RECOLLEMENTS GENERATED BY IDEMPOTENTS AND APPLICATION TO SINGULARITY CATEGORIES

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Abstract. In this note I report on an ongoing work joint with Martin Kalck, which generalises and improves a construction of Thanhoffer de Völcsey and Van den Bergh.

Key Words: Recollement, Singularity category, Non-commutative resolution.

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In [15] Thanhoffer de Völcsey and Van den Bergh showed that the stable category of maximal Cohen–Macaulay modules over a local complete commutative Gorenstein algebra with isolated singularity can be realized as the triangle quotient of the perfect derived category by the finite-dimensional category of a certain nice dg algebra constructed from the given Gorenstein algebra. We generalises and improves their construction by studying recollements of derived categories generated by idempotents.

1. Recollements generated by idempotents

Let k be a field, let A be a k-algebra and $e \in A$ be an idempotent. Let $\mathcal{D}(A)$ denote the (unbounded) derived category of the category of right modules over A. This is a triangulated category with shift functor Σ being the shift of complexes. Consider the following *standard diagram*

$$
\begin{tikzcd}
\mathcal{D}(A/AeA) \arrow[r, "i_*=i_!"] & \mathcal{D}(A) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}(eAe) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"]
\end{tikzcd}
\tag{1.1}
$$

where

$$
\begin{aligned}
i^* &= ? \overset{L}{\otimes}_A A/AeA, & j_! &= ? \overset{L}{\otimes}_{eAe} eA,\\
i_* &= \mathsf{RHom}_{A/AeA}(A/AeA, ?), & j^! &= \mathsf{RHom}_A(eA, ?),\\
i_! &= ? \overset{L}{\otimes}_{A/AeA} A/AeA, & j^* &= ? \overset{L}{\otimes}_A Ae,\\
i^! &= \mathsf{RHom}_A(A/AeA, ?), & j_* &= \mathsf{RHom}_{eAe}(Ae, ?).
\end{aligned}
$$

One asks when this diagram is a *recollement* ([3]), *i.e.* the following conditions hold

(1) $(i^*, i_* = i_!, i^!)$ and $(j_!, j^! = j^*, j_*)$ are adjoint triples;

(2r) $j_!$ and j_* are fully faithful;

(2l) $i_* = i_!$ is fully faithful;

The detailed version of this paper will be submitted for publication elsewhere.

(3) $j^*i_* = 0$;

(4) for every object M of $\mathcal{D}(A)$ there are two triangles

$$i_!i^!M \longrightarrow M \longrightarrow j_*j^*M \longrightarrow \Sigma i_!i^!M$$

and

$$j_!j^!M \longrightarrow M \longrightarrow i_*i^*M \longrightarrow \Sigma j_!j^!M \ ,$$

where the four morphisms starting from and ending at M are the units and counits.

This type of recollements attracts considerable attention, see for example [6, 8, 7, 14]. The conditions (1) and (3) are easy to check, and it is known that (2r) holds (by applying [11, Proposition 3.2] to eA). However, in general (2l) is not necessarily true, as seen from the next example.

Example 1. Let A be the finite-dimensional algebra given by the quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \end{tikzcd}$$

with relation $\alpha\beta = 0$. Take the idempotent $e = e_1$, the trivial path at the vertex 1. Then the associated functor $i_* : \mathcal{D}(A/AeA) \to \mathcal{D}(A)$ is not fully faithful. Indeed, $i_*(A/AeA)$ is the simple A-module at vertex 2, which has non-vanishing self-extensions in degree 2, while as an A/AeA-module A/AeA has no self-extensions.

Theorem 2. ([8]) *The following conditions are equivalent*

(i) *the standard diagram* (1.1) *is a recollement,*

(ii) *the homomorphism $A \to A/AeA$ is a* homological epimorphism*,* i.e. *the functor $i_* : \mathcal{D}(A/AeA) \to \mathcal{D}(A)$ is fully faithful,*

(iii) *the ideal AeA is a* stratifying ideal*,* i.e. *the counit $Ae \overset{L}{\otimes}_{eAe} eA \to A$ induces an isomorphism $Ae \overset{L}{\otimes}_{eAe} eA \cong AeA$.*

In general, to make the standard diagram (1.1) a recollement, one needs to replace A/AeA by a dg (=differential graded) algebra, which, in some sense, enhances A/AeA. For dg algebras and their derived categories, we refer to [13]. We remark that a k-algebra can be viewed as a dg k-algebra concentrated in degree 0.

