ON A GENERALIZATION OF COSTABLE TORSION THEORY

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Abstract. E. P. Armendariz characterized a stable torsion theory in [1]. R. L. Bernhardt dualised a part of characterizations of stable torsion theory in Theorem1.1 of [3], as follows. Let ($\mathcal{T}$,$\mathcal{F}$) be a hereditary torsion theory for Mod-R such that every torsionfree module has a projective cover. Then the following are equivalent. (1) $\mathcal{F}$ is closed under taking projective covers. (2) every projective module splits. In this paper we generalize and characterize this by using torsion theory. In the remainder of this paper we study a dualization of Eckman and Shopf's Theorem and a generalization of Wu and Jans's Theorem.

1. INTRODUCTION

Throughout this paper R is a right perfect ring with identity. Let Mod-R be the categories of right R-modules. For $M \in$ Mod-R we denote by $[0 \to K(M) \to P(M) \overset{\pi_M}{\to} M \to 0\ ]$ the projective cover of M, where $P(M)$ is projective and $\ker\pi_M$ is small in $P(M)$. A subfunctor of the identity functor of Mod-R is called a preradical. For a preradical σ, $\mathcal{T}_\sigma := \{M \in \text{Mod-}R\ ;\ \sigma(M) = M\}$ is the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma := \{M \in \text{Mod-}R\ ;\ \sigma(M) = 0\}$ is the class of σ-torsionfree right R-modules. A right R-module M is called σ-projective if the functor $\text{Hom}_R(M,\ \ )$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathcal{F}_\sigma$. A preradical σ is idempotent[radical] if $\sigma(\sigma(M)) = \sigma(M)[\sigma(M/\sigma(M)) = 0]$ for a module M, respectively. A preradical σ is called epi-preserving if $\sigma(M/N) = (\sigma(M)+N)/N$ holds for any module M and any submodule N of M. For a preradical σ, a short exact sequence $[0 \to K_\sigma(M) \to P_\sigma(M) \overset{\pi^\sigma_M}{\to} M \to 0]$ is called σ-projective cover of a module M if $P_\sigma(M)$ is σ-projective, $K_\sigma(M)$ is σ-torsion free and $K_\sigma(M)$ is small in $P_\sigma(M)$. If σ is an idempotent radical and a module M has a projective cover, then M has a σ-projective cover and it is given $K_\sigma(M) = K(M)/\sigma(K(M)), P_\sigma(M) = P(M)/\sigma(K(M))$. For $X, Y \in$ Mod-R we call an epimorphism $g \in \text{Hom}_R(X, Y)$ a minimal epimorphism if $g(H) \subsetneqq Y$ holds for any proper submodule H of X. It is well known that a minimal epimorphism is an epimorphism having a small kernel. For a preradical σ we say that M is a σ-coessential extension of X if there exists a minimal epimorphism $h : M \twoheadrightarrow X$ with $\ker h \in \mathcal{F}_\sigma$.

For a module M, $P_\sigma(M)$ is a σ-coessential extension of M. We say that a subclass $\mathcal{C}$ of Mod-R is closed under taking σ-coessential extensions if : for any minimal epimorphism $f : M \twoheadrightarrow X$ with $\ker f \in \mathcal{F}_\sigma$ if $X \in \mathcal{C}$ then $M \in \mathcal{C}$. For the sake of simplicity we say that M is a σ-coessential extension of M/N if N is a σ-torsionfree small submodule of M. We say that a subclass $\mathcal{C}$ of Mod-R is closed under taking σ-coessential extensions if : if $M/N \in \mathcal{C}$ then $M \in \mathcal{C}$ for any σ-torsion free small submodule N of any module M.

The final version of this paper will be submitted for publication elsewhere.

We say that a subclass $\mathcal{C}$ of Mod-R is closed under taking $\mathcal{F}_\sigma$-factor modules if : if $M \in \mathcal{C}$ and N is a σ-torsionfree submodule of M then $M/N \in \mathcal{C}$.

