DIMENSIONS OF DERIVED CATEGORIES

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Abstract. Several years ago, Bondal, Rouquier and Van den Bergh introduced the notion of the dimension of a triangulated category, and Rouquier proved that the bounded derived category of coherent sheaves on a separated scheme of finite type over a perfect field has finite dimension. In this paper, we study the dimension of the bounded derived category of finitely generated modules over a commutative Noetherian ring. The main result of this paper asserts that it is finite over a complete local ring containing a field with perfect residue field.

1. Introduction

The notion of the dimension of a triangulated category has been introduced by Bondal, Rouquier and Van den Bergh [4, 14]. Roughly speaking, it measures how quickly the category can be built from a single object. The dimensions of the bounded derived category of finitely generated modules over a Noetherian ring and that of coherent sheaves on a Noetherian scheme are called the *derived dimensions* of the ring and the scheme, while the dimension of the singularity category (in the sense of Orlov [12]; the same as the stable derived category in the sense of Buchweitz [5]) is called the *stable dimension.* These dimensions have been in the spotlight in the studies of the dimensions of triangulated categories.

The importance of the notion of derived dimension was first recognized by Bondal and Van den Bergh [4] in relation to representability of functors. They proved that smooth proper commutative/non-commutative varieties have finite derived dimension, which yields that every contravariant cohomological functor of finite type to vector spaces is representable.

As to upper bounds, the derived dimension of a ring is at most its Loewy length [14]. In particular, Artinian rings have finite derived dimension. Christensen, Krause and Kussin [6, 9] showed that the derived dimension is bounded above by the global dimension, whence rings of finite global dimension are of finite derived dimension. In relation to a conjecture of Orlov [13], a series of studies by Ballard, Favero and Katzarkov [1, 2, 3] gave in several cases upper bounds for derived and stable dimensions of schemes. For instance, they obtained an upper bound of the stable dimension of an isolated hypersurface singularity by using the Loewy length of the Tjurina algebra. On the other hand, there are a lot of triangulated categories having infinite dimension. The dimension of the derived category of perfect complexes over a ring (respectively, a quasi-projective scheme) is infinite unless

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the ring has finite global dimension (respectively, the scheme is regular) [14]. It has turned out by work of Oppermann and Št'ovíček [11] that over a Noetherian algebra (respectively, a projective scheme) all proper thick subcategories of the bounded derived category of finitely generated modules (respectively, coherent sheaves) containing perfect complexes have infinite dimension. However, these do not apply for the finiteness of the derived dimension of a non-regular Noetherian ring of positive Krull dimension.

As a main result of the paper [14], Rouquier gave the following theorem.

Theorem 1 (Rouquier)**.** *Let X be a separated scheme of finite type over a perfect field. Then the bounded derived category of coherent sheaves on X has finite dimension.*

Applying this theorem to an affine scheme, one obtains:

Corollary 2. *Let R be a commutative ring which is essentially of finite type over a perfect field k. Then the bounded derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$ of finitely generated R-modules has finite dimension, and so does the singularity category $\mathsf{D}_{\mathsf{Sg}}(R)$ of R.*

The main purpose of this paper is to study the dimension and generators of the bounded derived category of finitely generated modules over a commutative Noetherian ring. We will give lower bounds of the dimensions over general rings under some mild assumptions, and over some special rings we will also give upper bounds and explicit generators. The main result of this paper is the following theorem. (See Definition 5 for the notation.)

Main Theorem. *Let R be either a complete local ring containing a field with perfect residue field or a ring that is essentially of finite type over a perfect field. Then there exist a finite number of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ of R and an integer $m \geq 1$ such that*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_n \rangle_m.$$

In particular, $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$ and $\mathsf{D}_{\mathsf{Sg}}(R)$ have finite dimension.

In Rouquier's result stated above, the essential role is played, in the affine case, by the Noetherian property of the enveloping algebra $R \otimes_k R$. The result does not apply to a complete local ring, since it is in general far from being (essentially) of finite type and therefore the enveloping algebra is non-Noetherian. Our methods not only show finiteness of dimensions over a complete local ring but also give a ring-theoretic proof of Corollary 2.

2. Preliminaries

This section is devoted to stating our convention, giving some basic notation and recalling the definition of the dimension of a triangulated category.

We assume the following throughout this paper.

