INTRODUCTION TO REPRESENTATION THEORY OF COHEN-MACAULAY MODULES AND THEIR DEGENERATIONS

YUJI YOSHINO

Abstract. This is a quick introduction to the theory of representation theory of Cohen-Macaulay modules and their degenerations.

1. Representation theory of Cohen-Macaulay modules.

Let k be a field and let R be commutative noetherian complete local k-algebra with unique maximal ideal $\mathfrak{m}$. We assume $k \cong R/\mathfrak{m}$ naturally. Then it is known that there is a regular local k-subalgebra T of R such that R is a module-finite T-algebra. (Cohen's structure theorem for complete local rings.) Note that T is isomorphic to a formal power series ring over k.

Definition 1. (1) R is called a **Cohen-Macaulay ring** (a CM ring for short) if R is free as a T-module.

(2) A finitely generated R-module M is called a **Cohen-Macaulay module** over R, or a maximal Cohen-Macaulay module (a CM module or an MCM module for short) if M is free as a T-module.

Given a CM module M, since $M \cong T^n$ for some $n \geq 0$, we have a k-algebra homomorphism $R \to \mathrm{End}_T(M) \cong T^{n\times n}$, which is a matrix-representation of R over T.

In the following we always assume that R **is a CM complete local k-algebra**. We denote by $\mathrm{mod}(R)$ (res. $\mathrm{CM}(R)$) the category of finitely generated R-modules (resp. CM modules over R) and R-homomorphisms.

$$\mathrm{CM}(R) := \{ \text{ CM modules over } R\} \subseteq \mathrm{mod}(R) := \{\text{finitely generated } R\text{-modules }\}$$

Since R is complete, $\mathrm{mod}(R)$ and $\mathrm{CM}(R)$ are Krull-Schmidt categories. Note that $\mathrm{CM}(R)$ is a resolving subcategory of $\mathrm{mod}(R)$ in the following sense: Suppose there is an exact sequence $0 \to L \to M \to N \to 0$ in $\mathrm{mod}(R)$.

(i) If $L, N \in \mathrm{CM}(R)$ then $M \in \mathrm{CM}(R)$.

(ii) If $M, N \in \mathrm{CM}(R)$ then $L \in \mathrm{CM}(R)$.

Let d be the Krull-dimension of the ring R (so that we can take $T = k[[t_1, \dots, t_d]]$ on which R is finite). If $d = 1$ and if R is reduced, then CM modules are just torsion-free modules. If $d = 2$ and if R is normal, then CM modules are nothing but reflexive modules. In general, if $d \geq 3$ and if R is normal, then $\mathrm{CM}(R) \subseteq \{\text{reflexive modules}\}$ but this is not necessarily an equality. If R is regular (i.e. gl-dim$R < \infty$) then all CM modules over R are free.

Let $K_R := \mathrm{Hom}_T(R, T)$ and call it the canonical module of R. Since R is a CM ring, $K_R \in \mathrm{CM}(R)$. For any $X \in \mathrm{mod}(R)$, we have a natural isomorphism $\mathrm{Hom}_R(X, K_R) \cong$

$\mathrm{Hom}_T(X,T)$. It follows that $\mathrm{Hom}_R(-,K_R)$ gives duality $\mathrm{CM}(R) \to \mathrm{CM}(R)^{op}$. Grothendieck's local duality theorem claims the existence of natural isomorphisms

$$\mathrm{Ext}^i_R(M,K_R) \cong \mathrm{Hom}_R(H^{d-i}_{\mathfrak{m}}(M), E_R(k)) \quad (\forall i \in \mathbb{N})$$

whenever R is a CM complete ring and $M \in \mathrm{mod}(R)$. Thus it is easy to see the following

Lemma 2. *The following are equivalent for* $M \in \mathrm{mod}(R)$*:*

(1) $M \in \mathrm{CM}(R)$,
(2) $\mathrm{Ext}^i_R(M,K_R) = 0 \ \ (\forall i > 0)$,
(3) $H^j_{\mathfrak{m}}(M) = 0 \ \ (\forall j < d)$,
(4) $\mathrm{Ext}^i_R(k,M) = 0 \ \ (\forall i < d)$.