Theorem 3. ([12]) *Let A and $e \in A$ be as above. There is a dg k-algebra B with a homomorphism of dg algebras $f : A \to B$ and a recollement of derived categories*

$$\begin{tikzcd} \mathcal{D}(B) \arrow[r, "i_*=i_!"] & \mathcal{D}(A) \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}(eAe) \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \ ,$$

such that

(a) *the adjoint triples* $(i^*, i_* = i_!, i^!)$ *and* $(j_!, j^! = j^*, j_*)$ *are given by*

$$\begin{array}{ll} i^* = ? \overset{L}{\otimes}_A B, & j_! = ? \overset{L}{\otimes}_{eAe} eA, \\ i_* = \mathsf{RHom}_B(B, ?), & j^! = \mathsf{RHom}_A(eA, ?), \\ i_! = ? \overset{L}{\otimes}_B B, & j^* = ? \overset{L}{\otimes}_A Ae, \\ i^! = \mathsf{RHom}_A(B, ?), & j_* = \mathsf{RHom}_{eAe}(Ae, ?), \end{array}$$

where B *is considered as a left* A*-module and as a right* A*-module via the homomorphism* f*;*

(b) *the degree* i *component* B^i *of* B *vanishes for* $i > 0$*;*

(c) *the* 0*-th cohomology* $H^0(B)$ *of* B *is isomorphic to* A/AeA*.*

As a consequence of the recollement, there is a triangle equivalence

$$\mathsf{per}(B) \cong (K^b(\mathsf{proj}\, A)/\,\mathsf{thick}(eA))^{\omega}.$$

Here $\mathsf{per}(B)$ *is the smallest triangulated subcategory of* $\mathcal{D}(B)$ *which contains* B *and which is closed under taking direct summands,* $K^b(\mathsf{proj}\, A)$ *is the homotopy category of bounded complexes of finitely generated projective* A*-modules,* $\mathsf{thick}(eA)$ *is the smallest triangulated subcategory of* $K^b(\mathsf{proj}\, A)$ *which contains* eA *and which is closed under taking direct summands, and* $()^{\omega}$ *denotes the idempotent completion.*

Assume further that A/AeA *is finite-dimensional and that each simple* A/AeA*-module has finite projective dimension over* A*. Then*

(d) $H^i(B)$ *is finite-dimensional over* k *for any* $i \in \mathbb{Z}$*, equivalently,* $\mathsf{per}(B)$ *is Hom-finite,* i.e. $\mathsf{Hom}(M, N)$ *is finite-dimensional over* k *for any* $M, N \in \mathsf{per}(B)$*,*

(e) $\mathcal{D}_{fd}(B) \subseteq \mathsf{per}(B)$*, here* $\mathcal{D}_{fd}(B)$ *denotes the full subcategory of* $\mathcal{D}(B)$ *consisting of those objects whose total cohomology is finite-dimensional over* k*,*

(f) $\mathsf{per}(B)$ *has a* t*-structure whose heart is* $\mathsf{fdmod}\,{-A/AeA}$*, the category of finite-dimensional modules over* A/AeA*,*

(g) *if moreover there is a quasi-isomorphism from a dg algebra* $\tilde{A} = (\widehat{kQ}, d)$ *to* A*, where* Q *is a graded quiver concentrated in non-positive degrees and* $d : \widehat{kQ} \to \widehat{kQ}$ *is a continuous* k*-linear differential satisfying the graded Leibniz rule and* $d(\widehat{\mathfrak{m}}) \subseteq \widehat{\mathfrak{m}}^2$*, such that* e *is the image of a sum* $\tilde{e}$ *of some trivial paths of* Q*, then* B *is quasi-isomorphic to* $\tilde{A}/\overline{\tilde{A}\tilde{e}\tilde{A}}$*. Here* $\widehat{kQ}$ *is the completion of the path algebra* kQ *with respect to the* $\mathfrak{m}$*-adic topology in the category of graded algebras for the ideal* $\mathfrak{m}$ *of* kQ *generated by all arrows, and* $\overline{\tilde{A}\tilde{e}\tilde{A}}$ *is the closure of* $\tilde{A}\tilde{e}\tilde{A}$ *under the* $\widehat{\mathfrak{m}}$*-adic topology for the ideal* $\widehat{\mathfrak{m}}$ *of* $\widehat{kQ}$ *generated by all arrows.*

Thanks to the following lemma due to Keller, Theorem 3 (g) becomes practical when the global dimension of A is 2.

Lemma 4. *Let* $A = \widehat{kQ'}/\overline{(R)}$ *be of global dimension* 2*, where* Q' *is a finite (ordinary) quiver and* R *is a finite set of minimal relations. Let* Q *be the graded quiver obtained from* Q' *by adding an arrow* ρ_r *of degree* -1 *from the source of* r *to the target of* r *for* *each relation $r \in R$. Let d be the unique continuous k-linear automorphism of $\widehat{kQ}$ which satisfies the graded Leibniz rule and which takes ρ_r to r for each relation $r \in R$. Then there is a quasi-isomorphism from $(\widehat{kQ}, d)$ to A.*

Example 5. Let A be as in Example 1. Let Q be the graded quiver

$$\begin{tikzcd} 1 \arrow[r, shift left, "\alpha"] & 2 \arrow[l, shift left, "\beta"] \arrow[loop right, dotted, "\rho"] \end{tikzcd}$$

where α and β are in degree 0 and ρ is in degree -1. Let d be the unique continuous k-linear automorphism of $\widehat{kQ}$ which satisfies the graded Leibniz rule and which takes ρ to $\alpha\beta$. Then the obvious map from $(\widehat{kQ}, d)$ to A is a quasi-isomorphism.

