SHARP BOUNDS FOR HILBERT COEFFICIENTS OF PARAMETERS

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Abstract. Let A be a Noetherian local ring with $d = \dim A > 0$. This paper shows that the Hilbert coefficients $\{\mathrm{e}_Q^i(A)\}_{1 \le i \le d}$ of parameter ideals Q have uniform bounds if and only if A is a generalized Cohen-Macaulay ring. The uniform bounds are huge; the sharp bound for $\mathrm{e}_Q^2(A)$ in the case where A is a generalized Cohen-Macaulay ring with $\dim A \ge 3$ is given.

Key Words: commutative algebra, generalized Cohen-Macaulay local ring, Hilbert coefficient, Castelnuovo-Mumford regularity.

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1. Introduction

This is based on [5] a joint work with Shiro Goto.

The purpose of this paper is to study the problem of when the Hilbert coefficients of parameter ideals in a Noetherian local ring have uniform bounds, and when this is the case, to ask for their sharp bounds.

To state the problem and the results also, let us fix some notation. In what follows, let A be a commutative Noetherian local ring with maximal ideal $\mathfrak{m}$ and $d = \dim A > 0$ denotes the Krull dimension of A. For simplicity, we assume that the residue class field $A/\mathfrak{m}$ of A is infinite. Let $\ell_A(M)$ denote, for an A-module M, the length of M. Then for each $\mathfrak{m}$-primary ideal I in A, we have integers $\{\mathrm{e}_I^i(A)\}_{0 \le i \le d}$ such that the equality

$$\ell_A(A/I^{n+1}) = \mathrm{e}_I^0(A)\binom{n+d}{d} - \mathrm{e}_I^1(A)\binom{n+d-1}{d-1} + \cdots + (-1)^d \mathrm{e}_I^d(A)$$

holds true for all $n \gg 0$, which we call the Hilbert coefficients of A with respect to I.

With this notation our first purpose is to study the problem of when the sets

$$\Lambda_i(A) = \{\mathrm{e}_Q^i(A) \mid Q \text{ is a parameter ideal in } A\}$$

are finite for all $1 \le i \le d$.

Then the first main result is stated as follows. We say that our local ring is a generalized Cohen-Macaulay ring, if the local cohomology modules $\mathrm{H}_\mathfrak{m}^i(A)$ are finitely generated for all $i \ne d$.

Theorem 1. *Let A be a commutative Noetherian local ring with $d = \dim A \ge 2$. Then the following conditions are equivalent.*

(1) *A is a generalized Cohen-Macaulay ring.*
(2) *The set $\Lambda_i(A)$ is finite for all $1 \le i \le d$.*

The detailed version of this paper has been submitted for publication elsewhere.

Although the finiteness problem of $\Lambda_i(A)$ is settled affirmatively, we need to ask for the sharp bounds for the values of $\mathrm{e}_Q^i(A)$ of parameter ideals Q, which is our second purpose of the present research. Let $h^i(A) = \ell_A(\mathrm{H}_{\mathfrak{m}}^i(A))$ for each $i \in \mathbb{Z}$.

When A is a generalized Cohen-Macaulay ring with $d = \dim A \geq 2$, one has the inequalities

$$0 \geq \mathrm{e}_Q^1(A) \geq -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$$

for every parameter ideal Q in A ([9, Theorem 8], [3, Lemma 2.4]), where the equality $\mathrm{e}_Q^1(A) = -\sum_{i=1}^{d-1} \binom{d-2}{i-1} h^i(A)$ holds true if and only if Q is a standard parameter ideal in A ([10, Korollar 3.2], [4, Theorem 2.1]), provided $\operatorname{depth} A > 0$. The reader may consult [2] for the characterization of local rings which contain parameter ideals Q with $\mathrm{e}_Q^1(A) = 0$. Thus the behavior of the first Hilbert coefficients $\mathrm{e}_Q^1(A)$ for parameter ideals Q are rather satisfactorily understood.

The second purpose is to study the natural question of how about $\mathrm{e}_Q^2(A)$. First, we will show that in the case where $\dim A = 2$ and $\operatorname{depth} A > 0$, even though A is not necessarily a generalized Cohen-Macaulay ring, the inequality

$$-h^1(A) \leq \mathrm{e}_Q^2(A) \leq 0$$

holds true for every parameter ideal Q in A. We will also show that $\mathrm{e}_Q^2(A) = 0$ if and only if the ideal Q is generated by a system a, b of parameters which forms a d-sequence in A in the sense of C. Huneke [7]. When A is a generalized Cohen-Macaulay ring with $\dim A \geq 3$, we shall show that the inequality

$$-\sum_{j=2}^{d-1} \binom{d-3}{j-2} h^j(A) \leq \mathrm{e}_Q^2(A) \leq \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$$

holds true for every parameter ideal Q (Theorem 13). The following theorem which is the second main result of this paper shows that the upper bound $\mathrm{e}_Q^2(A) \leq \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$ is sharp, clarifying when the equality $\mathrm{e}_Q^2(A) = \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$ holds true.

Theorem 2. *Suppose that A is a generalized Cohen-Macaulay ring with $d = \dim A \geq 3$ and $\operatorname{depth} A > 0$. Let Q be a parameter ideal in A. Then the following two conditions are equivalent.*

(1) $\mathrm{e}_Q^2(A) = \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A)$.