2. COSTABLE TORSION THEORY

Lemma 1. *Let σ be an idempotent radical. For a module M and its submodule N, consider the following diagram with exact rows.*

$$\begin{array}{ccccccc} 0 \to K_\sigma(M) & \to & P_\sigma(M) & \overset{f}{\to} & M & \to & 0 \\ & & & & \downarrow_j & & \\ 0 \to K_\sigma(M/N) & \to & P_\sigma(M/N) & \underset{g}{\to} & M/N & \to & 0, \end{array}$$

where f and g are epimorphisms associated with the σ-projective covers and j is the canonical epimorphism. Since g is a minimal epimorphism, there exists an epimorphism $h : P_\sigma(M) \to P_\sigma(M/N)$ induced by the σ-projectivity of $P_\sigma(M)$ such that $jf = gh$. Then the following conditions hold.

(1) If M is a σ-coessential extension of M/N, then $h : P_\sigma(M) \to P_\sigma(M/N)$ is an isomorphism.

(2) Moreover if σ is epi-preserving and $h : P_\sigma(M) \to P_\sigma(M/N)$ is an isomorphism, then M is a σ-coessential extension of M/N.

Proof. (1): Let $N \in \mathcal{F}_\sigma$ be a small submodule of a module M. Since jf is an epimorphism and g is a minimal epimorphism, h is also an epimorphism. Since $j(f(\ker h)) = g(h(\ker h)) = g(0) = 0$, it follows that $f(\ker h) \subseteq \ker j = N \in \mathcal{F}_\sigma$, and so $f(\ker h) \in \mathcal{F}_\sigma$. Let $f|_{\ker h}$ be the restriction of f to $\ker h$. Then it follows that $\ker(f|_{\ker h}) = \ker h \cap \ker f = \ker h \cap K_\sigma(M) \subseteq K_\sigma(M) \in \mathcal{F}_\sigma$. Consider the exact sequence $0 \to \ker f|_{\ker h} \to \ker h \to f(\ker h) \to 0$. Since $\mathcal{F}_\sigma$ is closed under taking extensions, it follows that $\ker h \in \mathcal{F}_\sigma$. As $P_\sigma(M/N)$ is σ-projective, the exact sequence $0 \to \ker h \to P_\sigma(M) \to P_\sigma(M/N) \to 0$ splits, and so there exists a submodule L of $P_\sigma(M)$ such that $P_\sigma(M) = L \oplus \ker h$. So it follows that $f(P_\sigma(M)) = f(L) + f(\ker h)$. As $f(\ker h) \subseteq N$ and $f(P_\sigma(M)) = M$, $M = f(L) + N$. Since N is small in M, it follows that $M = f(L)$. As f is a minimal epimorphism, it follows that $P_\sigma(M) = L$ and $\ker h = 0$, and so $h : P_\sigma(M) \simeq P_\sigma(M/N)$, as desired.

(2): Suppose that $h : P_\sigma(M) \simeq P_\sigma(M/N)$. By the commutativity of the above diagram with h, it follows that $h(f^{-1}(N)) \subseteq K_\sigma(M/N) \in \mathcal{F}_\sigma$. Since h is an isomorphism, $f^{-1}(N) \in \mathcal{F}_\sigma$. As $f|_{f^{-1}(N)} : f^{-1}(N) \to N \to 0$ and σ is an epi-preserving preradical, it follows that $N \in \mathcal{F}_\sigma$. Next we will show that N is small in M. Let K be a submodule of M such that $M = N + K$. If $f^{-1}(K) \subsetneqq P_\sigma(M)$, then $h(f^{-1}(K)) \subsetneqq P_\sigma(M/N)$ as h is an isomorphism. Since $g(h(f^{-1}(K))) = j(f(f^{-1}(K))) = j(K) = (K + N)/N = M/N$ and g is a minimal epimorphism, this is a contradiction. Thus it holds that $f^{-1}(K) = P_\sigma(M)$, and so $K = f(f^{-1}(K)) = f(P_\sigma(M)) = M$. Thus it follows that N is small in M. □

We call a preradical t σ-costable if $\mathcal{F}_t$ is closed under taking σ-projective covers. Now we characterize σ-costable preradicals.

Theorem 2. *Let t be a radical and σ be an idempotent radical. Consider the following conditions.*

(1) t is σ-costable.

(2) $P/t(P)$ is σ-projective for any σ-projective module P.

