ALGEBRAIC STRATIFICATIONS OF DERIVED MODULE CATEGORIES AND DERIVED SIMPLE ALGEBRAS

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ABSTRACT. In this note I will survey on some recent progress in the study of recollements of derived module categories.

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The notion of recollement of triangulated categories was introduced in [5] as an analogue of short exact sequence of modules or groups. In representation theory of algebras it provides us with reduction techniques, which have proved very useful, for example, in

- proving conjectures on homological dimensions, see [9];
- computing homological invariants, see [11, 12];
- classifying t-structures, see [14].

In this note I will survey on some recent progress in the study of recollements of derived module categories.

1. RECOLLEMENTS

Let k be a field. For a k-algebra A denote by $\mathcal{D}(A) = \mathcal{D}(\mathsf{Mod}\, A)$ the (unbounded) derived category of the category $\mathsf{Mod}\, A$ of right A-modules. The objects of $\mathcal{D}(A)$ are complexes of right A-modules. The category $\mathcal{D}(A)$ is triangulated with shift functor Σ being the shift of complexes. See [10] for a nice introduction on derived categories.

A *recollement* of derived module categories is a diagram of derived module categories and triangle functors

$$\begin{array}{ccccc} & \overset{i^*}{\longleftarrow} & & \overset{j_!}{\longleftarrow} & \\ \mathcal{D}(B) & \xrightarrow{i_*=i_!} & \mathcal{D}(A) & \xrightarrow{j^!=j^*} & \mathcal{D}(C), \\ & \underset{i^!}{\longleftarrow} & & \underset{j_*}{\longleftarrow} & \end{array} \tag{1.1}$$

where A, B and C are k-algebras, such that

(1) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^! = j^*, j_*)$ are adjoint triples;
(2) $j_!$, i_* and j_* are fully faithful;
(3) $j^* i_* = 0$;

The detailed /final/ version of this paper will be /has been/ submitted for publication elsewhere.

(4) for every object M of $\mathcal{D}(A)$ there are two triangles

$$i_!i^!M \longrightarrow M \longrightarrow j_*j^*M \longrightarrow \Sigma i_!i^!M$$

and

$$j_!j^!M \longrightarrow M \longrightarrow i_*i^*M \longrightarrow \Sigma j_!j^!M\ ,$$

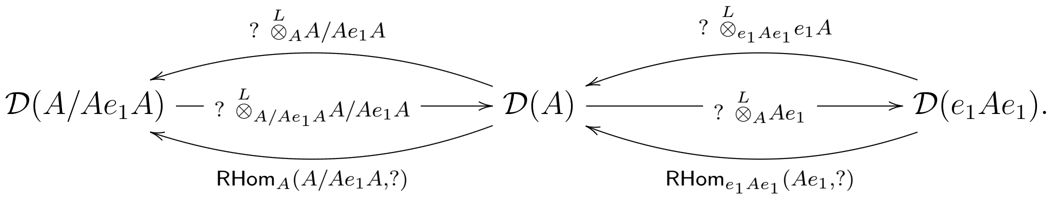
where the four morphisms starting from and ending at M are the units and counits.

Necessary and sufficient conditions under which such a recollement exists were discussed in [13, 16].

Example 1. Let A be the path algebra of the Kronecker quiver

$$1 \rightrightarrows 2\ .$$

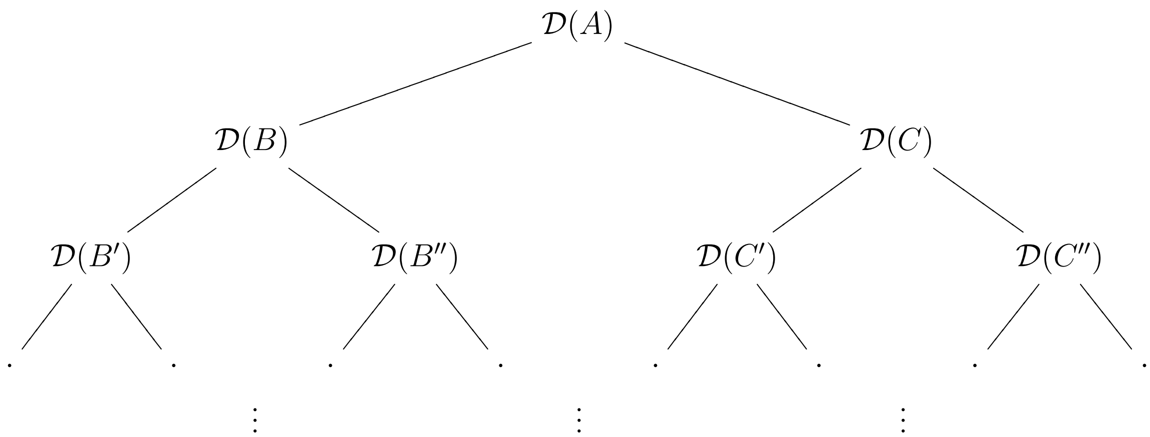
The trivial path e_1 at 1 is an idempotent of A and e_1A is a projective A-module. The following diagram is a recollement



Note that both e_1Ae_1 and A/Ae_1A are isomorphic to k.

