded subalgebra H of $\mathrm{HH}^*(A)$.

(Fg1) H is a commutative Noetherian algebra with $H^0 = \mathrm{HH}^0(A)$.
(Fg2) $\mathrm{Ext}^*_A(A/\mathrm{rad}\, A, A/\mathrm{rad}\, A)$ is a finitely generated H-module.

In [19], Solberg showed that the finiteness conditions are equivalent to the following condition.

(Fg) $\mathrm{HH}^*(A)$ is Noetherian and $\mathrm{Ext}^*_A(A/\mathrm{rad}A, A/\mathrm{rad}A)$ is a finitely generated $\mathrm{HH}^*(A)$-module.

In [4], under the finiteness condition **(Fg)**, some geometric properties of the support variety and some representation theoretic properties are related. In particular, the following theorem hold.

Theorem 3 ([4, Theorem 2.5]). *Suppose that A satisfies* **(Fg)**.

(a) *A is Gorenstein, that is, A has finite injective dimension both as a left A-module and as a right A-module.*

(b) *The following are equivalent for an A-module M.*
  (i) *The variety of M is trivial.*
  (ii) *The projective dimension of M is finite.*
  (iii) *The injective dimension of M is finite.*

There are some papers which deal with the finiteness condition **(Fg)** as follows.

(1) In [2], Bergh and Oppermann show that a codimension n quantum complete intersection satisfies **(Fg)** if and only if all the commutators q_{ij} are roots of unity.

Definition 4. Let n be integer with $n \geq 1$, a_i integer with $a_i \geq 2$ for $1 \leq i \leq n$, and q_{ij} a non-zero element in k for every $1 \leq i < j \leq n$. A codimension n quantum complete intersection is defined by

$$k\langle x_1, \ldots, x_n \rangle / I$$

where I generated by

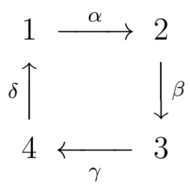
$$x_i^{a_i}, x_j x_i - q_{ij} x_i x_j \quad \text{for } 1 \leq i < j \leq n.$$

(2) In [5], Erdmann and Solberg gave the necessary and sufficient conditions on a Koszul algebra for it to satisfy **(Fg)**.

Theorem 5 ([5, Theorem 1.3])**.** *Let A be a finite dimensional Koszul algebra over an algebraically closed field, and let $E(A) = \operatorname{Ext}_A^*(A/\operatorname{rad} A, A/\operatorname{rad} A)$. A satisfies* **(Fg)** *if and only if $Z_{gr}(E(A))$ is Noetherian and $E(A)$ is a finitely generated $Z_{gr}(E(A))$-module.*

(3) In [8], Furuya and Snashall provided examples of (D, A)-stacked monomial algebras which are not self-injective but satisfy **(Fg)**.

Example 6. ([8, Example 3.2]) Let Q be the quiver



and I the ideal of kQ generated by

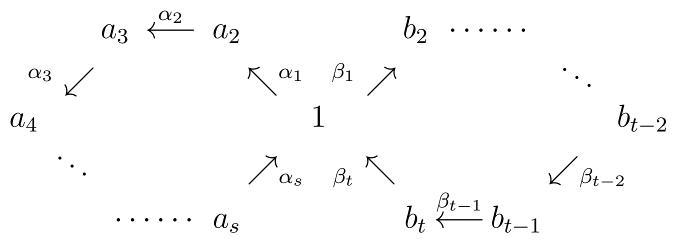
$$\alpha\beta\gamma\delta\alpha\beta, \gamma\delta\alpha\beta\gamma\delta.$$

Then, $A = kQ/\mathcal{I}$ is not self-injective but satisfies **(Fg)**.

(4) In [15], Schroll and Snashall show that **(Fg)** hold for the principal block of the Heche algebra $H_q(S_5)$ with $q = -1$.

4. Quiver algebras defined by two cycles and a quantum-like relation

In this section, we consider the quiver algebras $A_q = kQ/I_q$ defined by the quiver Q as follows:



and the ideal I_q of kQ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}$$

for $a, b \geq 2$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$, and q is non-zero element in k. Paths are written from right to left.

In the case $s = t = 1$, A_q is called a quantum complete intersection (cf. [1]). In this case, when $a = b = 2$, the Hochschild cohomology ring $\mathrm{HH}^*(A_q)$ of A_q was described by Buchweitz, Green, Madsen and Solberg [3] for any $q \in k$. Moreover, in the case where $s = t = 1$, $a, b \geq 2$, Bergh and Erdmann [1] determined $\mathrm{HH}^*(A_q)$ if q is not a root of unity. And in the same case, Bergh and Oppermann [2] show that A_q satisfies **(Fg)** if and only if q is a root of unity. In [4], Erdmann, Holloway, Snashall, Solberg and Taillefer describe that if an algebra A satisfies **(Fg)** then $\mathrm{HH}^*(A)$ is a finitely generated k-algebra. Therefore, we consider the case where $s \geq 2$ or $t \geq 2$.

In this paper, we determine the Hochschild cohomology ring of A_q modulo nilpotence $\mathrm{HH}^*(A_q)/\mathcal{N}$ and show that A_q satisfies **(Fg)** if and only if q is a root of unity.