(3) For any module M consider the following commutative diagram, then $t(P_\sigma(M))$ is contained in $kerf$.

$$\begin{array}{ccc} P_\sigma(M) & \xrightarrow{h} & M \to 0 \\ \downarrow_f & & \downarrow_j \\ P_\sigma(M/t(M)) & \underset{g}{\to} & M/t(M) \to 0, \end{array}$$

where j is a canonical epimorphism, h and g are epimorphisms associated with their projective covers and f is a morphism induced by σ-projectivity of $P_\sigma(M)$.

(4) $\mathcal{F}_t$ is closed under taking σ-coessential extensions.

(5) For any σ-projective module P such that $t(P) \in \mathcal{F}_\sigma$, $t(P)$ is a direct sumand of P.

Then $(1) \Leftarrow (5) \Longleftarrow (2) \Longleftrightarrow (1) \Longleftrightarrow (3), (4) \Longrightarrow (1)$ *hold. Moreover if $\mathcal{F}_t$ is closed under taking $\mathcal{F}_\sigma$-*factor *modules, then all conditions are equivalent.*

Proof. $(1) \to (2)$: Let P be a σ-projective module. Since $P/t(P) \in \mathcal{F}_t$, it follows that $P_\sigma(P/t(P)) \in \mathcal{F}_t$ by the assumption. Consider the following commutative diagram.

$$\begin{array}{ccc} & & P \\ & f \swarrow & \downarrow h \\ 0 \to K_\sigma(P/t(P)) \to P_\sigma(P/t(P)) & \underset{g}{\to} & P/t(P) \to 0, \end{array}$$

where h is a canonical epimorphism, g is an epimorphism associated with the σ-projective cover of $P/t(P)$ and f is a morphism induced by σ-projectivity of $P_\sigma(P/t(P))$. Since $f(t(P)) \subseteq t(P_\sigma(P/t(P))) = 0$, f induces $f' : P/t(P) \to P_\sigma(P/t(P))$ $(x + t(P) \longmapsto f(x))$. Thus for $x \in P$, $h(x) = gf(x) = gf'h(x)$. So the above exact sequence splits. Therefore $P/t(P)$ is a direct summand of σ-projective module $P_\sigma(P/t(P))$, and so $P/t(P)$ is also a σ-projective module, as desired.

$(2) \to (5)$: Let P be σ-projective and $t(P) \in \mathcal{F}_\sigma$. By the assumption $P/t(P)$ is σ-projective. Thus the sequence $(0 \to t(P) \to P \to P/t(P) \to 0)$ splits, and so $t(P)$ is a direct summand of P.

$(5) \to (1)$: Let M be in $\mathcal{F}_t$. Consider the exact sequence $0 \to K_\sigma(M) \to P_\sigma(M) \underset{f}{\to} M \to 0$. Since $f(t(P_\sigma(M))) \subseteq t(M) = 0$, $K_\sigma(M) = \ker f \supseteq t(P_\sigma(M))$. As $K_\sigma(M) \in \mathcal{F}_\sigma$, $t(P_\sigma(M)) \in \mathcal{F}_\sigma$. Since $P_\sigma(M)$ is σ-projective, $t(P_\sigma(M))$ is a direct summand of $P_\sigma(M)$ by the assumption. Thus there exists a submodule K of $P_\sigma(M)$ such that $P_\sigma(M) = t(P_\sigma(M)) \oplus K$. Since $K_\sigma(M) = \ker f \supseteq t(P_\sigma(M))$, $P_\sigma(M) = K_\sigma(M) + K$. As $K_\sigma(M)$ is small in $P_\sigma(M)$, $P_\sigma(M) = K$. Thus $t(P_\sigma(M)) = 0$, as desired.

$(1) \to (3)$: Consider the following commutative diagram.

$$\begin{array}{ccc} P_\sigma(M) & \xrightarrow{h} & M \to 0 \\ f \downarrow & & \downarrow j \\ P_\sigma(M/t(M)) & \xrightarrow{g} & M/t(M) \to 0, \end{array}$$

where j is a canonical epimorphism, h and g are epimorphisms associated with their projective covers and f is a morphism induced by σ-projectivity of $P_\sigma(M)$. As g is a minimal epimorphism, f is an epimorphism. By the assumption $P_\sigma(M/t(M)) \in \mathcal{F}_t$, and so $f(t(P_\sigma(M))) \subseteq t(P_\sigma(M/t(M))) = 0$. Hence $t(P_\sigma(M)) \subseteq \ker f$.