2. Algebraic stratifications of derived module categories

Let A be an algebra. An *algebraic stratification* of $\mathcal{D}(A)$ is a sequence of iterated non-trivial recollements of derived module categories. It can be depicted as a binary tree as below, where each edge represents an adjoint triple of triangle functors and each hook represents a recollement



The leaves of the tree are the *simple factors* of the stratification. The following questions are basic:

(a) Does every derived module category admit a finite algebraic stratification?
(b) Do two finite algebraic stratifications of a derived module category have the same number of simple factors? Do they have the same simple factors (up to triangle equivalence and up to reordering)?
(c) Which derived module categories occur as simple factors of some algebraic stratifications?

The question (c) will be discussed in the next section. The questions (a) and (b) ask for a Jordan–Hölder type result for derived module categories. For general (possibly infinite-dimensional) algebras the answers are negative. Below we give some (counter-)examples.

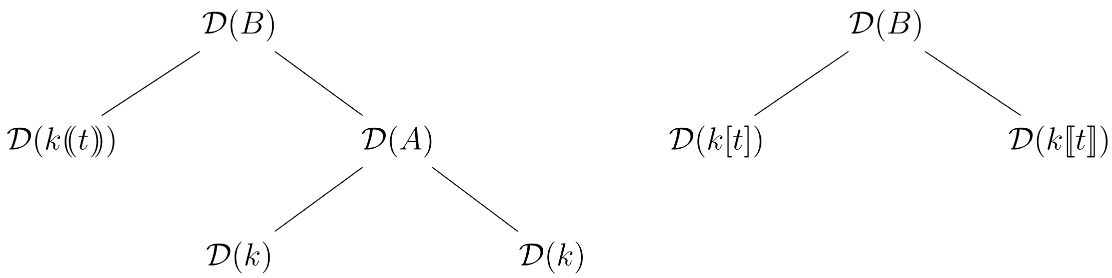
Example 2. ([2]) Let $A = \prod_{\mathbb{N}} k$. Then $\mathcal{D}(A)$ does not admit a finite algebraic stratification.

Example 3. ([6]) Let A be as in Example 1. Let V be a regular simple A-module, namely, V corresponds to one of the following representations of the Kronecker quiver

$$k \underset{\lambda}{\overset{1}{\rightrightarrows}} k \ \ (\lambda \in k), \qquad k \underset{1}{\overset{0}{\rightrightarrows}} k \ .$$

Let $\varphi : A \to A_V$ be the corresponding universal localisation. Then $T = A \oplus A_V/\varphi(A)$ is an (infinitely generated) tilting A-module. We refer to [6] for the unexplained notions.

Let $B = \mathsf{End}_A(T)$. Then there are two algebraic stratifications of $\mathcal{D}(B)$ of length 3 and 2 respectively :



Examples of this type are systematically studied in [7].

Notice that the algebra B in the preceding example is infinite-dimensional. For finite-dimensional algebras, the questions (a) and (b) are open. For piecewise hereditary algebras the answers to them are positive. Recall that a finite-dimensional algebra is *piecewise hereditary* if it is derived equivalent to a hereditary abelian category.

Theorem 4. ([1, 3]) *Let A be a piecewise hereditary algebra. Then any algebraic stratification of $\mathcal{D}(A)$ has the same set (with multiplicities) of simple factors: they are precisely the derived categories of the endomorphism algebras of the simple A-modules.*

3. Derived simple algebras

An algebra is said to be *derived simple* if its derived category does not admit any non-trivial recollements of derived module categories. For example, the field k is derived simple. Derived simple algebras are precisely those algebras whose derived categories occur as simple factors of some algebraic stratifications.

Example 5. ([17, 4]) Let $n \in \mathbb{N}$. Let A be the algebra given by the quiver

$$1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

with relations $(\alpha\beta)^n = 0 = (\beta\alpha)^n$ or with relations $(\alpha\beta)^n\alpha = 0 = \beta(\alpha\beta)^n$. Then A is derived simple.

Example 6. ([8]) There are finite-dimensional derived simple algebras of finite global dimension. In [8], Happel constructed a family of finite-dimensional algebras A_m ($m \in \mathbb{N}$) such that

- the global dimension of A_m is $6m - 3$,
- A_m is derived simple.