Convention 3. (1) All subcategories are full and closed under isomorphisms.
(2) All rings are associative and with identities.
(3) A Noetherian ring, an Artinian ring and a module mean a right Noetherian ring, a right Artinian ring and a right module, respectively.
(4) All complexes are cochain complexes.

We use the following notation.

Notation 4. (1) Let $\mathcal{A}$ be an abelian category.

(a) For a subcategory $\mathcal{X}$ of $\mathcal{A}$, the smallest subcategory of $\mathcal{A}$ containing $\mathcal{X}$ which is closed under finite direct sums and direct summands is denoted by $\mathsf{add}_{\mathcal{A}}\, \mathcal{X}$.

(b) We denote by $\mathsf{C}(\mathcal{A})$ the category of complexes of objects of $\mathcal{A}$. The derived category of $\mathcal{A}$ is denoted by $\mathsf{D}(\mathcal{A})$. The left bounded, the right bounded and the bounded derived categories of $\mathcal{A}$ are denoted by $\mathsf{D}^+(\mathcal{A}), \mathsf{D}^-(\mathcal{A})$ and $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$, respectively. We set $\mathsf{D}^{\varnothing}(\mathcal{A}) = \mathsf{D}(\mathcal{A})$, and often write $\mathsf{D}^{\star}(\mathcal{A})$ with $\star \in \{\varnothing, +, -, \mathrm{b}\}$ to mean $\mathsf{D}^{\varnothing}(\mathcal{A})$, $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$ and $\mathsf{D}^{\mathrm{b}}(\mathcal{A})$.

(2) Let R be a ring. We denote by $\mathsf{Mod}\, R$ and $\mathsf{mod}\, R$ the category of R-modules and the category of finitely generated R-modules, respectively. For a subcategory $\mathcal{X}$ of $\mathsf{mod}\, R$ (when R is Noetherian), we put $\mathsf{add}_R\, \mathcal{X} = \mathsf{add}_{\mathsf{mod}\, R}\, \mathcal{X}$.

The concept of the dimension of a triangulated category has been introduced by Rouquier [14]. Now we recall its definition.

Definition 5. Let $\mathcal{T}$ be a triangulated category.

(1) A triangulated subcategory of $\mathcal{T}$ is called *thick* if it is closed under direct summands.

(2) Let $\mathcal{X}, \mathcal{Y}$ be two subcategories of $\mathcal{T}$. We denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of all objects M that admit exact triangles

$$X \to M \to Y \to \Sigma X$$

with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. We denote by $\langle \mathcal{X} \rangle$ the smallest subcategory of $\mathcal{T}$ containing $\mathcal{X}$ which is closed under finite direct sums, direct summands and shifts. For a non-negative integer n, we define the subcategory $\langle \mathcal{X} \rangle_n$ of $\mathcal{T}$ by

$$\langle \mathcal{X} \rangle_n = \begin{cases} \{0\} & (n = 0), \\ \langle \mathcal{X} \rangle & (n = 1), \\ \langle \langle \mathcal{X} \rangle * \langle \mathcal{X} \rangle_{n-1} \rangle & (2 \le n < \infty). \end{cases}$$

Put $\langle \mathcal{X} \rangle_\infty = \bigcup_{n \ge 0} \langle \mathcal{X} \rangle_n$. When the ground category $\mathcal{T}$ should be specified, we write $\langle \mathcal{X} \rangle_n^{\mathcal{T}}$ instead of $\langle \mathcal{X} \rangle_n$. For a ring R and a subcategory $\mathcal{X}$ of $\mathsf{D}(\mathsf{Mod}\, R)$, we put $\langle \mathcal{X} \rangle_n^R = \langle \mathcal{X} \rangle_n^{\mathsf{D}(\mathsf{Mod}\, R)}$.

(3) The *dimension* of $\mathcal{T}$, denoted by $\dim \mathcal{T}$, is the infimum of the integers d such that there exists an object $M \in \mathcal{T}$ with $\langle M \rangle_{d+1} = \mathcal{T}$.

3. Upper bounds

The aim of this section is to find explicit generators and upper bounds of dimensions for derived categories in several cases.

We observe that the dimensions of the bounded derived categories of finitely generated modules over quotient singularities are at most their (Krull) dimensions, particularly that they are finite.

