# GRADED FROBENIUS ALGEBRAS AND QUANTUM BEILINSON ALGEBRAS

KENTA UEYAMA

Abstract. Frobenius algebras are one of the important classes of algebras studied in representation theory of finite dimensional algebras. In this article, we will study when given graded Frobenius Koszul algebras are graded Morita equivalent. As applications, we apply our results to quantum Beilinson algebras.

*Key Words:* Frobenius Koszul algebras, quantum Beilinson algebras, graded Morita equivalence.

2010 *Mathematics Subject Classification*: Primary 16W50, 16S37; Secondary 16D90.

## 1. Introduction

This is based on a joint work with Izuru Mori.

Classification of Frobenius algebras is an active project in representation theory of finite dimensional algebras. This article tries to answer the question when given graded Frobenius Koszul algebras are graded Morita equivalent, that is, they have equivalent graded module categories.

This problem is related to classification of quasi-Fano algebras. It is known that every finite dimensional algebra of global dimension 1 is a path algebra of a finite acyclic quiver up to Morita equivalence, so such algebras can be classified in terms of quivers. As an obvious next step, it is interesting to classify finite dimensional algebras of global dimension 2 or higher. Recently, Minamoto introduced a nice class of finite dimensional algebras of finite global dimension, called (quasi-)Fano algebras [2], which are a very interesting class of algebras to study and classify. It was shown that, for a graded Frobenius Koszul algebra $A$, we can define another algebra $\nabla A$, called the quantum Beilinson algebra associated to $A$, and with some additional assumptions, $\nabla A$ turns out to be a quasi-Fano algebra. Moreover, it was shown that two graded Frobenius algebras $A, A'$ are graded Morita equivalent if and only if $\nabla A, \nabla A'$ are isomorphic as algebras, so classifying graded Frobenius (Koszul) algebra up to graded Morita equivalence is related to classifying quasi-Fano algebras up to isomorphism (see [3] for details).

In addition, this problem is related to the study of AS-regular algebras which are the most important class of algebras in noncommutative algebraic geometry (see [8]).

Our main theorem (Theorem 9) is as follows. For every co-geometric Frobenius Koszul algebra $A$, we define another graded algebra $\overline{A}$, and see that if two co-geometric Frobenius Koszul algebras $A, A'$ are graded Morita equivalent, then $\overline{A}, \overline{A'}$ are isomorphic as graded algebras. Unfortunately, the converse does not hold in general. On the other hand, the converse is also true for many co-geometric Frobenius Koszul algebras of Gorenstein parameter $-3$.

The detailed version of this paper will be submitted for publication elsewhere.

## 2. Frobenius Koszul algebras

Throughout this paper, we fix an algebraically closed field $k$ of characteristic 0, and we assume that all vector spaces and algebras are over $k$ unless otherwise stated. In this paper, a graded algebra means a connected graded algebra finitely generated in degree 1, that is, every graded algebra can be presented as $A = T(V)/I$ where $V$ is a finite dimensional vector space, $T(V)$ is the tensor algebra on $V$ over $k$, and $I$ is a homogeneous two-sided ideal of $T(V)$. We denote by $\operatorname{GrMod} A$ the category of graded right $A$-modules. Morphisms in $\operatorname{GrMod} A$ are right $A$-module homomorphisms preserving degrees. We say that two graded algebras $A$ and $A'$ are graded Morita equivalent if $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$.

For a graded module $M \in \operatorname{GrMod} A$ and an integer $n \in \mathbb{Z}$, we define the truncation $M_{\geq n} := \bigoplus_{i \geq n} M_i \in \operatorname{GrMod} A$ and the shift $M(n) \in \operatorname{GrMod} A$ by $M(n)_i := M_{n+i}$ for $i \in \mathbb{Z}$. For $M, N \in \operatorname{GrMod} A$, we write

$$\underline{\operatorname{Hom}}_A(M, N) := \bigoplus_{n \in \mathbb{Z}} \operatorname{Hom}_{\operatorname{GrMod} A}(M, N(n)).$$

We denote by $V^*$ the dual vector space of a vector space $V$. If $M$ is a graded right (resp. left) module over a graded algebra $A$, then we denote by $M^* := \underline{\operatorname{Hom}}_k(M, k)$ the dual graded vector space of $M$ by abuse of notation, i.e. $(M^*)_i := (M_{-i})^*$. Note that $M^*$ has a graded left (resp. right) $A$-module structure.