In [13] and [14], we determined the ring structure of $\mathrm{HH}^*(A_q)$ by means of generators and Yoneda product. By this ring structure of $\mathrm{HH}^*(A_q)$, we have the following results.

Theorem 7. *In the case where q is a root of unity, $\mathrm{HH}^*(A_q)$ is finitely generated as a k-algebra.*

Theorem 8. *In the case where q is a root of unity, $\mathrm{HH}^*(A_q)/\mathcal{N}$ is isomorphic to the polynomial ring of two variables. In the case $s, t \geq 2$, $r \geq 1$, we have*

$$\mathrm{HH}^*(A_q)/\mathcal{N} \cong \begin{cases} k[W^{2r}_{0,0,0}, W^{2r}_{2r,0,0}] & \text{if } s,t \geq 2, \bar{a} \neq 0, \bar{b} \neq 0, \\ k[W^{2r}_{0,0,0}, W^{2}_{2,0,0}] & \text{if } s,t \geq 2, \bar{a} = 0, \bar{b} \neq 0, \\ k[W^{2}_{0,0,0}, W^{2r}_{2r,0,0}] & \text{if } s,t \geq 2, \bar{a} \neq 0, \bar{b} = 0, \\ k[W^{2}_{0,0,0}, W^{2}_{2,0,0}] & \text{if } s,t \geq 2, \bar{a} = \bar{b} = 0, \end{cases}$$

where for any integer z, $\bar{z}$ is the remainder when we divide z by r, and for $n \geq 1$,

$$W^{2n}_{0,l,l'} := X^{sl}Y^{tl'}e^{2n}_{b(1)} + \sum_{j=2}^{t} Y^{j-1}X^{sl}Y^{t(l'-1)+t-j+1}e^{2n}_{b(j)} \text{ for } 0 \leq l \leq a-1 \text{ and } 0 \leq l' \leq b,$$

$$W^{2n}_{2n,l,l'} := X^{sl}Y^{tl'}e^{2n}_{a(1)} + \sum_{i=2}^{s} X^{s(l-1)+i-1}Y^{tl'}X^{s-i+1}e^{2n}_{a(i)} \text{ for } 0 \leq l \leq a \text{ and } 0 \leq l' \leq b-1.$$

In the case where $s = 1$ or $t = 1$, we have similar results.

Theorem 9. *In the case where q is not a root of unity,* $\mathrm{HH}^*(A_q)$ *is not a finitely generated k-algebra.*

Theorem 10. *In the case where q is not a root of unity,* $\mathrm{HH}^*(A_q)/\mathcal{N} \cong k$.

There exists an example of our algebra A_q which is not self-injective, monomial or Koszul. Moreover this example of A_q have no stratifying ideal.

Example 11. In the case where $s = 2$, $t = 1$ and $a = b = 2$, A_q is not self-injective, monomial or Koszul. Moreover A_q have no stratifying ideal.

Therefore A_q is new example of a class of algebras for which the Hochschild cohomology ring modulo nilpotence is finitely generated as a k-algebra.

Next, we give the necessary and sufficient condition for A to satisfy **(Fg)**. Now, we consider the case where q is an r-th root of unity for $r \geq 1$, $s, t \geq 2$ and $\bar{a}, \bar{b} \neq 0$.

Let φ: $\mathrm{HH}^*(A_q) \to E(A_q) := \oplus_{n \geq 0} \mathrm{Ext}^n_{A_q}(A_q/\mathrm{rad}\, A_q, A_q/\mathrm{rad}\, A_q)$ be a homomorphism of graded rings given by $\varphi(\eta) = \eta \otimes_{A_q} A_q/\mathrm{rad}\, A_q$. Then it is easy to see that $E(A_q)^n := \mathrm{Ext}^n_{A_q}(A_q/\mathrm{rad}\, A_q, A_q/\mathrm{rad}\, A_q) \simeq \coprod_{l=0}^{n} k e_1^n \oplus \coprod_{j=2}^{t} k e^n_{b(j)} \oplus \coprod_{i=2}^{s} k e^n_{a(i)}$, and that the image of φ is precisely the graded ring $k[x, y]$ where $x := \sum_{j=1}^{t} e^{2r}_{b(j)}$ and $y := \sum_{i=1}^{s} e^{2r}_{a(i)}$ in degree $2r$. Then, we have the following proposition.

Proposition 12. $E(A_q)$ *is a finitely generated left $k[x, y]$-module.*

In the other cases, we have same results as Proposition 12. Then we have the following immediate consequence of Proposition 12.

Theorem 13. *In the case where $s \geq 2$ or $t \geq 2$, if q is a root of unity then A_q satisfies* **(Fg)**.

By [2], Theorem 9 and 13, we have the necessary and sufficient condition for A_q to satisfy **(Fg)**.

Theorem 14. A_q *satisfies* **(Fg)** *if and only if q is a root of unity.*

Remark 15. By Theorem 2.5 in [4] and Theorem 14, in the case where q is a root of unity, we have the following properties

(1) A_q is Gorenstein.
(2) The support variety of an A_q-module M is trivial if and only if the projective dimension of M is finite.

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