Proposition 6. *Let S be either the polynomial ring $k[x_1, \dots, x_n]$ or the formal power series ring $k[[x_1, \dots, x_n]]$ over a field k. Let G be a finite subgroup of the general linear* *group* $\mathrm{GL}_n(k)$*, and assume that the characteristic of* k *does not divide the order of* G*. Let* $R = S^G$ *be the invariant subring. Then* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,R) = \langle S \rangle_{n+1}$ *holds, and hence one has*

$$\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,R) \le n = \dim R < \infty.$$

For a commutative ring R, we denote the set of minimal prime ideals of R by $\operatorname{Min} R$. As is well-known, $\operatorname{Min} R$ is a finite set whenever R is Noetherian. Also, we denote by $\lambda(R)$ the infimum of the integers $n \ge 0$ such that there is a filtration

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = R$$

of ideals of R with $I_i/I_{i-1} \cong R/\mathfrak{p}_i$ for $1 \le i \le n$, where $\mathfrak{p}_i \in \operatorname{Spec} R$. If R is Noetherian, then such a filtration exists and $\lambda(R)$ is a non-negative integer.

Proposition 7. *Let* R *be a Noetherian commutative ring.*

(1) *Suppose that for every* $\mathfrak{p} \in \operatorname{Min} R$ *there exist an* $R/\mathfrak{p}$*-complex* $G(\mathfrak{p})$ *and an integer* $n(\mathfrak{p}) \ge 0$ *such that* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,R/\mathfrak{p}) = \langle G(\mathfrak{p}) \rangle_{n(\mathfrak{p})}$*. Then* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,R) = \langle G \rangle_n$ *holds, where* $G = \bigoplus_{\mathfrak{p} \in \operatorname{Min} R} G(\mathfrak{p})$ *and* $n = \lambda(R) \cdot \max\{\, n(\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Min} R \,\}$*.*

(2) *There is an inequality*

$$\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,R) \le \lambda(R) \cdot \sup\{\, \dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,R/\mathfrak{p}) + 1 \mid \mathfrak{p} \in \operatorname{Min} R \,\} - 1.$$

Let R be a commutative Noetherian ring. We set

$$\ell\ell(R) = \inf\{\, n \ge 0 \mid (\operatorname{rad} R)^n = 0 \,\},$$
$$r(R) = \min\{\, n \ge 0 \mid (\operatorname{nil} R)^n = 0 \,\},$$

where $\operatorname{rad} R$ and $\operatorname{nil} R$ denote the Jacobson radical and the nilradical of R, respectively. The first number is called the *Loewy length* of R and is finite if (and only if) R is Artinian, while the second one is always finite. Let $R_{\mathrm{red}} = R/\operatorname{nil} R$ be the associated reduced ring. When R is reduced, we denote by $\overline{R}$ the integral closure of R in the total quotient ring Q of R. Let C_R denote the *conductor* of R, i.e., C_R is the set of elements $x \in Q$ with $x\overline{R} \subseteq R$. We can give an explicit generator and an upper bound of the dimension of the bounded derived category of finitely generated modules over a one-dimensional complete local ring.

Proposition 8. *Let* R *be a Noetherian commutative complete local ring of Krull dimension one with residue field* k*. Then it holds that* $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,R) = \langle \overline{R_{\mathrm{red}}} \oplus k \rangle_{r(R)\cdot(2\,\ell\ell(R_{\mathrm{red}}/C_{R_{\mathrm{red}}})+2)}$*. In particular,*

$$\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\,R) \le r(R) \cdot (2\,\ell\ell(R_{\mathrm{red}}/C_{R_{\mathrm{red}}}) + 2) - 1 < \infty.$$

Let R be a commutative Noetherian local ring of Krull dimension d with maximal ideal $\mathfrak{m}$. We denote by $e(R)$ the multiplicity of R, that is, $e(R) = \lim_{n\to\infty} \frac{d!}{n^d} \ell_R(R/\mathfrak{m}^{n+1})$. Recall that a *numerical semigroup* is defined as a subsemigroup H of the additive semigroup $\mathbb{N} = \{0, 1, 2, \dots\}$ containing 0 such that $\mathbb{N} \backslash H$ is a finite set. For a numerical semigroup H, let $c(H)$ denote the *conductor* of H, that is,

$$c(H) = \max\{\, i \in \mathbb{N} \mid i - 1 \notin H \,\}.$$

For a real number α, put $\lceil \alpha \rceil = \min\{\, n \in \mathbb{Z} \mid n \ge \alpha \,\}$. Making use of the above proposition, one can get an upper bound of the dimension of the bounded derived category of finitely generated modules over a *numerical semigroup ring* $k[[H]]$ over a field k, in terms of the conductor of the semigroup and the multiplicity of the ring.