(2) *There exist elements $a_1, a_2, \cdots, a_d \in A$ such that*
   - (a) $Q = (a_1, a_2, \cdots, a_d)$,
   - (b) *the sequence $a_1, a_2, \cdots, a_d$ is a d-sequence in A, and*
   - (c) $Q{\cdot}\mathrm{H}_{\mathfrak{m}}^j(A/(a_1, a_2, \cdots, a_k)) = (0)$ *for all $j \geq 1$ and $k \geq 0$ with $j + k \leq d - 2$.*

When this is the case, we furthermore have the following :

(i) $(-1)^i{\cdot}\mathrm{e}_Q^i(A) = \sum_{j=1}^{d-i} \binom{d-i-1}{j-1} h^j(A)$ *for $3 \leq i \leq d-1$ and*

(ii) $\mathrm{e}_Q^d(A) = 0$.

At this moment we do not know the sharp uniform bound for $\mathrm{e}_Q^3(A)$ for parameter ideals Q in a generalized Cohen-Macaulay ring A with $\dim A \geq 3$.

Let us briefly note how this paper is organized. We shall prove Theorem 1 in Section 2. Theorem 2 will be proven in Section 4. Section 3 is devoted to some preliminary steps for the proof of Theorem 2. We will closely study in Section 3 the problem of when $\mathrm{e}_Q^2(A) = 0$ in the case where $\dim A = 2$.

In what follows, unless otherwise specified, for each $\mathfrak{m}$-primary ideal I in A, we put

$$\mathrm{R}(I) = A[It], \quad \mathrm{R}'(I) = A[It, t^{-1}], \quad \text{and} \quad \mathrm{G}(I) = \mathrm{R}'(I)/t^{-1}\mathrm{R}'(I),$$

where t is an indeterminate over A. Let $\mathcal{M} = \mathfrak{m}R + R_+$ be the unique graded maximal ideal in $R = \mathrm{R}(I)$. We denote by $\mathrm{H}^i_{\mathcal{M}}(*)$ $(i \in \mathbb{Z})$ the $i^{\underline{th}}$ local cohomology functor of $\mathrm{R}(I)$ with respect to $\mathcal{M}$. Let L be a graded R-module. For each $n \in \mathbb{Z}$ let $[\mathrm{H}^i_{\mathcal{M}}(L)]_n$ stand for the homogeneous component of $\mathrm{H}^i_{\mathcal{M}}(L)$ with degree n. We denote by $L(\alpha)$, for each $\alpha \in \mathbb{Z}$, the graded R-module whose grading is given by $[L(\alpha)]_n = L_{\alpha+n}$ for all $n \in \mathbb{Z}$.

2. Proof of Theorem 1

In this section, we shall prove Theorem 1.

The heart of the proof of the implication (1) $\Rightarrow$ (2) is, in the case where A is a generalized Cohen-Macaulay ring, the existence of uniform bounds of the Castelnuovo-Mumford regularity $\mathrm{reg}\,\mathrm{G}(Q)$ of the associated graded rings $\mathrm{G}(Q)$ of parameter ideals Q. So, let us briefly recall the definition of the Castelnuovo-Mumford regularity.

Let Q be a parameter ideal in A and let

$$\mathrm{R}(Q) = A[Qt], \quad \mathrm{R}'(Q) = A[Qt, t^{-1}], \quad \text{and} \quad \mathrm{G}(Q) = \mathrm{R}'(Q)/t^{-1}\mathrm{R}'(Q)$$

respectively, denote the Rees algebra, the extended Rees algebra, and the associated graded ring of Q. Let $\mathcal{M} = \mathfrak{m}R + R_+$ be the unique graded maximal ideal in $R = \mathrm{R}(Q)$. For each $i \in \mathbb{Z}$ let

$$\mathrm{a}_i(\mathrm{G}(Q)) = \max\{n \in \mathbb{Z} \mid [\mathrm{H}^i_{\mathcal{M}}(\mathrm{G}(Q))]_n \neq (0)\}$$

and put

$$\mathrm{reg}\mathrm{G}(Q) = \max\{\mathrm{a}_i(\mathrm{G}(Q)) + i \mid i \in \mathbb{Z}\},$$

which we call the Castelnuovo-Mumford regularity of the graded ring $\mathrm{G}(Q)$.

Let us now note the following result of Linh and Trung [8], which gives a uniform bound for $\mathrm{reg}\,\mathrm{G}(Q)$ for parameter ideals Q in a generalized Cohen-Macaulay ring.

Theorem 3 ([8], Theorem 2.3)**.** *Suppose that A is a generalized Cohen-Macaulay ring and let Q be a parameter ideal in A. Then*

(1) $\mathrm{reg}\,\mathrm{G}(Q) \leq \max\{\mathrm{I}(A) - 1, 0\}$*, if* $d = 1$.

(2) $\mathrm{reg}\,\mathrm{G}(Q) \leq \max\{(4\mathrm{I}(A))^{(d-1)!} - \mathrm{I}(A) - 1, 0\}$*, if* $d \geq 2$.

Thus, the following result is the key for our proof of the implication (1) $\Rightarrow$ (2) in Theorem 1, where $h_i(A) = \ell_A(\mathrm{H}^i_{\mathfrak{m}}(A))$ and $\mathrm{I}(A) = \sum_{j=0}^{d-1} \binom{d-1}{j} h^j(A)$.

Theorem 4. *Suppose that A is a generalized Cohen-Macaulay ring. Let Q be a parameter ideal in A and put $r = \mathrm{reg}\,\mathrm{G}(Q)$. Then*

(1) $|\mathrm{e}_Q^1(A)| \leq \mathrm{I}(A)$.