Let $A$ be a graded algebra, and $\tau \in \operatorname{Aut}_k A$ a graded algebra automorphism. For a graded right $A$-module $M \in \operatorname{GrMod} A$, we define a new graded right $A$-module $M_\tau \in \operatorname{GrMod} A$ by $M_\tau = M$ as a graded vector space with the new right action $m * a := m\tau(a)$ for $m \in M$ and $a \in A$. If $M$ is a graded $A$-$A$ bimodule, then $M_\tau$ is also a graded $A$-$A$ bimodule by this new right action. The rule $M \mapsto M_\tau$ is a $k$-linear autoequivalence for $\operatorname{GrMod} A$.

A graded algebra $A$ is called quadratic if $A \cong T(V)/(R)$ where $R \subseteq V \otimes_k V$ is a subspace and $(R)$ is the ideal of $T(V)$ generated by $R$. If $A = T(V)/(R)$ is a quadratic algebra, then we define the dual graded algebra by $A^! := T(V^*)/(R^\perp)$ where

$$R^\perp := \{\lambda \in V^* \otimes_k V^* \cong (V \otimes_k V)^* \mid \lambda(r) = 0 \text{ for all } r \in R\}.$$

Clearly, $A^!$ is again a quadratic algebra and $(A^!)^! \cong A$ as graded algebras.

We now recall the definitions of Koszul algebras and graded Frobenius algebras. Frobenius algebras are one of the main classes of algebras of study in representation theory of finite dimensional algebras.

**Definition 1.** Let $A$ be a connected graded algebra, and suppose $k \in \operatorname{GrMod} A$ has a minimal free resolution of the form

$$\cdots \to \bigoplus_{j=1}^{r_i} A(-s_{ij}) \to \cdots \to \bigoplus_{j=1}^{r_0} A(-s_{0j}) \to k \to 0.$$

The complexity of $A$ is defined by

$$c_A := \inf\{d \in \mathbb{R}^+ \mid r_i \leq c i^{d-1} \text{for some constant } c > 0, i \gg 0\}.$$

We say that $A$ is Koszul if $s_{ij} = i$ for all $1 \leq j \leq r_i$ and all $i \in \mathbb{N}$.

It is known that if $A$ is Koszul, then $A$ is quadratic, and its dual graded algebra $A^!$ is also Koszul, which is called the Koszul dual of $A$.

**Definition 2.** A graded algebra $A$ is called a graded Frobenius algebra of Gorenstein parameter $\ell$ if $A^* \cong {}_\nu A(-\ell)$ as graded $A$-$A$ bimodules for some graded algebra automorphism $\nu \in \operatorname{Aut}_k A$, called the Nakayama automorphism of $A$. We say that $A$ is graded symmetric if $A^* \cong A(-\ell)$ as graded $A$-$A$ bimodules.

A skew exterior algebra

$$A = k\langle x_1, \ldots, x_n\rangle/(\alpha_{ij}x_ix_j + x_jx_i, x_i^2)$$

where $\alpha_{ij} \in k$ such that $\alpha_{ij}\alpha_{ji} = \alpha_{ii} = 1$ for $1 \le i, j \le n$ is a typical example of a Frobenius Koszul algebra.

At the end of this section, we give an interesting result about graded Morita equivalence of graded skew exterior algebras. It is known that every (ungraded) Frobenius algebra which is Morita equivalent to symmetric algebra is symmetric. The situation in the graded case is different as the following theorem shows.