Now recall that R is called an **isolated singularity** if $R_{\mathfrak{p}}$ is a regular local ring for each prime $\mathfrak{p} \neq \mathfrak{m}$. It is not hard to prove the following

Lemma 3. *Let R be a CM local ring as above. The R is an isolated singularity if and only if* $\mathrm{Ext}^1_R(M,N)$ *is of finite length for each* $M, N \in \mathrm{CM}(R)$.

Definition 4. A CM local ring R is said to be **of finite CM representation type** if $\mathrm{CM}(R)$ has only a finite number of isomorphism classes of indecomposable modules.

The first celebrated result about finiteness of CM representation type was due to M. Auslander.

Theorem 5. [Auslander, 1986] *Let R be a CM complete local ring. If R is of finite CM representation type, then R is an isolated singularity.*

We prove this theorem by using an idea of Huneke and Leuschke [6]. By virtue of Lemma 3 it is enough to prove the following:

(*) Let $a_1, a_2, a_3, \dots$ be any countable sequence of elements in $\mathfrak{m}$ and let $M, N \in \mathrm{CM}(R)$ be any indecomposable CM modules. Then there is an integer n such that
$a_1 a_2 \cdot \dots \cdot a_n \mathrm{Ext}^1_R(M,N) = 0$.

Actually this will imply that a power of $\mathfrak{m}$ annihilates $\mathrm{Ext}^1_R(M,N)$, hence the length of $\mathrm{Ext}^1_R(M,N)$ is finite. To prove $(*)$, take a $\sigma \in \mathrm{Ext}^1_R(M,N)$ that corresponds to a short exact sequence $\sigma : 0 \to N \to E_0 \to M \to 0$. Now assume the corresponding sequence to $a_1 a_2 \cdots a_n \sigma \in \mathrm{Ext}^1_R(M,N)$ is $0 \to N \to E_n \to M \to 0$ for any integer n. Note that each E_n is a direct sum of indecomposable CM modules and the multiplicity (or the rank if it is defined) $e(E_n)$ is constantly equal to $e(M) + e(N)$. Therefore the possibilities of such E_n are finite, and hence there are integers n and $r > 0$ such that $E_n \cong E_{n+r}$. By definition, there is a commutative diagram with exact rows:

$$\begin{array}{rccccccccc}
a_1 \cdots a_n \sigma : & 0 & \longrightarrow & N & \xrightarrow{\ j\ } & E_n & \longrightarrow & M & \longrightarrow & 0 \\
 & & & {\scriptstyle b := a_{n+1} \cdots a_{n+r}} \downarrow & & \downarrow & & \downarrow {\scriptstyle =} & & \\
a_1 \cdots a_{n+r} \sigma \ : & 0 & \longrightarrow & N & \longrightarrow & E_{n+r} & \longrightarrow & M & \longrightarrow & 0,
\end{array}$$

where the first square is a push-out. Hence,

$$0 \longrightarrow N \xrightarrow{\binom{j}{b}} E_n \oplus N \longrightarrow E_{n+r} \longrightarrow 0$$

is exact. Since $E_n \cong E_{n+r}$, Miyata's theorem forces that $\binom{j}{b}$ is a split monomorphism. Then one can see that j is also a split monomorphism. ($pj + qb = 1_N$ in the local ring $\mathrm{End}_R(N)$.) Hence $a_1 \cdots a_n \sigma = 0$ as an element of $\mathrm{Ext}^1_R(M, N)$. $\square$

By a similar idea to the proof above, Huneke and Leuschke [7] was able to prove the following theorem which had been conjectured by F.-O.Schreyer in 1987.

Theorem 6. [Huneke-Leuschke 2003] *Let R be a CM complete local ring and assume that R is of countable CM representation type (i.e.* $\mathrm{CM}(R)$ *has only a countable number of isomorphism classes of indecomposable modules). Then the singular locus of R has at most one-dimension, i.e. $R_\mathfrak{p}$ is regular for each prime $\mathfrak{p}$ with* $\dim R/\mathfrak{p} > 1$.