(2) $|\mathrm{e}_Q^i(A)| \leq 3 \cdot 2^{i-2}(r+1)^{i-1}\mathrm{I}(A)$ *for* $2 \leq i \leq d$.

Proof. See [5, Section 2]. $\square$

Therefore, thanks to the uniform bounds [8, Theorem 2.3] of $\operatorname{reg} \mathrm{G}(Q)$ for parameter ideals Q in a generalized Cohen-Macaulay ring A, we readily get the finiteness in the set $\Lambda_i(A)$ for all $1 \le i \le d$.

We are now in a position to finish the proof of Theorem 1.

Proof of Theorem 1. We may assume that A is complete. Also we may assume A is not unmixed, because $\Lambda_1(A)$ is a finite set (cf. [2, Proposition 4.2]). Let U denote the unmixed component of the ideal (0) in A. We put $B = A/U$ and $t = \dim_A U$ ($\le d-1$). We must show that B is a generalized Cohen-Macaulay ring and $t = 0$.

Let Q be a parameter ideal in A. We then have

$$\ell_A(A/Q^{n+1}) = \ell_A(B/Q^{n+1}B) + \ell_A(U/Q^{n+1} \cap U)$$

for all integers $n \ge 0$. Therefore, the function $\ell_A(U/Q^{n+1} \cap U)$ is a polynomial in $n \gg 0$ with degree t and there exist integers $\{\mathrm{s}_Q^i(U)\}_{0 \le i \le t}$ with $\mathrm{s}_Q^0(U) = \mathrm{e}_Q^0(U)$ such that

$$\ell_A(U/Q^{n+1} \cap U) = \sum_{i=0}^{t} (-1)^i \mathrm{s}_Q^i(U) \binom{n+t-i}{t-i}$$

for all $n \gg 0$, whence

$$\ell_A(A/Q^{n+1}) = \sum_{i=0}^{d} (-1)^i \mathrm{e}_Q^i(B) \binom{n+d-i}{d-i} + \sum_{i=0}^{t} (-1)^i \mathrm{s}_Q^i(U) \binom{n+t-i}{t-i}.$$

Consequently

$$(-1)^{d-i}\mathrm{e}_Q^{d-i}(A) = \begin{cases} (-1)^{d-i}\mathrm{e}_Q^{d-i}(B) + (-1)^{t-i}\mathrm{s}_Q^{t-i}(U) & \text{if} \quad 0 \le i \le t, \\ (-1)^{d-i}\mathrm{e}_Q^{d-i}(B) & \text{if} \quad t+1 \le i \le d. \end{cases}$$

Therefore, if $t < d-1$, we have $\mathrm{e}_Q^1(A) = \mathrm{e}_Q^1(B)$, so that $\Lambda_1(B) = \Lambda_1(A)$ is a finite set. If $t = d-1$, we get $-\mathrm{e}_Q^1(A) = -\mathrm{e}_Q^1(B) + \mathrm{s}_Q^0(U)$. Since $\mathrm{e}_Q^1(A), \mathrm{e}_Q^1(B) \le 0$ and $\mathrm{s}_Q^0(U) = \mathrm{e}_Q^0(U) \ge 1$, $\Lambda_1(B)$ is a finite set also in this case. Thus the set $\Lambda_1(B)$ is finite in any case, so that the ring B a generalized Cohen-Macaulay ring.

We now assume that $t \ge 1$ and choose a system $a_1, a_2, \cdots, a_d$ of parameters in A so that $(a_{t+1}, a_{t+2}, \cdots, a_d)U = (0)$. Let $\ell \ge 1$ be an integer such that $\mathfrak{m}^\ell$ is standard for the ring B and choose integers $n \ge \ell$. We look at parameter ideals $Q = (a_1^n, a_2^n, \cdots, a_d^n)$ of A. Then

$$(-1)^{d-t}\mathrm{e}_Q^{d-t}(B) = \sum_{j=1}^{t} \binom{t-1}{j-1} h^j(B)$$

by [10, Korollar 3.2], which is independent of the integers $n \ge \ell$. Therefore, since

$$\mathrm{s}_Q^0(U) = \mathrm{e}^0_{(a_1^n, a_2^n, \cdots, a_t^n)}(U) = n^t \cdot \mathrm{e}^0_{(a_1, a_2, \cdots, a_t)}(U) \ge n^t,$$

we see

$$\begin{aligned} (-1)^{d-t}\mathrm{e}_Q^{d-t}(A) &= (-1)^{d-t}\mathrm{e}_Q^{d-t}(B) + \mathrm{s}_Q^0(U) \\ &= \sum_{j=1}^{t} \binom{t-1}{j-1} h^j(B) + n^t \cdot \mathrm{e}^0_{(a_1, a_2, \cdots, a_t)}(U) \ge n^t, \end{aligned}$$

whence the set $\Lambda_{d-t}(A)$ cannot be finite. Thus $t = 0$ and A a generalized Cohen-Macaulay ring. $\square$

3. The second Hilbert coefficients $\mathrm{e}_Q^2(A)$ of parameters

In this section we study the second Hilbert coefficients $\mathrm{e}_Q^2(A)$ of parameter ideals Q. The purpose is to find the sharp bound for $\mathrm{e}_Q^2(A)$. The bound $|\mathrm{e}_Q^2(A)| \leq 3(r+1)\mathrm{I}(A)$ given by Theorem 1 is too huge in general and far from the sharp bound.

Let us begin with the following.