Corollary 9. *Let k be a field and H be a numerical semigroup. Let R be the numerical semigroup ring $k[[H]]$, that is, the subring $k[[t^h | h \in H]]$ of $S = k[[t]]$. Then $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle S \oplus k \rangle_{2\lceil \frac{c(H)}{e(R)} \rceil + 2}$ holds. Hence*

$$\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \leq 2 \left\lceil \frac{c(H)}{e(R)} \right\rceil + 1.$$

4. Finiteness

In this section, we consider finiteness of the dimension of the bounded derived category of finitely generated modules over a complete local ring. Let R be a commutative algebra over a field k. Rouquier [14] proved the finiteness of the dimension of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$ when R is an affine k-algebra, where the fact that the enveloping algebra $R \otimes_k R$ is Noetherian played a crucial role. The problem in the case where R is a complete local ring is that one cannot hope that $R \otimes_k R$ is Noetherian. Our methods instead use the completion of the enveloping algebra, that is, the complete tensor product $R \widehat{\otimes}_k R$, which is a Noetherian ring whenever R is a complete local ring with coefficient field k.

Let R and S be commutative Noetherian complete local rings with maximal ideals $\mathfrak{m}$ and $\mathfrak{n}$, respectively. Suppose that they contain fields and have the same residue field k, i.e., $R/\mathfrak{m} \cong k \cong S/\mathfrak{n}$. Then Cohen's structure theorem yields isomorphisms

$$R \cong k[[x_1, \dots, x_m]]/(f_1, \dots, f_a),$$
$$S \cong k[[y_1, \dots, y_n]]/(g_1, \dots, g_b).$$

We denote by $R \widehat{\otimes}_k S$ the *complete tensor product* of R and S over k, namely,

$$R \widehat{\otimes}_k S = \varprojlim_{i,j} (R/\mathfrak{m}^i \otimes_k S/\mathfrak{n}^j).$$

For $r \in R$ and $s \in S$, we denote by $r \widehat{\otimes} s$ the image of $r \otimes s$ by the canonical ring homomorphism $R \otimes_k S \to R \widehat{\otimes}_k S$. Note that there is a natural isomorphism

$$R \widehat{\otimes}_k S \cong k[[x_1, \dots, x_m, y_1, \dots, y_n]]/(f_1, \dots, f_a, g_1, \dots, g_b).$$

Details of complete tensor products can be found in [15, Chapter V].

Recall that a ring extension $A \subseteq B$ is called *separable* if B is projective as a $B \otimes_A B$-module. This is equivalent to saying that the map $B \otimes_A B \to B$ given by $x \otimes y \mapsto xy$ is a split epimorphism of $B \otimes_A B$-modules.

Now, let us prove our main theorem.

Theorem 10. *Let R be a Noetherian complete local commutative ring containing a field with perfect residue field. Then there exist a finite number of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \operatorname{Spec} R$ and an integer $m \geq 1$ such that*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_n \rangle_m.$$

Hence one has $\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) < \infty$.

Sketch of proof. We use induction on the Krull dimension $d := \dim R$. If $d = 0$, then R is an Artinian ring, and the assertion follows from [14, Proposition 7.37]. Assume $d \geq 1$. By [10, Theorem 6.4], we have a sequence

$$0 = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n = R$$

of ideals of R such that for each $1 \leq i \leq n$ one has $I_i/I_{i-1} \cong R/\mathfrak{p}_i$ with $\mathfrak{p}_i \in \operatorname{Spec} R$. Then every object X of $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R)$ possesses a sequence

$$0 = XI_0 \subseteq XI_1 \subseteq \cdots \subseteq XI_n = X$$

of R-subcomplexes. Decompose this into exact triangles

$$XI_{i-1} \to XI_i \to XI_i/XI_{i-1} \to \Sigma XI_{i-1},$$

in $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R)$, and note that each XI_i/XI_{i-1} belongs to $\mathsf{D}^{\mathrm{b}}(\operatorname{\mathsf{mod}} R/\mathfrak{p}_i)$. Hence one may assume that R is an integral domain. By [16, Definition-Proposition (1.20)], we can take a formal power series subring $A = k[[x_1, \ldots, x_d]]$ of R such that R is a finitely generated A-module and that the extension $Q(A) \subseteq Q(R)$ of the quotient fields is finite and separable.