(Proof) Let $\{M_i \mid i = 1, 2, \ldots\}$ be a complete list of isomorphism classes of indecomposable CM modules, and set

$$\Lambda = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathfrak{p} = \mathrm{Ann}_R\, \mathrm{Ext}^1_R(M_i, M_j) \ \text{ for some } \ i, j \ \text{ and } \ \dim R/\mathfrak{p} = 1\},$$

which is a countable set of prime ideals. Let J be an ideal defining the singular locus of $\mathrm{Spec}(R)$ and we want to show $\dim R/J \leq 1$. Assume contrarily $\dim R/J \geq 2$. If $\mathfrak{p} \in \Lambda$ then, since $(M_i)_\mathfrak{p}$ is not free, we have $J \subseteq \mathfrak{p}$. Thus $J \subseteq \bigcap_{\mathfrak{p}\in\Lambda} \mathfrak{p}$. By countable prime avoidance, there is an $f \in \mathfrak{m} \backslash \bigcup_{\mathfrak{p}\in\Lambda} \mathfrak{p}$, and we can find a prime $\mathfrak{q}$ so that $\mathfrak{q} \supseteq J + fR$ and $\dim R/\mathfrak{q} = 1$. Set $X_i = \Omega^i_R(R/\mathfrak{q})$ the ith syzygy for $i \geq 0$. Then $X_i \in \mathrm{CM}(R)$ if $i \geq d$ and one can show that $\mathrm{Ann}_R\, \mathrm{Ext}^1_R(X_d, X_{d+1}) = \mathfrak{q}$. The CM modules X_d and X_{d+1} is a direct sum of indecomposables as $X_d \cong \bigoplus_{u=1}^r M_{i_u}$ and $X_{d+1} \cong \bigoplus_{v=1}^s M_{j_u}$. Thus since $\mathfrak{q} = \bigcap_{u,v} \mathrm{Ann}_R\, \mathrm{Ext}^1_R(M_{i_u}, M_{j_v})$, we have $\mathfrak{q} = \mathrm{Ann}_R\, \mathrm{Ext}^1_R(M_{i_u}, M_{j_v})$ for some u, v. Thus $\mathfrak{q} \in \Lambda$, but this is a contradiction for $f \in \mathfrak{q}$. $\square$

Auslander's original proof of Theorem 5 uses AR-sequences.

Definition 7. A non-split short exact sequence $0 \to N \to E \xrightarrow{p} M \to 0$ in $\mathrm{CM}(R)$ is called an **AR-sequence** (ending in M) if

(1) M and N are indecomposable,
(2) if $f : X \to M$ is any morphism in $\mathrm{CM}(R)$ that is not a splitting epimorphism, then f factors through p.

We say that the category $\mathrm{CM}(R)$ admits AR-sequences if, for any indecomposable $M \in \mathrm{CM}(R)$, there is an AR-sequence ending in M.

M.Auslander proved the following theorems.

Theorem 8. *Let R be a CM complete local ring and assume that R is of finite CM representation type. Then* $\mathrm{CM}(R)$ *admits AR-sequences.*

Theorem 9. *Let R be a CM complete local ring. Then* $\mathrm{CM}(R)$ *admits AR-sequences if and only if R is an isolated singularity.*

The most difficult part of the proofs of Theorems 8 and 9 is to show the implication "being isolated singularity ⇒ admitting AR-sequences". This implication follows from the following isomorphism which is called the Auslander-Reiten duality :

Theorem 10. *Assume that a CM complete local ring R is an isolated singularity of dimension d. Then, for any $M, N \in \mathrm{CM}(R)$, there is a natural isomorphism*

$$\mathrm{Ext}_R^d(\underline{\mathrm{Hom}}_R(N,M), K_R) \cong \mathrm{Ext}_R^1(M, \mathrm{Hom}_R(\Omega_R^d \mathrm{tr}(N), K_R)).$$