Lemma 5. *Suppose that $d = 2$ and $\operatorname{depth} A > 0$. Let $Q = (x, y)$ be a parameter ideal in A and assume that x is superficial with respect to Q. Then*

$$\mathrm{e}_Q^2(A) = -\ell_A\left(\frac{[(x^\ell) : y^\ell] \cap Q^\ell}{(x^\ell)}\right) \leq 0$$

for all $\ell \gg 0$.

Proof. Let $\ell \gg 0$ be an integer which is sufficiently large and put $I = Q^\ell$. Let $G = \mathrm{G}(I)$ and $R = \mathrm{R}(I)$ be the associated graded ring and the Rees algebra of I, respectively. We put $\mathcal{M} = \mathfrak{m}R + R_+$. Then $[\mathrm{H}^i_{\mathcal{M}}(G)]_n = (0)$ for all integers $i \in \mathbb{Z}$ and $n > 0$, thanks to [6, Lemma 2.4]. We put $a = x^\ell$ and $b = y^\ell$. Then the element a remains superficial with respect to I and the equality $I^2 = (a, b)I$ holds true, whence $\mathrm{a}_2(G) < 0$.

We furthermore have the following.

Claim 6. $[\mathrm{H}^i_{\mathcal{M}}(R)]_0 \cong [\mathrm{H}^i_{\mathcal{M}}(G)]_0$ *as A-modules for all $i \in \mathbb{Z}$. Hence $\mathrm{H}^0_{\mathcal{M}}(G) = (0)$, so that $f = at \in R$ is G-regular.*

Proof of Claim 6. Let $L = R_+$ and apply the functors $\mathrm{H}^i_{\mathcal{M}}(*)$ to the following canonical exact sequences

$$0 \to L \to R \xrightarrow{p} A \to 0 \quad \text{and} \quad 0 \to L(1) \to R \to G \to 0,$$

where p denotes the projection, and get the exact sequences

$$(1) \quad \cdots \to \mathrm{H}^{i-1}_{\mathfrak{m}}(A) \to \mathrm{H}^i_{\mathcal{M}}(L) \to \mathrm{H}^i_{\mathcal{M}}(R) \to \mathrm{H}^i_{\mathfrak{m}}(A) \to \cdots \quad \text{and}$$

$$(2) \quad \cdots \to \mathrm{H}^{i-1}_{\mathcal{M}}(G) \to \mathrm{H}^i_{\mathcal{M}}(L)(1) \to \mathrm{H}^i_{\mathcal{M}}(R) \to \mathrm{H}^i_{\mathcal{M}}(G) \to \mathrm{H}^{i+1}_{\mathcal{M}}(L)(1) \to \cdots$$

of local cohomology modules. Then by exact sequence (2) we get the isomorphism

$$[\mathrm{H}^i_{\mathcal{M}}(L)]_{n+1} \cong [\mathrm{H}^i_{\mathcal{M}}(R)]_n$$

for $n \geq 1$, because $[\mathrm{H}^{i-1}_{\mathcal{M}}(G)]_n = [\mathrm{H}^i_{\mathcal{M}}(G)]_n = (0)$ for $n \geq 1$, while we have the isomorphism

$$[\mathrm{H}^i_{\mathcal{M}}(L)]_{n+1} \cong [\mathrm{H}^i_{\mathcal{M}}(R)]_{n+1}$$

for $n \geq 1$, thanks to exact sequence (1). Hence $[\mathrm{H}^i_{\mathcal{M}}(R)]_n \cong [\mathrm{H}^i_{\mathcal{M}}(R)]_{n+1}$ for $n \geq 1$, which implies $[\mathrm{H}^i_{\mathcal{M}}(R)]_n = (0)$ for all $i \in \mathbb{Z}$ and $n \geq 1$, because $[\mathrm{H}^i_{\mathcal{M}}(R)]_n = (0)$ for $n \gg 0$. Thus by exact sequence (1) we get $[\mathrm{H}^i_{\mathcal{M}}(L)(1)]_n = (0)$ for all $i \in \mathbb{Z}$ and $n \geq 0$, so that by exact sequence (2) we see $[\mathrm{H}^i_{\mathcal{M}}(R)]_0 \cong [\mathrm{H}^i_{\mathcal{M}}(G)]_0$ as A-modules for all $i \in \mathbb{Z}$. Considering the case where $i = 1$ in exact sequence (2), we have the embedding

$$0 \to \mathrm{H}^0_{\mathcal{M}}(G) \to \mathrm{H}^1_{\mathcal{M}}(L)(1),$$

so that $[\mathrm{H}^0_{\mathcal{M}}(G)]_0 = (0)$, because $[\mathrm{H}^1_{\mathcal{M}}(L)(1)]_0 = [\mathrm{H}^0_{\mathcal{M}}(L)]_1 = (0)$. Hence $\mathrm{H}^0_{\mathcal{M}}(G) = (0)$, so that f is G-regular, because $(0) :_G f$ is finitely graded. $\square$

Thanks to Serre's formula (cf. [1, Theorem 4.4.3]), Claim 6 shows that

$$\mathrm{e}^2_Q(A) = \sum_{i=0}^{2}(-1)^i \ell_A([\mathrm{H}^i_{\mathcal{M}}(G)]_0) = -\ell_A([\mathrm{H}^1_{\mathcal{M}}(G)]_0),$$

since $a_2(G) < 0$. Therefore to prove

$$\mathrm{e}^2_Q(A) = -\ell_A\left(\frac{[(x^\ell) : y^\ell] \cap Q^\ell}{(x^\ell)}\right),$$

it is suffices to check that

$$[\mathrm{H}^1_{\mathcal{M}}(G)]_0 \cong \frac{[(a) : b] \cap I}{(a)}$$

as A-modules.