Claim 1. *We have natural isomorphisms*

$$\begin{aligned}
R &\cong k[[x]][t]/(f(x,t)) = k[[x,t]]/(f(x,t)),\\
S := R \otimes_A R &\cong k[[x]][t,t']/(f(x,t), f(x,t')) = k[[x,t,t']]/(f(x,t), f(x,t')),\\
U := R \mathbin{\widehat{\otimes}_k} A &\cong k[[x,t,x']]/(f(x,t)),\\
T := R \mathbin{\widehat{\otimes}_k} R &\cong k[[x,t,x',t']]/(f(x,t), f(x',t')).
\end{aligned}$$

Here $x = x_1, \ldots, x_d$*,* $x' = x'_1, \ldots, x'_d$*,* $t = t_1, \ldots, t_n$*,* $t' = t'_1, \ldots, t'_n$ *are indeterminates over* k*, and* $f(x,t) = f_1(x,t), \ldots, f_m(x,t)$ *are elements of* $k[[x]][t] \subseteq k[[x,t]]$*. In particular, the rings* S, T, U *are Noetherian commutative complete local rings.*

There is a surjective ring homomorphism $\mu : S = R \otimes_A R \to R$ which sends $r \otimes r'$ to rr'. This makes R an S-module. Using Claim 1, we observe that μ corresponds to the map $k[[x,t,t']]/(f(x,t), f(x,t')) \to k[[x,t]]/(f(x,t))$ given by $t' \mapsto t$. Taking the kernel, we have an exact sequence

$$0 \to I \to S \xrightarrow{\mu} R \to 0$$

of finitely generated S-modules. Along the injective ring homomorphism $A \to S$ sending $a \in A$ to $a \otimes 1 = 1 \otimes a \in S$, we can regard A as a subring of S. Note that S is a finitely generated A-module. Put $W = A \setminus \{0\}$. This is a multiplicatively closed subset of A, R and S, and one can take localization $(-)_W$.

Claim 2. *The* S_W*-module* R_W *is projective.*

There are ring epimorphisms

$$\begin{aligned}
\alpha &: U \to R, \quad r \mathbin{\widehat{\otimes}} a \mapsto ra,\\
\beta &: T \to S, \quad r \mathbin{\widehat{\otimes}} r' \mapsto r \otimes r',\\
\gamma &: T \to R, \quad r \mathbin{\widehat{\otimes}} r' \mapsto rr'.
\end{aligned}$$

Identifying the rings R, S, T and U with the corresponding residue rings of formal power series rings made in Claim 1, we see that α, β are the maps given by $x' \mapsto x$, and γ

is the map given by $x' \mapsto x$ and $t' \mapsto t$. Note that $\gamma = \mu\beta$. The map α is naturally a homomorphism of (R, A)-bimodules, and β, γ are naturally homomorphisms of (R, R)-bimodules. The ring R has the structure of a finitely generated U-module through α. The Koszul complex on the U-regular sequence $x' - x$ gives a free resolution of the U-module R:

$$0 \to U \to U^{\oplus d} \to U^{\oplus\binom{d}{2}} \to \cdots \to U^{\oplus\binom{d}{2}} \to U^{\oplus d} \xrightarrow{x'-x} U \xrightarrow{\alpha} R \to 0. \tag{4.1}$$

This is an exact sequence of (R, A)-bimodules. Since the natural homomorphisms

$$A \cong k[[x']] \to k[[x']][x,t]/(f(x,t)),$$
$$k[[x']][x,t]/(f(x,t)) \to k[[x']][[x,t]]/(f(x,t)) \cong U$$

are flat, so is the composition. Therefore U is flat as a right A-module. The exact sequence (4.1) gives rise to a chain map $\eta : F \to R$ of U-complexes, where

$$F = (0 \to U \to U^{\oplus d} \to U^{\oplus\binom{d}{2}} \to \cdots \to U^{\oplus\binom{d}{2}} \to U^{\oplus d} \xrightarrow{x'-x} U \to 0)$$

is a complex of finitely generated free U-modules. By Claim 1, we have isomorphisms

$$\begin{aligned} U \otimes_A R &\cong U \otimes_A A[t]/(f(x,t)) \cong U[t']/(f(x',t')) \cong U[[t']]/(f(x',t')) \\ &\cong (k[[x,t,x']]/(f(x,t)))[[t']]/(f(x',t')) \cong k[[x,t,x',t']]/(f(x,t), f(x',t')) \cong T. \end{aligned}$$