Now we discuss some generalities about stable categories. For this let R be a CM complete local ring of dimension d. We denote by $\underline{\mathrm{CM}}(R)$ the stable category of $\mathrm{CM}(R)$. By definition, $\underline{\mathrm{CM}}(R)$ is the factor category $\mathrm{CM}(R)/[R]$. Recall that the objects of $\underline{\mathrm{CM}}(R)$ is CM modules over R, and the morphisms of $\underline{\mathrm{CM}}(R)$ are elements of $\underline{\mathrm{Hom}}_R(M,N) := \mathrm{Hom}_R(M,N)/P(M,N)$ for $M, N \in \underline{\mathrm{CM}}(R)$, where $P(M,N)$ denotes the set of morphisms from M to N factoring through projective R-modules. For a CM module M we denote it by $\underline{M}$ to indicate that it is an object of $\underline{\mathrm{CM}}(R)$.

Since R is a complete local ring, note that $\underline{M}$ is isomorphic to $\underline{N}$ in $\underline{\mathrm{CM}}(R)$ if and only if $M \oplus P \cong N \oplus Q$ in $\mathrm{CM}(R)$ for some projective (hence free) R-modules P and Q.

For any R-module M, we denote the first syzygy module of M by $\Omega_R M$. We should note that $\Omega_R M$ is uniquely determined up to isomorphism as an object in the stable category. The nth syzygy module $\Omega_R^n M$ is defined inductively by $\Omega_R^n M = \Omega_R(\Omega_R^{n-1} M)$, for any nonnegative integer n.

We say that R is a Gorenstein ring if $K_R \cong R$. If R is Gorenstein, then it is easy to see that the syzygy functor $\Omega_A : \underline{\mathrm{CM}}(R) \to \underline{\mathrm{CM}}(R)$ is an autoequivalence. Hence, in particular, one can define the cosyzygy functor Ω_R^{-1} on $\underline{\mathrm{CM}}(R)$ which is the inverse of Ω_R. We note from [3, 2.6] that $\underline{\mathrm{CM}}(R)$ is a triangulated category with shifting functor $[1] = \Omega_R^{-1}$. In fact, if there is an exact sequence $0 \to L \to M \to N \to 0$ in $\mathrm{CM}(R)$, then we have the following commutative diagram by taking the pushout:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L & \longrightarrow & P & \longrightarrow & \Omega^{-1}L & \longrightarrow & 0, \end{array}$$

where P is projective (hence free). We define the triangles in $\underline{\mathrm{CM}}(R)$ are the sequences

$$\underline{L} \longrightarrow \underline{M} \longrightarrow \underline{N} \longrightarrow \underline{L}[1]$$

obtained in such a way.

Now we remark one of the fundamental dualities called the *Auslander-Reiten-Serre duality*, which essentially follows from Theorem 10.

Theorem 11. *Let R be a Gorenstein complete local ring of dimension d. Suppose that R is an isolated singularity. Then, for any $\underline{X}, \underline{Y} \in \underline{\mathrm{CM}}(R)$, we have a functorial isomorphism*

$$\mathrm{Ext}_R^d(\underline{\mathrm{Hom}}_R(X,Y), R) \cong \underline{\mathrm{Hom}}_R(Y, X[d-1]).$$

Therefore the triangulated category $\underline{\mathrm{CM}}(R)$ is a $(d-1)$-Calabi-Yau category.

2. Degenerations of modules

Let us recall the definition of degeneration of finitely generated modules over a noetherian algebra, which is given in [12].

Let R be an associative k-algebra where k is any field. We take a discrete valuation ring (V, tV, k) which is a k-algebra and t is a prime element. We denote by K the quotient field of V. We denote by $\mathrm{mod}(R)$ the category of all finitely generated left R-modules and R-homomorphisms as before. Then we have the natural functors

$$\mathrm{mod}(R) \xleftarrow{\;r\;} \mathrm{mod}(R \otimes_k V) \xrightarrow{\;\ell\;} \mathrm{mod}(R \otimes_k K),$$

where $r = - \otimes_V V/tV$ and $\ell = - \otimes_V K$. ("r" for residue, and "ℓ" for localization.)

Definition 12. For modules $M, N \in \mathrm{mod}(R)$, we say that M **degenerates to** N if there exist a discrete valuation ring (V, tV, k) which is a k-algebra and a module $Q \in \mathrm{mod}(R \otimes_k V)$ that is V-flat such that $\ell(Q) \cong M \otimes_k K$ and $r(Q) \cong N$.