Let $\overline{A} = A/(a)$ and $\overline{I} = I\overline{A}$. Then $G/fG \cong \mathrm{G}(\overline{I})$, because $f = at$ is G-regular (cf. Claim 6). We now look at the exact sequence

$$0 \to \mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I})) \to \mathrm{H}^1_{\mathcal{M}}(G)(-1) \xrightarrow{f} \mathrm{H}^1_{\mathcal{M}}(G)$$

of local cohomology modules which is induced from the exact sequence

$$0 \to G(-1) \xrightarrow{f} G \to \mathrm{G}(\overline{I}) \to 0$$

of graded G-modules. Then, since $[\mathrm{H}^1_{\mathcal{M}}(G)]_n = (0)$ for all $n \geq 1$, we have an isomorphism

$$[\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I}))]_1 \cong [\mathrm{H}^1_{\mathcal{M}}(G)]_0$$

of A-modules and the vanishing $[\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I})]_n = (0)$ for $n \geq 2$.

Look now at the homomorphism

$$\rho : \frac{[(a) : b] \cap I}{(a)} \to [\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I}))]_1$$

of A-modules defined by $\rho(\overline{x}) = \overline{\overline{x}t}$ for each $x \in [(a) : b] \cap I$, where $\overline{x}$ and $\overline{\overline{x}t}$ denote the images of x in $\overline{A}$ and $\overline{x}t \in [\mathrm{R}(\overline{I})]_1$ in $\mathrm{G}(\overline{I})$, respectively. We will show that the map ρ is an isomorphism. Take $\varphi \in [\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I}))]_1$ and write $\varphi = \overline{\overline{x}t}$ with $x \in I$. Since $[\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(I))]_2 = (0)$, we have $bt \cdot \overline{\overline{x}t} = \overline{\overline{bx}t^2} = 0$ in $\mathrm{G}(\overline{I})$, whence $bx \in [(a) + I^3] \cap I^2 = [(a) \cap I^2] + I^3 = aI + bI^2$ (recall that $I^2 = (a,b)I$ and that a is super-regular with respect to I). So, we write $bx = ai + bj$ with $i \in I$ and $j \in I^2$. Then, since $b(x - j) = ai \in (a)$, we have $x - j \in [(a) : b] \cap I$, whence $\varphi = \overline{\overline{x}t} = \overline{\overline{(x - j)}t}$. Thus the map ρ is surjective.

To show that the map ρ is injective, take $x \in [(a) : b] \cap I$ and suppose that $\rho(\overline{x}) = \overline{\overline{x}t} = 0$ in $\mathrm{G}(\overline{I})$. Then

$$x \in [(a) : b] \cap [(a) + I^2] = (a) + [((a) : b) \cap I^2].$$

To conclude that $x \in (a)$, we need the following.

Claim 7. *Let $n \geq 2$ be an integer. Then $[(a) : b] \cap I^n \subseteq (a) + [((a) : b) \cap I^{n+1}]$.*

Proof of Claim 7. Take $y \in [(a):b] \cap I^n$. Then, since $by \in (a)$, we see $bt \cdot \overline{\overline{y}t^n} = \overline{\overline{by}t^{n+1}} = 0$ in $\mathrm{G}(\overline{I})$. Hence $\overline{\overline{y}t^n} \in [\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I}))]_n$, because bt is a homogeneous parameter for the graded ring $\mathrm{G}(\overline{I})$. Recall now that $n \geq 2$, whence $[\mathrm{H}^0_{\mathcal{M}}(\mathrm{G}(\overline{I}))]_n = (0)$, so that $\overline{\overline{y}t^n} = 0$. Thus $y \in (a) + I^{n+1}$, whence $y \in (a) + [((a):b) \cap I^{n+1}]$, as claimed. □

Since $x \in (a) + [((a):b) \cap I^2]$, thanks to Claim 7, we get $x \in (a) + I^{n+1}$ for all $n \geq 1$, whence $x \in (a)$, so that the map ρ is injective. Thus

$$[\mathrm{H}^1_{\mathcal{M}}(G)]_0 \cong \frac{[(a):b] \cap I}{(a)}$$

as A-modules. □

Theorem 8. *Suppose that $d = 2$ and $\operatorname{depth} A > 0$. Let $Q = (x, y)$ be a parameter ideal in A and assume that x is superficial with respect to Q. Then*

$$-h^1(A) \leq \mathrm{e}^2_Q(A) \leq 0$$

and the following three conditions are equivalent.

(1) $\mathrm{e}^2_Q(A) = 0$.
(2) *x, y forms a d-sequence in A.*
(3) *x^ℓ, y^ℓ forms a d-sequence in A for all integers $\ell \geq 1$.*

Proof. By Lemma 5 we have

$$\mathrm{e}^2_Q(A) = -\ell_A\left(\frac{[(x^\ell):y^\ell] \cap (x,y)^\ell}{(x^\ell)}\right) \leq 0$$

for all integers $\ell \gg 0$. To show that $-h^1(A) \leq \mathrm{e}^2_Q(A)$, we may assume that $\mathrm{H}^1_{\mathfrak{m}}(A)$ is finitely generated. Take the integer $\ell \gg 0$ so that the system $a = x^\ell$, $b = y^\ell$ of parameters of A is standard. Then since

$$\frac{[(a):b] \cap Q^\ell}{(a)} \subseteq \frac{(a):b}{(a)} \cong \mathrm{H}^0_{\mathfrak{m}}(A/(a)) \cong \mathrm{H}^1_{\mathfrak{m}}(A),$$

we get $-h^1(A) \leq \mathrm{e}^2_Q(A)$.