Note from [17, Exercise 10.6.2] that $R \otimes^{\mathbf{L}}_A R$ is an object of $\mathsf{D}^-(R\text{-}\mathsf{Mod}\text{-}R) = \mathsf{D}^-(\mathsf{Mod}\, R \otimes_k R)$. (Here, R-Mod-R denotes the category of (R, R)-bimodules, which can be identified with $\mathsf{Mod}\, R \otimes_k R$.) There are isomorphisms

$$\begin{aligned} R \otimes^{\mathbf{L}}_A R &\cong F \otimes_A R \\ &\cong (0 \to U \otimes_A R \to (U \otimes_A R)^{\oplus d} \to \cdots \to (U \otimes_A R)^{\oplus d} \xrightarrow{x'-x} U \otimes_A R \to 0) \\ &\cong (0 \to T \to T^{\oplus d} \to \cdots \to T^{\oplus d} \xrightarrow{x'-x} T \to 0) =: C \end{aligned}$$

in $\mathsf{D}^-(\mathsf{Mod}\, R \otimes_k R)$. Note that C can be regarded as an object of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, T)$. Taking the tensor product $\eta \otimes_A R$, one gets a chain map $\lambda : C \to S$ of T-complexes. Thus, one has a commutative diagram

$$\begin{array}{ccccccc} K & \longrightarrow & C & \longrightarrow & R & \xrightarrow{\delta} & \Sigma K \\ \downarrow & & \lambda\downarrow & & \| & & \xi\downarrow \\ I & \longrightarrow & S & \xrightarrow{\mu} & R & \xrightarrow{\varepsilon} & \Sigma I \end{array}$$

of exact triangles in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, T)$.

Claim 3. *There exists an element $a \in W$ such that $\delta \cdot (1 \mathbin{\widehat{\otimes}} a) = 0$ in $\operatorname{Hom}_{\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, T)}(R, \Sigma K)$. One can choose it as a non-unit element of A, if necessary.*

Let $a \in W$ be a non-unit element of A as in Claim 3. Since we regard R as a T-module through the homomorphism γ, we have an exact sequence $0 \to R \xrightarrow{1 \widehat{\otimes} a} R \to R/(a) \to 0$.

The octahedral axiom makes a diagram in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, T)$

$$\begin{array}{ccccccc}
R & \xrightarrow{1\widehat{\otimes} a} & R & \longrightarrow & R/(a) & \longrightarrow & \Sigma R \\
\| & & \downarrow{\scriptstyle\delta} & & \downarrow & & \| \\
R & \xrightarrow{0} & \Sigma K & \longrightarrow & \Sigma K \oplus \Sigma R & \longrightarrow & \Sigma R \\
\downarrow & & \| & & \downarrow & & \downarrow \\
R & \xrightarrow{\delta} & \Sigma K & \longrightarrow & \Sigma C & \longrightarrow & \Sigma R \\
\downarrow & & \downarrow & & \| & & \downarrow \\
R/(a) & \longrightarrow & \Sigma K \oplus \Sigma R & \longrightarrow & \Sigma C & \longrightarrow & \Sigma R/(a)
\end{array}$$

with the bottom row being an exact triangle. Rotating it, we obtain an exact triangle

$$K \oplus R \to C \to R/(a) \to \Sigma(K \oplus R)$$

in $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, T)$. The exact functor $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, T) \to \mathsf{D}^{-}(\mathsf{Mod}\, R \otimes_k R)$ induced by the canonical ring homomorphism $R \otimes_k R \to T$ sends this to an exact triangle

$$K \oplus R \to R \otimes^{\mathbf{L}}_A R \to R/(a) \to \Sigma(K \oplus R) \tag{4.2}$$

in $\mathsf{D}^{-}(\mathsf{Mod}\, R \otimes_k R)$. As R is a local domain and a is a non-zero element of the maximal ideal of R, we have $\dim R/(a) = d-1 < d$. Hence one can apply the induction hypothesis to the ring $R/(a)$, and sees that