(3) $\to$ (1) : Let M be in $\mathcal{F}_t$. By the above commutative diagram, f is an identity. Thus by the assumption $t(P_\sigma(M)) \subseteq \ker f = 0$, as desired.

(1) $\to$ (4) : Let $N \in \mathcal{F}_\sigma$ be a small submodule of a module M such that $M/N \in \mathcal{F}_t$. By the assumption $P_\sigma(M/N) \in \mathcal{F}_t$. By Lemma1, $P_\sigma(M/N) \simeq P_\sigma(M)$, and so $P_\sigma(M) \in \mathcal{F}_t$. Consider the sequence $0 \to K_\sigma(M) \to P_\sigma(M) \to M \to 0$. Since $\mathcal{F}_t$ is closed under taking $\mathcal{F}_\sigma$-factor modules, it follows that $M \in \mathcal{F}_t$, as desired.

(4) $\to$ (1) : Since $P_\sigma(M)$ is σ-coessential extension of a module M in $\mathcal{F}_t$, $\mathcal{F}_t$ is closed under taking σ-projective covers. □

Remark 3. It is well known that t is epi-preserving if and only if t is a radical and $\mathcal{F}_t$ is closed under taking factor modules. Therefore if t is epi-preserving and σ be an idempotent radical, then all conditions in Theorem 2 are equivalent.

Next if σ is identity, then the following corollary holds. The following have the another characterization of Theorem1.1 of [3].

Corollary 4. *For a radical t the following conditions except (4) are equivalent. Moreover if t is an epi-preserving preradical, then all conditions are equivalent.*

(1) t is costable, that is, $\mathcal{F}_t$ is closed under taking projective covers.

(2) $P/t(P)$ is projective for any projective module P.

(3)
$$\begin{array}{ccc} P(M) & \overset{h}{\to} & M \to 0 \\ \downarrow_f & & \downarrow_j \\ P(M/t(M)) & \underset{g}{\to} & M/t(M) \to 0, \end{array}$$

where j is a canonical epimorphism, h and g are epimorphisms associated with their projective covers and f is induced by the projectivity of $P(M)$. Then $t(P(M))$ is contained in $\ker f$.

(4) $\mathcal{F}_t$ is closed under taking coessential extensions.

(5) For any projective module P, $t(P)$ is a direct summand of P.

3. DUALIZATION OF ECKMAN & SHOPF'S THEOREM

In [8] we state a torsion theoretic generalization of Eckman & Shopf's Theorem, as follows. Let σ be a left exact radical and $0 \to M \to E$ be a exact sequence of Mod-R. Then the following conditions from (1) to (4) are equivalent. (1) E is σ-injective and σ-essential extension of M. (2) E is minimal in $\{Y \in \text{Mod-}R | M \hookrightarrow Y \text{ and } Y \text{ is } \sigma\text{-injective}\}$. (3) E is maximal in $\{Y \in \text{Mod-}R | M \hookrightarrow Y \text{ and } Y \text{ is } \sigma\text{-essential extension of } M\}$. (4) E is isomorphic to $E_\sigma(M)$, where $\sigma(E(M)/M) = E_\sigma(M)/M$. Here we dualised this.

Lemma 5. *If P is σ-projective, then $P_\sigma(P)$ is isomorphic to P.*

Theorem 6. *Let $P \overset{f}{\to} M \to 0$ be a exact sequence of Mod-R. Let σ is an idempotent radical. Consider the following conditions, then the implications* (1) $\iff$ (3) *and* (1) $\implies$ (2) *hold. Moreover if σ is an epi-preserving preradical, then all conditions are equivalent.*

(1) P is σ-projective and $P \overset{f}{\twoheadrightarrow} M$ is a σ-coessential extension of M.

(2) P is a minimal σ-projective extension of M(i.e. P is σ-projective and if I is σ-projective and $P \overset{h}{\twoheadrightarrow} I, I \twoheadrightarrow M$, then h is an isomorphism.).

(3) P is a maximal σ-coessential extension of M (i.e. $P \overset{f}{\twoheadrightarrow} M$ is σ-coessential extension of M and if there exists an epimorphism $I \overset{h}{\twoheadrightarrow} P$ and $I \overset{h}{\twoheadrightarrow} P \twoheadrightarrow M$ is σ-coessential of M, then h is an isomorphism.).