Let $e = e_1$. The associated dg algebra B as in Theorem 3 is (quasi-isomorphic to) the dg algebra $k[\rho]$ with ρ in degree -1 and with vanishing differential.

2. Application to singularity categories

Let k be a field, and let R be a *Iwanaga–Gorenstein k-algebra*, *i.e.* R is left and right noetherian as a ring and R has finite injective dimension both as left R-module and as right R-module. Let $\mathsf{mod}\, R$ denote the category of finitely generated right R-modules. On the one hand, one defines the *singularity category*

$$\mathcal{D}_{sg}(R) := \mathcal{D}^b(\mathsf{mod}\, R)/K^b(\mathsf{proj}\, R),$$

which measures the complexity of the singularity of R. ($K^b(\mathsf{proj}\, R)$ is considered as the smooth part.) On the other hand, one defines the category $\mathsf{MCM}(R)$ of maximal Cohen–Macaulay R-modules

$$\mathsf{MCM}(R) := \{M \in \mathsf{mod}\, R \mid \mathsf{Ext}^i_R(M, R) = 0 \text{ for any } i > 0\}.$$

The following nice result of Buchweitz relates these categories.

Theorem 6. ([4]) $\mathsf{MCM}(R)$ *is a Frobenius category whose full subcategory of projective-injective objects is precisely* $\mathsf{proj}\, R$. *Moreover, the embedding* $\mathsf{MCM}(R) \to \mathsf{mod}\, R$ *induces a triangle equivalence from the stable category* $\underline{\mathsf{MCM}}(R)$ *to the singularity category* $\mathcal{D}_{sg}(R)$.

Let $M_1, \ldots, M_r \in \mathsf{MCM}(R)$ be pairwise non-isomorphic non-projective R-modules and let $M = R \oplus M_1 \oplus \ldots \oplus M_r$. Let $A = \mathsf{End}_R(M)$ and $e = id_R$ considered as an element of A. Then $R = eAe$ and $A/AeA = \mathsf{End}_{\underline{\mathsf{MCM}}(R)}(M)$. For example, the ring $R = k[x]/x^2$ has a unique simple module S, and letting $M = R \oplus S$ we obtain that $A = \mathsf{End}_R(M)$ is the algebra given in Example 1.

There is always an embedding of $K^b(\mathsf{proj}\, R)$ into $K^b(\mathsf{proj}\, A)$ with essential image being $\mathsf{thick}(eA)$. If the following condition is satisfied

(c1) A has finite global dimension,

then A becomes a *non-commutative/categorical resolution* of R. The condition (c1) has an interesting consequence: the object M generates $\underline{\mathsf{MCM}}(R)$ as a triangulated category.

Cluster-tilting theory comes into the story because cluster-tilting objects are closely related to Van den Bergh's non-commutative crepant resolutions [16], see [10].

The triangle quotient $K^b(\mathsf{proj}\, A)/\mathsf{thick}(eA)$ measures the difference between the resolution and the smooth part of the singularity, see [5]. So $K^b(\mathsf{proj}\, A)/\mathsf{thick}(eA)$ is in some sense a 'categorical exceptional locus'. A natural question is: how is $K^b(\mathsf{proj}\, A)/\mathsf{thick}(eA)$ related to $\mathcal{D}_{sg}(R)$?

Consider the following condition

(c2) $\underline{\mathsf{MCM}}(R)$ is Hom-finite.

Theorem 7. ([12]) *Keep the above notations and assume that (c1) and (c2) hold. There is a dg algebra B with a morphism $f : A \to B$ such that f induces a triangle equivalence*

$$\mathsf{per}(B) \cong (K^b(\mathsf{proj}\, A)/\mathsf{thick}(eA))^\omega.$$

Moreover, B satisfies the following properties:

(a) $B^i = 0$ *for any* $i > 0$,

(b) $H^0(B) \cong A/AeA$,

(c) $\mathcal{D}_{fd}(B) \subseteq \mathsf{per}(B)$,

(d) $\mathsf{per}(B)$ *is Hom-finite,*

(e) *there is a triangle equivalence*

$$\mathcal{D}_{sg}(R)^\omega \cong (\mathsf{per}(B)/\mathcal{D}_{fd}(B))^\omega.$$

Theorem 7 (a–d) are obtained by applying Theorem 3, and part (e) needs more work. This theorem was proved by Thanhoffer de Völcsey and Van den Bergh in [15] for R being a local complete commutative Gorenstein k-algebra with isolated singularity. As an application, they proved the following result, which was independently proved by Amiot, Iyama and Reiten.

Theorem 8. ([2, 15]) *Let $d \in \mathbb{N}$. Let $G \subset SL_d(k)$ be a finite subgroup, acting naturally on $S = k[\![x_1, \ldots, x_d]\!]$ and let $R = S^G$ be the ring of invariants. Then $\underline{\mathsf{MCM}}(R)$ is a generalized $(d-1)$-cluster category in the sense of Amiot* [1] *and Guo* [9].

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