**Proposition 3.** [7] *Every skew exterior algebra is graded Morita equivalent to a graded symmetric skew exterior algebra.*

For example, a 3-dimensional skew exterior algebra

$$A = k\langle x, y, z\rangle/(\alpha yz + zy, \beta zx + xz, \gamma xy + yx, x^2, y^2, z^2)$$

is graded Morita equivalent to a symmetric skew exterior algebra

$$A = k\langle x, y, z\rangle/(\sqrt[3]{\alpha\beta\gamma}yz + zy, \sqrt[3]{\alpha\beta\gamma}zx + xz, \sqrt[3]{\alpha\beta\gamma}xy + yx, x^2, y^2, z^2).$$

## 3. Co-geometric Frobenius Koszul algebras

In order to state our main result, let us define a co-geometric algebra (see [4] for details).

**Definition 4.** [4] Let $A = T(V)/I$ be a graded algebra. We say that $N \in \operatorname{GrMod} A$ is a co-point module if $N$ has a free resolution of the form

$$\cdots \to A(-2) \to A(-1) \to A \to N \to 0\ .$$

For a graded algebra $A = T(V)/I$, we can define the pair $\mathcal{P}^!(A) = (E, \sigma)$ consisting of the set $E \subseteq \mathbb{P}(V)$ and the map $\sigma : E \to E$ as follows:

- $E := \{p \in \mathbb{P}(V) \mid N_p := A/pA \in \operatorname{GrMod} A \text{ is a co-point module}\}$, and
- the map $\sigma : E \to E$ is defined by $\Omega N_p(1) = N_{\sigma(p)}$.

Meanwhile, for a geometric pair $(E, \sigma)$ consists of a closed subscheme $E \subseteq \mathbb{P}(V)$ and an automorphism $\sigma \in \operatorname{Aut}_k E$, we can define the algebra $\mathcal{A}^!(E, \sigma)$ as follows:

$$\mathcal{A}^!(E, \sigma) := (T(V^*)/(R))^! \quad \text{where } R := \{f \in V^* \otimes_k V^* \mid f(p, \sigma(p)) = 0, \forall p \in E\}.$$

**Definition 5.** [4] A graded algebra $A = T(V)/I$ is called co-geometric if $A$ satisfies the following conditions:

- $\mathcal{P}^!(A)$ consisting of a closed subscheme $E \subseteq \mathbb{P}(V)$ and an automorphism $\sigma \in \operatorname{Aut}_k E$,
- $A^!$ is noetherian, and
- $A \cong \mathcal{A}^!(\mathcal{P}^!(A))$.

**Example 6.** [4] Let $A = k\langle x, y\rangle/(\alpha xy + yx, x^2, y^2)$ be a 2-dimensional skew exterior algebra. Then for any point $p = (a, b) \in \mathbb{P}(V) = \mathbb{P}^1$, $N_p = A/(ax + by)A$ has a free resolution of the form

$$\cdots \longrightarrow A(-2) \xrightarrow{(\alpha^2 ax+by)\cdot} A(-1) \xrightarrow{(\alpha ax+by)\cdot} A \xrightarrow{(ax+by)\cdot} N_p \longrightarrow 0 \; .$$

Since $\Omega N_p(1) = A/(\alpha ax + by)A$, it follow that

$$\mathcal{P}^!(A) = (\mathbb{P}^1, \sigma), \text{ where } \sigma(a, b) := (\alpha a, b).$$

In fact, $A$ is co-geometric.