The module Q, regarded as a bimodule ${}_RQ_V$, is a flat family of R-modules with parameter in V. At the closed point in the parameter space $\mathrm{Spec}V$, the fiber of Q is N, which is a meaning of the isomorphism $r(Q) \cong N$. On the other hand, the isomorphism $\ell(Q) \cong M \otimes_k K$ means that the generic fiber of Q is essentially given by M.

Example 13. Let $R = k[[x, y]]/(x^2)$, where k is a field. In this case, a pair of matrices

$$(\varphi, \psi) = \left(\begin{pmatrix} x & y^2 \\ 0 & x \end{pmatrix}, \begin{pmatrix} x & -y^2 \\ 0 & x \end{pmatrix} \right)$$

over $k[[x, y]]$ is a matrix factorization of x^2, giving a CM R-module N that is isomorphic to an ideal $I = (x, y^2)R$. Thus there is a periodic free resolution of N;

$$\cdots \longrightarrow R^2 \xrightarrow{\;\psi\;} R^2 \xrightarrow{\;\varphi\;} R^2 \xrightarrow{\;\psi\;} R^2 \xrightarrow{\;\varphi\;} R^2 \longrightarrow N \longrightarrow 0.$$

Now we deform the matices to

$$(\Phi, \Psi) = \left(\begin{pmatrix} x + ty & y^2 \\ -t^2 & x - ty \end{pmatrix}, \begin{pmatrix} x - ty & -y^2 \\ t^2 & x + ty \end{pmatrix} \right)$$

over $R \otimes_k V$. Since this is a matrix factorization of x^2 again, we have a free resolution

$$\cdots \xrightarrow{\;\Phi\;} (R \otimes_k V)^2 \xrightarrow{\;\Psi\;} (R \otimes_k V)^2 \xrightarrow{\;\Phi\;} (R \otimes_k V)^2 \longrightarrow Q \longrightarrow 0.$$

It is obvious to see that $r(Q) = Q/tQ \cong N$, since $\Phi \otimes_V V/tV = \varphi$. On the other hand, since t^2 is a unit in $R \otimes_k K$, we have $\Phi \otimes_V K \cong \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ after elementary transformations of matrices. Hence, $\ell(Q) = Q_t \cong R \otimes_k K$. As a conclusion, we see that R degenerates to $I = (x, y^2)R$!

Theorem 14 ([12])**.** *The following conditions are equivalent for finitely generated left R-modules M and N.*

(1) *M degenerates to N.*
(2) *There is a short exact sequence of finitely generated left R-modules*

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

such that the endomorphism ψ of Z is nilpotent, i.e. $\psi^n = 0$ for $n \gg 1$.

Example 15. In Example 13, we have an exact sequence

$$0 \longrightarrow \mathfrak{m} \xrightarrow{(-1, \frac{x}{y})} R \oplus \mathfrak{m} \xrightarrow{\binom{x}{y}} I \longrightarrow 0,$$

such that $\frac{x}{y} : \mathfrak{m} \to \mathfrak{m}$ is nilpotent, where $\mathfrak{m} = (x, y)R$.

By virtue of this theorem together with a theorem of Zwara [17, Theorem 1], we see that if R is a finite-dimensional algebra over k, then our definition of degeneration agrees with the classical (geometric) definition of degenerations using module varieties of R-module structures.

We prove here the implication (2) ⇒ (1).

Suppose that there is an exact sequence of finitely generated left R-modules

$$0 \to Z \xrightarrow{f=\binom{\phi}{\psi}} M \oplus Z \to N \to 0,$$

such that ψ is nilpotent. Considering a trivial exact sequence

$$0 \to Z \xrightarrow{j=\binom{0}{1}} M \oplus Z \to M \to 0,$$

we shall combine these two exact sequences along a $[0,1]$-interval. More precisely, let V be the discrete valuation ring $k[t]_{(t)}$, where t in an indeterminate over k, and consider a left $R \otimes_k V$-homomorphism

$$g = j \otimes t + f \otimes (1-t) = \begin{pmatrix} \phi \otimes (1-t) \\ 1 \otimes t + \psi \otimes (1-t) \end{pmatrix} : Z \otimes_k V \to (M \oplus Z) \otimes_k V.$$

We can easily show that g is a monomorphism.