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R/(a)) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_h \rangle^{R/(a)}_m$$

for some integer $m \geq 1$ and some prime ideals $\mathfrak{p}_1, \cdots, \mathfrak{p}_h$ of R that contain a. Now, let X be any object of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$. Applying the exact functor $X \otimes^{\mathbf{L}}_R -$ to (4.2) gives an exact triangle in $\mathsf{D}^{-}(\mathsf{Mod}\, R)$

$$(X \otimes^{\mathbf{L}}_R K) \oplus X \to X \otimes^{\mathbf{L}}_A R \to X \otimes^{\mathbf{L}}_R R/(a) \to \Sigma((X \otimes^{\mathbf{L}}_R K) \oplus X). \tag{4.3}$$

Note that $X \otimes^{\mathbf{L}}_R R/(a)$ is an object of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R/(a)) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_h \rangle^{R/(a)}_m$. As an object of $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$, the complex $X \otimes^{\mathbf{L}}_R R/(a)$ belongs to $\langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_h \rangle^{R}_m$. We observe from (4.3) that X is in $\langle R \oplus R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_h \rangle^{R}_{d+1+m}$. Thus we obtain $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle R \oplus R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_h \rangle_{d+1+m}$. (As R is a domain, the zero ideal of R is a prime ideal.) □

Now, we make sure that the proof of Theorem 10 also gives a ring-theoretic proof of the affine case of Rouquier's theorem. Actually, we obtain a more detailed result as follows. Recall that a commutative ring R is said to be *essentially of finite type* over a field k if R is a localization of a finitely generated k-algebra. Of course, every finitely generated k-algebra is essentially of finite type over k.

Theorem 11. (1) *Let R be a finitely generated algebra over a perfect field. Then there exist a finite number of prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n \in \operatorname{Spec} R$ and an integer $m \geq 1$ such that*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_n \rangle_m.$$

(2) *Let R be a commutative ring which is essentially of finite type over a perfect field. Then there exist a finite number of prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_n \in \operatorname{Spec} R$ and an integer $m \geq 1$ such that*

$$\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) = \langle R/\mathfrak{p}_1 \oplus \cdots \oplus R/\mathfrak{p}_n \rangle_m.$$

Now the following result due to Rouquier (cf. [14, Theorem 7.38]) is immediately recovered by Theorem 11(2).

Corollary 12 (Rouquier)**.** *Let R be a commutative ring essentially of finite type over a perfect field. Then the derived category $\mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R)$ has finite dimension.*

Remark 13. In Corollary 12, the assumption that the base field is perfect can be removed; see [7, Proposition 5.1.2]. We do not know whether we can also remove the perfectness assumption of the residue field in Theorem 10. It seems that the techniques in the proof of [7, Proposition 5.1.2] do not directly apply to that case.

5. Lower bounds

In this section, we will mainly study lower bounds for the dimension of the bounded derived category of finitely generated modules. We shall refine a result of Rouquier over an affine algebra, and also give a similar lower bound over a general commutative Noetherian ring.

Throughout this section, let R be a commutative Noetherian ring. First, we consider refining a result of Rouquier.

Theorem 14. *Let R be a finitely generated algebra over a field. Suppose that there exists $\mathfrak{p} \in \operatorname{Assh} R$ such that $R_\mathfrak{p}$ is a field. Then one has the inequality* $\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \geq \dim R$.

The following result of Rouquier [14, Proposition 7.16] is a direct consequence of Theorem 14.

Corollary 15 (Rouquier)**.** *Let R be a reduced finitely generated algebra over a field. Then $\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \geq \dim R$.*

Next, we try to extend Theorem 14 to non-affine algebras. We do not know whether the inequality in Theorem 14 itself holds over non-affine algebras; we can prove that a similar but slightly weaker inequality holds over them.

Now we can show the following result.

Theorem 16. *Let R be a ring of finite Krull dimension such that $R_\mathfrak{p}$ is a field for all $\mathfrak{p} \in \operatorname{Assh} R$. Then we have* $\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \geq \dim R - 1$.

Here is an obvious conclusion of the above theorem.

Corollary 17. *Let R be a reduced ring of finite Krull dimension. Then* $\dim \mathsf{D}^{\mathrm{b}}(\mathsf{mod}\, R) \geq \dim R - 1$.

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