**Example 7.** The algebras below are examples of co-geometric algebras.

- A Frobenius Koszul algebra of finite complexity and of Gorenstein parameter $-3$. For example, if $A = k\langle x, y, z\rangle$ with the defining relations

$$\begin{aligned} &\alpha x^2 - \gamma yz, \quad \alpha y^2 - \gamma zx, \quad \alpha z^2 - \gamma xy, \\ &\beta yz - \alpha zy, \quad \beta zx - \alpha xz, \quad \beta xy - \alpha yx. \end{aligned}$$

  for a generic choice of $\alpha, \beta, \gamma \in k$, then $A = \mathcal{A}^!(E, \sigma)$ is a Frobenius Koszul algebra of complexity 3 and of Gorenstein parameter $-3$ such that

$$E = \mathcal{V}(\alpha\beta\gamma(x^3 + y^3 + z^3) - (\alpha^3 + \beta^3 + \gamma^3)xyz) \subset \mathbb{P}^2$$

  is an elliptic curve and $\sigma \in \operatorname{Aut}_k E$ is the translation automorphism by the point $(\alpha, \beta, \gamma) \in E$.
- The skew exterior algebra.

Let $A = \mathcal{A}^!(E, \sigma)$ be a co-geometric Frobenius Koszul algebra of Gorenstein parameter $-\ell$ with the Nakayama automorphism $\nu \in \operatorname{Aut}_k A$. The restriction $\nu|_{A_1} = \tau|_V$ induces an automorphism $\nu \in \operatorname{Aut}_k \mathbb{P}(V)$. Moreover, $\nu \in \operatorname{Aut}_k \mathbb{P}(V)$ restricts to an automorphism $\nu \in \operatorname{Aut}_k E$ by abuse of notation (see [5] for details). We can now define a new graded algebra $\overline{A}$ as follows:

$$\overline{A} := \mathcal{A}^!(E, \nu\sigma^\ell).$$

**Example 8.** If $A = k\langle x, y, z\rangle$ with the defining relations

$$\begin{aligned} &x^2 + \beta xz, \quad zx + xz, \quad z^2, \\ &y^2 + \alpha yz, \quad zy + yz, \quad xy + yx - (\beta + \gamma)xz - (\alpha + \gamma)yz, \end{aligned}$$

where $\alpha, \beta, \gamma \in k, \alpha + \beta + \gamma \neq 0$, then $A = \mathcal{A}^!(E, \sigma)$ is a Frobenius Koszul algebra of complexity 3 and of Gorenstein parameter $-3$ such that

$$E = \mathcal{V}(x) \cup \mathcal{V}(y) \cup \mathcal{V}(x - y) \subset \mathbb{P}^2$$

is a union of three lines meeting at one point, and $\sigma \in \operatorname{Aut}_k E$ is given by

$$\begin{aligned} \sigma|_{\mathcal{V}(x)}(0, b, c) &= (0, b, \alpha b + c), \\ \sigma|_{\mathcal{V}(y)}(a, 0, c) &= (a, 0, \beta a + c), \\ \sigma|_{\mathcal{V}(x-y)}(a, a, c) &= (a, a, -\gamma a + c) \end{aligned}$$

In this case, $\nu \in \operatorname{Aut}_k E$ induced by the Nakayama automorphism $\nu \in \operatorname{Aut}_k A$ is given by

$$\nu(a,b,c) = (a,b,(\alpha+\gamma-2\beta)a + (\beta+\gamma-2\alpha)b + c)$$

It follows that $\overline{A} = \mathcal{A}^!(E, \nu\sigma^3)$ is $k\langle x,y,z\rangle$ with the defining relations

$$x^2 + (\alpha+\beta+\gamma)xz, \quad zx + xz, \quad z^2,$$
$$y^2 + (\alpha+\beta+\gamma)yz, \quad zy + yz, \quad xy + yx - 2(\alpha+\beta+\gamma)xz - 2(\alpha+\beta+\gamma)yz.$$

Our main result is as follows.

**Theorem 9.** [7] *Let $A, A'$ be co-geometric Frobenius Koszul algebras. Then*

$$\operatorname{GrMod} A \cong \operatorname{GrMod} A' \Longrightarrow \overline{A} \cong \overline{A'} \text{ as graded algebras.}$$

*In particular, let $A = \mathcal{A}^!(E,\sigma), A' = \mathcal{A}^!(E',\sigma')$ be Frobenius Koszul algebras of finite complexities and of Gorenstein parameter $-3$ such that $E \cong E'$. Suppose that $E = \mathbb{P}^2$ or $E$ is a reduced and reducible cubic in $\mathbb{P}^2$, then*

$$\operatorname{GrMod} A \cong \operatorname{GrMod} A' \Longleftrightarrow \overline{A} \cong \overline{A'} \text{ as graded algebras.}$$

## 4. Quantum Beilinson Algebras

Finally, we apply our results to quantum Beilinson algebras.