(4) P is isomorphic to $P_\sigma(M)$.

Proof. (1)→(2): Let P be σ-projective and $P \overset{f}{\twoheadrightarrow} M$ be a σ-coessential extension of M. Consider the following diagram.

$$\begin{array}{ccccccc} 0 \to & \ker h & \to & P & \overset{h}{\to} & I & \to 0 \\ & & & & \searrow_f & \downarrow_g & \\ & & & & & M, & \end{array}$$

where I is σ-projective, g and h are epimorphisms such that $gh = f$.

Since $\mathcal{F}_\sigma \ni f^{-1}(0) = h^{-1}(g^{-1}(0)) \supseteq h^{-1}(0)$, it follows that $\mathcal{F}_\sigma \ni h^{-1}(0) = \ker h$. As f is a minimal epimorphism and g is an epimorphism, h is also a minimal epimorphism. Since I is σ-projective, there exists a submodule L of P such that $P = \ker h \oplus L$ and $L \cong I$. As $\ker h$ is small in P, $P = L$, and so $P \cong I$.

(2)→(1): Let σ be an epi-preserving idempotent radical and P be a minimal σ-projective extension of M. Consider the following commutative diagram.

$$\begin{array}{ccccc} P_\sigma(P) & \overset{j}{\to} & P & \to & 0 \\ g\downarrow & & \downarrow f & & \\ P_\sigma(M) & \overset{h}{\to} & M & \to & 0, \end{array}$$

where h and j are epimorphisms associated with the projective covers of M and P respectively and g is an induced epimorphism by the σ-projectivity of $P_\sigma(P)$. Since P is σ-projective, j is an isomorphism by Lemma 4. As $P_\sigma(P)$ and $P_\sigma(M)$ are σ-projective, g is an isomorphism by the assumption. By Lemma 1, it follows that $P \overset{f}{\to} M \to 0$ is a σ-coessential extension of M.

(1)→(3): Let $I \overset{g}{\to} P$ be an epimorphism. Let $P \overset{f}{\twoheadrightarrow} M$ and $I \overset{h}{\twoheadrightarrow} M$ be σ-coessential extensions of M such that $fg = h$. Consider the following exact diagram.

$$\begin{array}{ccccc} & & I & & \\ & g \swarrow & \downarrow h & & \\ P & \underset{f}{\to} & M & \to & 0 \end{array}$$

Since f is a minimal epimorphism, g is an epimorphim. As h and f are minimal epimorphisms, g is a minimal epimorphim. Since $\mathcal{F}_\sigma \ni h^{-1}(0) = g^{-1}(f^{-1}(0)) \supseteq g^{-1}(0)$, it follows that $\mathcal{F}_\sigma \ni g^{-1}(0)$. Since P is σ-projective, $0 \to \ker g \to I \overset{g}{\to} P \to 0$ splits, and so there exists a submodule H of I such that $H \cong P$ and $I = \ker g \oplus H$. As $\ker g$ is small in I, $I = H \cong P$, as desired.

(3)→(1): We show that P is σ-projective. Since $P \overset{f}{\twoheadrightarrow} M$ is a σ-coessential extension of M by the assumption, an induced morphism $P_\sigma(P) \to P_\sigma(M)$ is an isomorphism by Lemma 1. Consider the following commutative diagram.

$$\begin{array}{ccccc} P_\sigma(P) & \to & P & \to & 0 \\ \downarrow & & \downarrow & & \\ P_\sigma(M) & \to & M & \to & 0. \end{array}$$

Since $P_\sigma(P) \simeq P_\sigma(M) \twoheadrightarrow M$ is a σ-coessential extension of M and $P \overset{f}{\twoheadrightarrow} M$ is a σ-coessential extension of M, it follows that $P_\sigma(P) \cong P$ by the assumption, and so P is σ-projective.

(1)→(4): By Lemma 1, $P_\sigma(P) \simeq P_\sigma(M)$. By Lemma 4, $P_\sigma(P) \simeq P$, and so $P \simeq P_\sigma(M)$ as desired.