Setting the cokernel of the monomorphism g as Q, we have an exact sequence in $\mathrm{mod} R \otimes_k V$:

$$0 \to Z \otimes_k V \xrightarrow{g} (Z \otimes_k V) \oplus (M \otimes_k V) \to Q \to 0.$$

Since $g \otimes_k V/tV = f$ is an injection and since one can easily show $\mathrm{Tor}_1^V(Q, V/tV) = 0$, we conclude that Q is flat over V and $Q/tQ \cong N$.

Finally note that the morphism $g \otimes_k V[\frac{1}{t}]$ is essentially the same as the morphism

$$Z \otimes_k V[\tfrac{1}{t}] \xrightarrow{\begin{pmatrix} s\phi \\ 1+s\psi \end{pmatrix}} M \otimes_k V[\tfrac{1}{t}] \oplus Z \otimes_k V[\tfrac{1}{t}],$$

where $s = \frac{1-t}{t} \in V[\frac{1}{t}]$. Note that $s\psi : Z \otimes_k V[\frac{1}{t}] \to Z \otimes_k V[\frac{1}{t}]$ is nilpotent as well as ψ, hence $1 + s\psi$ is an automorphism on $Z \otimes_k V[\frac{1}{t}]$. Therefore we have an isomorphism $Q[\frac{1}{t}] \cong M \otimes_k V[\frac{1}{t}]$. This completes the proof of the theorem. □

We remark from this proof that we can always take $k[t]_{(t)}$ as V in Definition 12.

We give an outline of the proof of (1) ⇒ (2). (See [12] for the detail.)

We can take Q in Definition 12 so that $M \otimes_k V \subseteq Q$. Then we have an exact sequence

$$0 \to Q/(M \otimes_k V) \xrightarrow{t} Q/(M \otimes_k tV) \longrightarrow Q/tQ \to 0$$

Setting $Z = Q/(M \otimes_k V)$, we can see that the middle term will be $M \oplus Z$ and the right term is N. □

Lemma 16. *If there is an exact sequence* $0 \to L \xrightarrow{i} M \xrightarrow{p} N \to 0$ *in* $\mathrm{mod}(R)$*, then* M *degenerates to* $L \oplus N$.

(Proof)

$$0 \longrightarrow L \xrightarrow{\binom{i}{0}} M \oplus L \xrightarrow{\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}} N \oplus L \longrightarrow 0$$

is exact where $0 : L \to L$ is of course nilpotent. □

Such a degeneration given as in the lemma will be called a degeneration by an extension. There is a degeneration which is not a degeneration by an extension. See the degeneration of Example 13.

In the rest we mainly treat the case when R is a commutative ring.

Remark 17. Let R be a commutative noetherian algebra over k, and suppose that a finitely generated R-module M degenerates to a finitely generated R-module N. Then:

(1) The modules M and N give the same class in the Grothendieck group, i.e. $[M] = [N]$ as elements of $K_0(\mathrm{mod}(R))$. This is actually a direct consequence of $0 \to Z \to M \oplus Z \to N \to 0$. In particular, rank M = rank N if the ranks are defined for R-modules. Furthermore, if $(R, \mathfrak{m})$ is a local ring, then $e(I, M) = e(I, N)$ for any $\mathfrak{m}$-primary ideal I, where $e(I, M)$ denotes the multiplicity of M along I.

(2) If L is an R-module of finite length, then we have the following inequalities of lengths for any integer i:

$$\begin{cases} \mathrm{length}_R(\mathrm{Ext}^i_R(L, M)) \leqq \mathrm{length}_R(\mathrm{Ext}^i_R(L, N)), \\ \mathrm{length}_R(\mathrm{Ext}^i_R(M, L)) \leqq \mathrm{length}_R(\mathrm{Ext}^i_R(N, L)). \end{cases}$$

In particular, when R is a local ring, then

$$\nu(M) \leqq \nu(N), \quad \beta_i(M) \leqq \beta_i(N) \ \text{ and } \ \mu^i(M) \leqq \mu^i(N) \ \ (i \geqq 0),$$

where ν, β_i and μ^i denote the minimal number of generators, the ith Betti number and the ith Bass number respectively.