**Definition 10.** [1], [6] Let $A$ be a graded Frobenius algebra of Gorenstein parameter $-\ell$. Then the quantum Beilinson algebra of $A$ is defined by

$$\nabla A := \begin{pmatrix} A_0 & A_1 & \cdots & A_{\ell-1} \\ 0 & A_0 & \cdots & A_{\ell-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_0 \end{pmatrix}.$$

**Theorem 11.** [3] *Let $A, A'$ be graded Frobenius algebras. Then*

$$\operatorname{GrMod} A \cong \operatorname{GrMod} A' \iff \nabla A \cong \nabla A' \text{ as algebras.}$$

By the above theorem, classifying graded Frobenius algebras up to graded Morita equivalence is the same as classifying quantum Beilinson algebras up to isomorphism.

Quasi-Fano algebras introduced by Minamoto [2] are one of the nice classes of a finite dimensional algebras of finite global dimensions (see [3], [6] for details).

**Definition 12.** A finite dimensional algebra $R$ is called quasi-Fano of dimension $n$ if $\operatorname{gldim} R = n$ and $w_R^{-1}$ is a quasi-ample two-sided tilting complex, that is, $h^i((w_R^{-1})^{\otimes^{\mathbb{L}}_R j}) = 0$ for all $i \neq 0$ and all $j \geq 0$, where $w_R := R^*[-n]$.

Let $A$ be a graded Frobenius Koszul algebra of Gorenstein parameter $-d$. Assume that $A$ has the Hilbert series

$$H_A(t) := \sum_i (\dim_k A_i) t^i = (1+t)^d$$

and that $A^!$ is noetherian. Then $\nabla A$ is a quasi-Fano algebra of dimension $d-1$.

In general, it is not easy to check if two algebras given as path algebras of quivers with relations are isomorphic as algebras by constructing an explicit algebra isomorphism. On the other hand, it is much easier to check if two graded algebras $T(V)/I$ and $T(V')/I'$ generated in degree 1 over $k$ are isomorphic as graded algebras since any such isomorphism is induced by the vector space isomorphism $V \to V'$. In this sense, our main result is useful for the classification of a class of finite dimensional algebras of global dimension 2, namely, quantum Beilinson algebras of global dimension 2.

Fix the Beilinson quiver

$$Q = \bullet \begin{array}{c} \xrightarrow{x_1} \\ \xrightarrow{y_1} \\ \xrightarrow{z_1} \end{array} \bullet \begin{array}{c} \xrightarrow{x_2} \\ \xrightarrow{y_2} \\ \xrightarrow{z_2} \end{array} \bullet$$

and let

$$B = kQ/I, \quad B' = kQ/I', \quad B'' = kQ/I''$$

be path algebras with relations

$$\begin{aligned} I &= (\alpha y_1 z_2 + z_1 y_2, \beta z_1 x_2 + x_1 z_2, \gamma x_1 y_2 + y_1 x_2, x_1 x_2, y_1 y_2, z_1 z_2) \\ I' &= (x_1 x_2 + \alpha' y_1 z_2, y_1 y_2 + \beta' z_1 x_2, z_1 z_2 + \gamma' x_1 y_2, z_1 y_2, x_1 z_2, y_1 x_2) \\ I'' &= (\alpha'' y_1 z_2 + z_1 y_2, \beta'' z_1 x_2 + x_1 z_2, \beta'' x_1 y_2 + y_1 x_2, x_1 x_2 + y_1 z_2, y_1 y_2, z_1 z_2) \end{aligned}$$