All these algebras have exactly two isomorphism classes of simple modules. For example, A_1 is given by the quiver

$$1 \begin{array}{c} \xrightarrow{\ \alpha\ } \\ \xleftarrow{\ \beta\ } \\ \xrightarrow[\ \gamma\ ]{} \end{array} 2$$

with relations $\beta\alpha = 0 = \gamma\beta$.

The classification of derived simple algebras turns out to be a wild problem. Besides those in the above examples, only a few families of algebras have been shown to be derived simple.

Theorem 7. *The following algebras are derived simple:*

(a) ([2]) *local algebras,*
(b) ([2]) *simple artinian algebras,*
(c) ([4]) *indecomposable commutative algebras,*
(d) ([15]) *blocks of finite group algebras.*

Sketch of the proof for (d): First recall that a block of an algebra is an indecomposable algebra direct summand.

Step 1: Let A, B and C be finite-dimensional algebras such that there is a recollement of the form (1.1). Then $i_*(B)$ and $j_!(C)$ has no self-extensions. Moreover, $i_*(B) \in \mathcal{D}^b(\mathsf{mod}\, A)$, $j_!(C) \in K^b(\mathsf{proj}\, A)$ and $i^*(A) \in K^b(\mathsf{proj}\, B)$. Here $\mathcal{D}^b(\mathsf{mod})$ denotes the bounded derived category of finite-dimensional modules and $K^b(\mathsf{proj})$ denotes the homotopy category of bounded complexes of finite-dimensional projective modules. They can be considered as triangulated subcategories of the (unbounded) derived category.

Step 2: Let A be a finite-dimensional symmetric algebra, *i.e.* $D(A) \cong A$ as A-A-bimodules. Here $D = \mathsf{Hom}_k(?, k)$ is the k-dual. Then for $M, N \in K^b(\mathsf{proj}\, A)$, we have

$$D\,\mathsf{Hom}_A(M, N) \cong \mathsf{Hom}_A(N, M).$$

Step 3: Let A be a finite-dimensional symmetric algebra satisfying the following condition

(#) for any finite-dimensional A-module M, the space $\bigoplus_{i\in\mathbb{Z}} \mathsf{Ext}^i_A(M, M)$ is infinite-dimensional.

Let $M \in \mathcal{D}^b(\mathsf{mod}\, A)$. Then either $M \in K^b(\mathsf{proj}\, A)$ or the space $\bigoplus_{i\in\mathbb{Z}} \mathsf{Hom}_A(M, \Sigma^i M)$ is infinite-dimensional.

Step 4: Let G be a finite group. Then the group algebra kG satisfies the condition (#). So each block of kG is a finite-dimensional indecomposable symmetric algebra satisfying the condition (#).

Step 5: Let A be a finite-dimensional indecomposable symmetric algebra satisfying the condition (#). Then A is derived simple.

To show this, suppose on the contrary that there is a non-trivial recollement of the form (1.1). Then there is a triangle

$$j_!j^!(A) \longrightarrow A \longrightarrow i_*i^*(A) \longrightarrow \Sigma j_!j^!(A). \tag{3.1}$$

By Steps 1 and 3, we know that $i_*(B) \in K^b(\mathsf{proj}\, A)$, which implies that $i_*i^*(A) \in K^b(\mathsf{proj}\, A)$, and hence $j_!j^!(A) \in K^b(\mathsf{proj}\, A)$ as well. For any $n \in \mathbb{Z}$ we have

$$\mathsf{Hom}_A(j_!j^!(A), \Sigma^n i_*i^*(A)) = \mathsf{Hom}_A(j^!(A), \Sigma^n j^*i_*i^*(A)) = 0, \tag{3.2}$$

where the first equality follows from the adjointness of $j_!$ and j^*, and the second one follows from the fact that $j^*i_* = 0$ (the third condition in the definition of a recollement). It then follows from the formula in Step 2 that for any $n \in \mathbb{Z}$

$$\mathsf{Hom}_A(i_*i^*(A), \Sigma^n j_!j^!(A)) = 0. \tag{3.3}$$

Taking $n = 1$, we see that the triangle (3.1) splits, and hence $A = j_!j^!(A) \oplus i_*i^*(A)$. The formulas (3.2) and (3.3) for $n = 0$ say that there are no morphisms between $j_!j^!(A)$ and $i_*i^*(A)$. Thus we have

$$A = \mathsf{End}_A(A) = \mathsf{End}_A(j_!j^!(A) \oplus i_*i^*(A)) = \mathsf{End}_A(j_!j^!(A)) \oplus \mathsf{End}_A(i_*i^*(A)),$$

contradicting the assumption that A is indecomposable. □

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