Let us consider the second assertion.

(1) ⇒ (3). Take an integer $N \geq 1$ so that

$$\mathrm{e}^2_Q(A) = -\ell_A\left(\frac{[(x^\ell):y^\ell] \cap (x,y)^\ell}{(x^\ell)}\right)$$

for all $\ell \geq N$ (cf. Lemma 5); hence

$$[(x^\ell):y^\ell] \cap (x,y)^\ell = (x^\ell).$$

Claim 9. $[(x^\ell):y^\ell] \cap (x,y)^\ell = (x^\ell)$ *for all $\ell \geq 1$.*

Proof of Claim 9. We may assume that $1 \leq \ell < N$. Take $\tau \in [(x^\ell):y^\ell] \cap (x,y)^\ell$. Then, since $y^N(x^{N-\ell}\tau) = y^{N-\ell}x^{N-\ell}(y^\ell\tau) \in (x^N)$, we have $x^{N-\ell}\tau \in [(x^N):y^N] \cap (x,y)^N = (x^N)$. Thus $\tau \in (x^\ell)$, because x is A-regular (recall that $\operatorname{depth} A > 0$ and x is superficial with respect to Q). □

Since x^ℓ is A-regular and $[(x^\ell) : y^\ell] \cap (x^\ell, y^\ell) = (x^\ell)$ by Claim 9, we readily see that x^ℓ, y^ℓ is a d-sequence in A.

(3) $\Rightarrow$ (2) This is clear.

(2) $\Rightarrow$ (1) It is well-known that $\mathrm{e}^2_{(x,y)}(A) = 0$, if $\operatorname{depth} A > 0$ and the system x, y of parameters forms a d-sequence in A; see Proposition 11 below. $\square$

Passing to the ring $A/\mathrm{H}^0_{\mathfrak{m}}(A)$, thanks to Theorem 8, we readily get the following.

Corollary 10. *Suppose that $d = 2$ and let Q be a parameter ideal in A. Then*

$$h^0(A) - h^1(A) \leq \mathrm{e}^2_Q(A) \leq h^0(A).$$

The results in the following proposition are, more or less, known.

Proposition 11. ([5, Proposition 3.4]) *Suppose that $d > 0$ and let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Let $G = \mathrm{G}(Q)$ and $R = \mathrm{R}(Q)$. Let $f_i = a_i t \in R$ for $1 \leq i \leq d$. Assume that the sequence $a_1, a_2, \cdots, a_d$ forms a d-sequence in A. Then we have the following, where $Q_i = (a_1, a_2, \cdots, a_i)$ for $0 \leq i \leq d$.*

(1) $\mathrm{e}^0_Q(A) = \ell_A(A/Q) - \ell_A([Q_{d-1} : a_d]/Q_{d-1})$.

(2) $(-1)^i \mathrm{e}^i_Q(A) = h^0(A/Q_{d-i}) - h^0(A/Q_{d-i-1})$ *for* $1 \leq i \leq d-1$ *and* $(-1)^d \mathrm{e}^d_Q(A) = h^0(A)$.

(3) $\ell_A(A/Q^{n+1}) = \sum_{i=0}^{d} (-1)^i \mathrm{e}^i_Q(A) \binom{n+d-i}{d-i}$ *for all* $n \geq 0$, *whence* $\ell_A(A/Q) = \sum_{i=0}^{d} (-1)^i \mathrm{e}^i_Q(A)$.

(4) $f_1, f_2, \cdots, f_d$ *forms a d-sequence in G.*

(5) $\mathrm{H}^0_{\mathcal{M}}(G) = [\mathrm{H}^0_{\mathcal{M}}(G)]_0 \cong \mathrm{H}^0_{\mathfrak{m}}(A)$, *where* $\mathcal{M} = \mathfrak{m}R + R_+$

(6) $[\mathrm{H}^i_{\mathcal{M}}(G)]_n = (0)$ *for all* $n > -i$ *and* $i \in \mathbb{Z}$, *whence* $\operatorname{reg} G = 0$.

Let us note one example of local rings A which are not generalized Cohen-Macaulay rings but every parameter ideal in A is generated by a system of parameters that forms a d-sequence in A.

Example 12. Let R be a regular local ring with the maximal ideal $\mathfrak{n}$ and $d = \dim R \geq 2$. Let $X_1, X_2, \cdots, X_d$ be a regular system of parameters of R. We put $\mathfrak{p} = (X_1, X_2, \cdots, X_{d-1})$ and $D = R/\mathfrak{p}$. Then D is a DVR. Let $A = R \ltimes D$ denote the idealization of D over R. Then A is a Noetherian local ring with the maximal ideal $\mathfrak{m} = \mathfrak{n} \times D$, $\dim A = d$, and $\operatorname{depth} A = 1$. We furthermore have the following.

(1) $\Lambda_i(A) = \{0\}$ for all $1 \leq i \leq d$ such that $i \neq d-1$.

(2) $\Lambda_0(A) = \{n \mid 0 < n \in \mathbb{Z}\}$ and $\Lambda_{d-1}(A) = \{(-1)^{d-1} n \mid 0 < n \in \mathbb{Z}\}$.