where $\alpha\beta\gamma \neq 0, 1, \alpha'\beta'\gamma' \neq 0, 1, \alpha''(\beta'')^2 \neq 0, 1$. Then $B, B', B''$ are the quantum Beilinson algebras of co-geometric Frobenius Koszul algebras $A, A', A''$ of Gorenstein parameter $-3$

$$\begin{aligned} A &= \mathcal{A}^!(E, \sigma) = k\langle x, y, z\rangle/(\alpha yz + zy, \beta zx + xz, \gamma xy + yx, x^2, y^2, z^2), \\ A' &= \mathcal{A}^!(E', \sigma') = k\langle x, y, z\rangle/(x^2 + \alpha' yz, y^2 + \beta' zx, z^2 + \gamma' xy, zy, xz, yx), \\ A'' &= \mathcal{A}^!(E'', \sigma'') = k\langle x, y, z\rangle/(\alpha'' yz + zy, \beta'' zx + xz, \beta'' xy + yx, x^2 + yz, y^2, z^2), \end{aligned}$$

where $E$ is a triangle and $\sigma \in \operatorname{Aut}_k E$ stabilizes each component, $E'$ is a triangle and $\sigma' \in \operatorname{Aut}_k E'$ circulates three components, and $E''$ is a union of a line and a conic meeting at two points and $\sigma'' \in \operatorname{Aut}_k E''$ stabilizes each component and two intersection points. Since

$$E \cong E' \not\cong E'',$$

we see that

$$B \not\cong B'', \quad B' \not\cong B''.$$

Moreover, it is not difficult to compute

$$\begin{aligned} \overline{A} &= \mathcal{A}^!(E, \nu\sigma^3) \\ &= k\langle x, y, z\rangle/(\alpha\beta\gamma yz + zy, \alpha\beta\gamma zx + xz, \alpha\beta\gamma xy + yx, x^2, y^2, z^2), \\ \overline{A'} &= \mathcal{A}^!(E', \nu'(\sigma')^3) \\ &= k\langle x, y, z\rangle/(yz + \alpha'\beta'\gamma' zy, zx + \alpha'\beta'\gamma' xz, xy + \alpha'\beta'\gamma' yx, x^2, y^2, z^2). \end{aligned}$$

Since $\overline{A}, \overline{A'}$ are skew exterior algebras, it is easy to check when they are isomorphic as graded algebras. Using theorems, the following are equivalent.

(1) $B \cong B'$ as algebras.
(2) $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$.
(3) $\overline{A} \cong \overline{A'}$ as graded algebras.
(4) $\alpha'\beta'\gamma' = (\alpha\beta\gamma)^{\pm 1}$.

## References

[1] X. W. Chen, Graded self-injective algebras "are" trivial extensions, *J. Algebra* **322** (2009), 2601–2606.
[2] H. Minamoto, Ampleness of two-sided tilting complexes, *Int. Math. Res. Not.*, to appear.
[3] H. Minamoto and I. Mori, The structure of AS-Gorenstein algebras, *Adv. Math.*, **226** (2011), 4061–4095.
[4] I. Mori, Co-point modules over Koszul algebras, *J. London Math. Soc.* **74** (2006), 639–656.
[5] ———, Co-point modules over Frobenius Koszul algebras, *Comm. Algebra* **36** (2008), 4659–4677.
[6] ———, B-cosntrcution and C-construction, preprint.
[7] I. Mori and K. Ueyama, Graded Morita equivalences for geometric AS-regular algebras, preprint.
[8] S. P. Smith, *Some finite dimensional algebras related to elliptic curves*, in "Representation theory of algebras and related topics" (Mexico City, 1994) CMS Conf. Proc. **19**, Amer. Math. Soc., Providence, RI (1996), 315–348.



Department of Information Science and Technology
Graduate School of Science and Technology
Shizuoka University
836 Ohya, Suruga-ku, Shizuoka 422-8529 JAPAN
*E-mail address*: `f5144004@ipc.shizuoka.ac.jp`