(4)→(1): It is clear. □

In Theorem 5, if $\sigma = 1$, then the following corollary is obtained.

Corollary 7. *Let $P \overset{f}{\to} M \to 0$ be a exact sequence of Mod-R. Then the following conditions are equivalent.*

(1) P is projective and $P \overset{f}{\twoheadrightarrow} M$ is a coessential extension of M(that is, $\ker f$ is small in M).

(2) P is a minimal projective extension of M(i.e. P is projective and if I is projective and $P \overset{h}{\twoheadrightarrow} I, I \twoheadrightarrow M$, then h is an isomorphism).

(3) P is a maximal coessential extension of M(i.e. $P \overset{f}{\twoheadrightarrow} M$ is coessential extension of M and if there exists an epimorphism $I \overset{h}{\twoheadrightarrow} P$ and $I \overset{h}{\twoheadrightarrow} P \twoheadrightarrow M$ is coessential of M, then h is an isomorphism.).

(4) P is isomorphic to $P(M)$.

4. A GENERALIZATION OF WU, JANS AND MIYASHITA'S THEOREM AND AZUMAYA'S THEOREM

In [8] we state a torsion theoretic generalization of Johnson and Wong's Theorem. Here we study a dualization of this. For a module M and N, we call M σ-N-projective if $\mathrm{Hom}_R(M, \;\;)$ preserves the exactness of the short exact sequence $0 \to K \to N \to N/K \to 0$ with $K \in \mathcal{F}_\sigma$.

Theorem 8. *Let M and N be modules. Consider the following conditions for an idempotent radical σ.*

(1) $\gamma(K_\sigma(M)) \subseteq K_\sigma(N)$ holds for any $\gamma \in Hom_R(P_\sigma(M), P_\sigma(N))$.

(2) M is σ-N-projective.

Then the implication (1)→(2) holds. If σ is epi-preserving, then the implication (2)→(1) holds.

Proof. (1)→(2): Let f be in $\mathrm{Hom}_R(M, N/K)$ with $K \in \mathcal{F}_\sigma$. Then there exists $h \in \mathrm{Hom}_R(P_\sigma(M), N)$ such that $f\pi_M^\sigma = nh$, where n is a canonical epimorphism from N to N/K. And there exists $\gamma \in \mathrm{Hom}_R(P_\sigma(M), P_\sigma(N))$ such that $h = \pi_N^\sigma \gamma$. So we have the following commutative diagramm.

$$\begin{array}{ccccc} & & P_\sigma(M) & \overset{\pi_M^\sigma}{\twoheadrightarrow} & M \\ & \gamma \swarrow & \downarrow h & & \downarrow f \\ P_\sigma(N) & \underset{\pi_N^\sigma}{\twoheadrightarrow} & N & \underset{n}{\to} & N/K \end{array}$$

By the assumption, γ induces $\gamma' : P_\sigma(M)/K_\sigma(M) \to P_\sigma(N)/K_\sigma(N)$, and so γ' induces $\gamma'' : M \to N$ such that $f = \gamma'' n$, as desired.

(2)→(1): Let σ be epi-preserving and $\gamma \in \mathrm{Hom}_R(P_\sigma(M), P_\sigma(N))$. We will show that $\gamma(K_\sigma(M)) \subseteq K_\sigma(N)$. We put $T = \gamma(K_\sigma(M)) + K_\sigma(N)$. Since $T \supseteq \gamma(K_\sigma(M))$, γ induces $\gamma' : M \simeq P_\sigma(M)/K_\sigma(M) \to P_\sigma(N)/\gamma(K_\sigma(M)) \to P_\sigma(N)/T \to N/\pi_N^\sigma(T)$ $(\pi_M^\sigma(x) \longleftrightarrow x + K_\sigma(M) \to \gamma(x) + \gamma(K_\sigma(M)) \to \gamma(x) + T \to \pi_N^\sigma(\gamma(x)) + \pi_N^\sigma(T))$. Let n_N be a canonical epimorphism from N to $N/\pi_N^\sigma(T)$. Since $\pi_N^\sigma(T) = \pi_N^\sigma(\gamma(K_\sigma(M)) + K_\sigma(N)) = \pi_N^\sigma(\gamma(K_\sigma(M))$, $K_\sigma(M) \in \mathcal{F}_\sigma$ and $\mathcal{F}_\sigma$ is closed under taking factor modules, it follows that $\pi_N^\sigma(T) \in \mathcal{F}_\sigma$. Since M is σ-N-projective, there exists $\beta : M \to N$ such that $\gamma' = n_N\beta$. Therefore we have the following commutative diagramm.