(3) After renumbering, every system of parameters in A forms a d-sequence.

The ring A is not a generalized Cohen-Macaulay ring, because $\mathrm{H}^1_{\mathfrak{m}}(A)$ ($\cong \mathrm{H}^1_{\mathfrak{n}}(D)$) is not a finitely generated A-module.

In the rest of Section 3 let us consider the bound for $\mathrm{e}^2_Q(Q)$ in higher dimensional cases. In the case where $\dim A \geq 3$ we have the following.

Theorem 13. *Suppose that A is a generalized Cohen-Macaulay ring with $d = \dim A \geq 3$. Let $Q = (a_1, a_2, \cdots, a_d)$ be a parameter ideal in A. Then*

$$-\sum_{j=2}^{d-1} \binom{d-3}{j-2} h^j(A) \leq \mathrm{e}^2_Q(A) \leq \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A).$$

We have $Q{\cdot}\mathrm{H}^j_{\mathfrak m}(A/(a_1,a_2,\cdots,a_k))=(0)$ *for all* $k\ge 0$ *and* $j\ge 1$ *with* $j+k\le d-2$, *if* $\mathrm{e}^2_Q(A)=\sum_{j=1}^{d-2}\binom{d-3}{j-1}h^j(A)$ *and if* $a_1,a_2,\cdots,a_d$ *forms a superficial sequence with respect to* Q.

Proof. See [5, Theorem 3.6]. □

The following result guarantees the implication (2) ⇒ (1) and the last assertion in Theorem 2.

Proposition 14. *Suppose that* A *is a generalized Cohen-Macaulay ring with* $d=\dim A\ge 3$ *and let* $Q=(a_1,a_2,\cdots,a_d)$ *be a parameter ideal in* A. *Assume that the sequence* $a_1,a_2,\cdots,a_d$ *forms a d-sequence in* A *and* $Q{\cdot}\mathrm{H}^j_{\mathfrak m}(A/(a_1,a_2,\cdots,a_k))=(0)$ *for all* $k\ge 0$ *and* $j\ge 1$ *with* $j+k\le d-2$. *Then*

$$(-1)^i\mathrm{e}^i_Q(A)=\sum_{j=1}^{d-i}\binom{d-i-1}{j-1}h^j(A)$$

for $2\le i\le d-1$ *and* $(-1)^d\mathrm{e}^d_Q(A)=h^0(A)$.

Proof. See [5, Proposition 3.7]. □

4. Proof of Theorem 2

The purpose of this section is to prove Theorem 2. Thanks to Proposition 11 and 14, we have only to show the following.

Theorem 15. *Suppose that* A *is a generalized Cohen-Macaulay ring with* $d=\dim A\ge 3$ *and* $\operatorname{depth} A>0$. *Let* Q *be a parameter ideal in* A *and assume that* $\mathrm{e}^2_Q(A)=\sum_{j=1}^{d-2}\binom{d-3}{j-1}h^j(A)$. *Then* Q *is generated by a system of parameters which forms a d-sequence in* A.

For each ideal $\mathfrak a$ in A ($\mathfrak a\ne A$) let $\mathrm{U}(\mathfrak a)$ denote the unmixed component of $\mathfrak a$. When $\mathfrak a=(a)$ with $a\in A$, we write $\mathrm{U}(\mathfrak a)$ simply by $\mathrm{U}(a)$. We have

$$\mathrm{U}(a)=\bigcup_{n\ge 0}\left[(a):_A\mathfrak m^n\right],$$

if A is a generalized Cohen-Macaulay ring with $\dim A\ge 2$ and a is a part of a system of parameters in A (cf. [11, Section 2]). The following result is the key in our proof of Theorem 15.

Proposition 16. *Suppose that* A *is a generalized Cohen-Macaulay ring with* $d=\dim A\ge 2$ *and* $\operatorname{depth} A>0$. *Let* $Q=(a_1,a_2,\cdots,a_d)$ *be a parameter ideal in* A. *Assume that* $a_d\mathrm{H}^1_{\mathfrak m}(A)=(0)$ *and that the sequence* $a_1,a_2,\cdots,a_{d-1}$ *forms a d-sequence in the generalized Cohen-Macaulay ring* $A/\mathrm{U}(a_d)$. *Then*

$$\mathrm{U}(a_1)\cap[Q+\mathrm{U}(a_d)]=(a_1).$$

Proof. See [5, Proposition 4.2]. □

We are now ready to prove Theorem 15.

Proof of the Theorem 15. We proceed by induction on d. Choose $a_1, a_2, \cdots, a_d \in A$ so that $Q = (a_1, a_2, \cdots, a_d)$ and for each $1 \le i \le d-2$, the $i+2$ elements $a_1, a_2, \cdots, a_i, a_{d-1}, a_d$ form a superficial sequence with respect to Q. We will show that there exist $b_2, b_3, \cdots, b_d \in A$ such that $b_1 = a_{d-1}, b_2, b_3, \cdots, b_d$ forms a d-sequence in A and $Q = (b_1, b_2, \cdots, b_d)$. We put $\overline{A} = A/(a_1)$, $\overline{Q} = Q\overline{A}$, and $C = \overline{A}/\mathrm{H}^0_{\mathfrak{m}}(\overline{A})$ $(= A/\mathrm{U}(a_1))$.

Suppose that $d = 3$. Then

$$\mathrm{e}^2_{QC}(C) = \mathrm{e}^2_{\overline{Q}}(\overline{A}) - h^0(\overline{A}) = \mathrm{e}^2_Q(A) - h^0(\overline{A}) = h^1(A) - h^0(\overline{A}) = 0,$$

because $h^1(A) = h^0(\overline{A})$ (recall that $Q\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$ by Proposition 13). Hence, thanks to Proposition 8, a_2, a_3 forms a d-sequence in C, because a_2 is superficial for the ideal $QC = (a_2, a_3)C$. Therefore, since $a_1\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$, we have

$$\mathrm{U}(a_2) \cap [Q + \mathrm{U}(a_1)] = (a_2),$$

by Proposition 16. Let $Q = (a_2, a_3, b_3)$ and $B = A/\mathrm{U}(a_2)$. Then since $\mathrm{e}^2_{QB}(B) = 0$, by Proposition 8 the sequence $b_2 = a_3, b_3$ forms a d-sequence in B, because b_2 is superficial for QB. Therefore, since $\mathrm{U}(a_2) \cap Q \subseteq \mathrm{U}(a_2) \cap [Q + \mathrm{U}(a_1)] = (a_2)$, the sequence b_2, b_3 forms a d-sequence in $A/(a_2)$, so that $b_1 = a_2, b_2, b_3$ forms a d-sequence in A, because b_1 is A-regular.

Assume that $d \ge 4$ and that our assertion holds true for $d-1$. Then, thanks to Theorem 13 and its proof, we have

$$\begin{aligned}
\mathrm{e}^2_Q(A) = \mathrm{e}^2_{\overline{Q}}(\overline{A}) = \mathrm{e}^2_{QC}(C) &\le \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(C) \\
&= \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(\overline{A}) \\
&= \sum_{j=1}^{d-2} \binom{d-3}{j-1} h^j(A) = \mathrm{e}^2_Q(A),
\end{aligned}$$

because $Q{\cdot}\mathrm{H}^j_{\mathfrak{m}}(A) = (0)$ for $1 \le j \le d-3$. Hence

$$\mathrm{e}^2_{QC}(C) = \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(C).$$

Therefore, because $QC = (\overline{a_2}, \overline{a_3}, \cdots, \overline{a_d})C$ and the sequence $\overline{a_2}, \overline{a_3}, \cdots, \overline{a_i}, \overline{a_{d-1}}, \overline{a_d}$ is superficial in the ideal QC for all $1 \le i \le d-2$ where $\overline{a_j}$ denotes the image of a_j in C, the hypothesis of induction on d yields that there exist $\gamma_2, \gamma_3, \cdots, \gamma_{d-1} \in C$ such that the sequence $\gamma_1 = \overline{a_{d-1}}, \gamma_2, \gamma_3, \cdots, \gamma_{d-1}$ forms a d-sequence in C and $QC = (\gamma_1, \gamma_2, \cdots, \gamma_{d-1})C$. Let us write $\gamma_j = \overline{c_j}$ for each $2 \le j \le d-1$ with $c_j \in Q$, where $\overline{c_j}$ denote the image of c_j in C. We put $\mathfrak{q} = (a_1, a_{d-1}, c_2, c_3, \cdots, c_{d-1})$. Then $\mathfrak{q}$ is a parameter ideal in A, $a_1\mathrm{H}^1_{\mathfrak{m}}(A) = (0)$, and $a_{d-1}, c_2, c_3, \cdots, c_{d-1}$ forms a d-sequence in C. Therefore

$$\mathrm{U}(a_{d-1}) \cap [Q + \mathrm{U}(a_1)] = \mathrm{U}(a_{d-1}) \cap [\mathfrak{q} + \mathrm{U}(a_1)] = (a_{d-1})$$

by Proposition 16, whence $\mathrm{U}(a_{d-1}) \cap Q = (a_{d-1})$.

Let $B = A/\mathrm{U}(a_{d-1})$. We then have

$$\mathrm{e}^2_{QB}(B) = \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(B)$$

for the same reason as for the equality $\mathrm{e}^2_{QC}(C) = \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(C)$ (in fact, to show $\mathrm{e}^2_{QC}(C) = \sum_{j=1}^{d-3} \binom{d-4}{j-1} h^j(C)$, we only need that a_1 is superficial with respect to Q). Therefore, by the hypothesis of induction on d, we may choose elements $\beta_2, \beta_3, \cdots, \beta_d \in B$ so that $QB = (\beta_2, \beta_3, \cdots, \beta_d)B$ and the sequence $\beta_2, \beta_3, \cdots, \beta_d$ forms a d-sequence in B. We put $b_1 = a_{d-1}$ and write $\beta_j = \overline{b_j}$ with $b_j \in Q$ for $2 \leq j \leq d$, where $\overline{b_j}$ denotes the image of b_j in B. We now put $\mathfrak{q}' = (b_1, b_2, \cdots, b_d)$. Then $\mathfrak{q}'$ is a parameter ideal in A and because $\mathrm{U}(b_1) \cap Q = (b_1)$, we get

$$Q \subseteq [\mathfrak{q}' + \mathrm{U}(b_1)] \cap Q = \mathfrak{q}' + [\mathrm{U}(b_1) \cap Q] \subseteq \mathfrak{q}' + (b_1) = \mathfrak{q}';$$

hence $Q = \mathfrak{q}'$. Thus the sequence $b_2, b_3, \cdots, b_d$ forms a d-sequence in $A/(b_1)$, so that $b_1, b_2, \cdots, b_d$ forms a d-sequence in A, because b_1 is A-regular. This complete the proof of Theorem 15 and that of Theorem 2 as well. $\square$

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