$$\begin{array}{ccccccc} & & & & M & & \\ & & & \beta \swarrow & \downarrow \gamma' & & \\ 0 \to \pi_N^\sigma(T) & \to & N & \underset{n_N}{\to} & N/\pi_N^\sigma(T) & \to & 0 \end{array}$$

·By the σ-projectivity of $P_\sigma(M)$, there exists $\alpha : P_\sigma(M) \to P_\sigma(N)$ such that $\pi_N^\sigma\alpha = \beta\pi_M^\sigma$. Thus we have the following commutative diagramm.

$$\begin{array}{ccccccc} 0 \to K_\sigma(M) & \to & P_\sigma(M) & \overset{\pi_M^\sigma}{\to} & M & \to & 0 \\ & & \downarrow_\alpha & & \downarrow_\beta & & \\ 0 \to K_\sigma(N) & \to & P_\sigma(N) & \underset{\pi_N^\sigma}{\to} & N & \to & 0 \end{array}$$

Thus by the commutativity of the above diagram, we have $\alpha(K_\sigma(M)) \subseteq K_\sigma(N)$.

We put $X = \{x \in P_\sigma(M) | \gamma(x) - \alpha(x) \in K_\sigma(N)\}$. We will show that $X + K_\sigma(M) = P_\sigma(M)$. For any $x \in P_\sigma(M)$ it follows that $\gamma'(\pi_M^\sigma(x)) = \pi_N^\sigma(\gamma(x)) + \pi_N^\sigma(T)$, $(n_N\beta)(\pi_M^\sigma(x)) = \beta(\pi_M^\sigma x) + \pi_N^\sigma(T)$ and $\gamma' = n_N\beta$, it follows that $\pi_N^\sigma(\gamma(x)) + \pi_N^\sigma(T) = \beta(\pi_M^\sigma x) + \pi_N^\sigma(T)$, and so $\pi_N^\sigma(\gamma(x)) - \beta(\pi_M^\sigma x) \in \pi_N^\sigma(T)$. Since $\pi_N^\sigma\alpha = \beta\pi_M^\sigma$, it follows that $\pi_N^\sigma(\gamma(x)) - \pi_N^\sigma(\alpha(x)) \in \pi_N^\sigma(T)$, and so $\gamma(x) - \alpha(x) \in T + (\pi_N^\sigma)^{-1}(0) = T + K_\sigma(N) = \gamma(K_\sigma(M)) + K_\sigma(N)$. Thus there exists $m \in K_\sigma(M)$ such that $\gamma(x) - \alpha(x) - \gamma(m) \in K_\sigma(N)$, and so $\gamma(x - m) - \alpha(x - m) \in \alpha(m) + K_\sigma(N) \subseteq \alpha(K_\sigma(M)) + K_\sigma(N) = K_\sigma(N)$. Therefore it follows that $x - m \in X$, and so $x \in K_\sigma(M) + X$. Thus we conclude that $P_\sigma(M) = K_\sigma(M) + X$. Since $K_\sigma(M)$ is small in $P_\sigma(M)$, it holds that $X = P_\sigma(M)$. Thus it follows that $\{x \in P_\sigma(M) | \gamma(x) - \alpha(x) \in K_\sigma(N)\} = P_\sigma(M)$. Thus if $x \in K_\sigma(M) (\subseteq P_\sigma(M))$, then $\gamma(x) - \alpha(X) \in K_\sigma(N)$, and so $\gamma(x) \in \alpha(x) + K_\sigma(N) \subseteq \alpha(K_\sigma(M)) + K_\sigma(N) = K_\sigma(N)$, and so it follows that $\gamma(K_\sigma(M)) \subseteq K_\sigma(N)$. □

In Theorem 7 we put $\sigma = 1$, then we have a generalization of Azumaya's Theorem in [2]. In Theorem 7 we put $M = N$ and $\sigma = 1$, then we have a generalization of Wu, Jans and Miyashita's Theorem in [9] and [5].

References

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