(3) We also have $\mathrm{pd}_R M \leq \mathrm{pd}_R N$, $\mathrm{depth}\,_R M \geq \mathrm{depth}\,_R N$ and similar inequalities like $\text{G-dim}_R M \leq \text{G-dim}_R N$. Roughly speaking, when there is a degeneration from M to N, then M is a better module than N.

Recall that a finitely generated R-module is called rigid if it satisfies $\mathrm{Ext}^1_R(N, N) = 0$.

Lemma 18. *Let R be a complete local k-algebra and let $M, N \in \mathrm{mod}(R)$. Assume that N is rigid. If M degenerates to N, then $M \cong N$.*

(Proof) From the sequence $0 \to Z \xrightarrow{\binom{\phi}{\psi}} M \oplus Z \to N \to 0$, we have an exact sequence

$$\mathrm{Ext}^1_R(N, Z) \xrightarrow{\binom{\phi}{\psi}} \mathrm{Ext}^1_R(N, M) \oplus \mathrm{Ext}^1_R(N, Z) \to \mathrm{Ext}^1_R(N, N),$$

where ψ is nilpotent and $\mathrm{Ext}^1_R(N, N) = 0$. Thus we have $\mathrm{Ext}^1_R(N, Z) = 0$. It follows the first sequence splits, and thus $M \oplus Z \cong N \oplus Z$. Since R is complete, it forces $M \cong N$. □

We recall the definition of the Fitting ideal of a finitely presented module. Suppose that a module M over a commutative ring R is given by a finitely free presentation

$$R^m \xrightarrow{\ C\ } R^n \longrightarrow M \longrightarrow 0,$$

where C is an $n \times m$-matrix with entries in R. Then recall that the ith Fitting ideal $\mathcal{F}_i^R(M)$ of M is defined to be the ideal $I_{n-i}(C)$ of R generated by all the $(n-i)$-minors of the matrix C. (We use the convention that $I_r(C) = R$ for $r \leqq 0$ and $I_r(C) = 0$ for $r > \min\{m, n\}$.) It is known that $\mathcal{F}_i^R(M)$ depends only on M and i, and independent of the choice of free presentation, and $\mathcal{F}_0^R(M) \subseteq \mathcal{F}_1^R(M) \subseteq \cdots \subseteq \mathcal{F}_n^R(M) = R$. The following lemma will be used to prove the theorem.

Lemma 19. *Let $f : A \to B$ be a ring homomorphism and let M be an A-module which possesses a finitely free presentation. Then $\mathcal{F}_i^B(M \otimes_A B) = f(\mathcal{F}_i^A(M))B$ for all $i \geqq 0$.*

(Proof) If M has a presentation $A^m \xrightarrow{C} A^n \to M \to 0$, then $M \otimes_A B$ has a presentation $B^m \xrightarrow{f(C)} B^n \to M \otimes_A B \to 0$. Thus $\mathcal{F}_i^B(M \otimes_A B) = I_{n-i}(f(C)) = f(I_{n-i}(C))B = f(\mathcal{F}_i^A(M))B$. $\square$

Theorem 20. [Y, 2011] *Let R be a noetherian commutative algebra over k, and M and N finitely generated R-modules. Suppose M degenerates to N. Then we have $\mathcal{F}_i^R(M) \supseteqq \mathcal{F}_i^R(N)$ for all $i \geqq 0$.*

(Proof) By the assumption there is a finitely generated $R \otimes_k V$-module Q such that $Q_t \cong M \otimes_k K$ and $Q/tQ \cong N$, where $V = k[t]_{(t)}$ and $K = k(t)$. Note that $R \otimes_k V \cong S^{-1}R[t]$ where $S = k[t] \backslash (t)$. Since Q is finitely generated, we can find a finitely generated $R[t]$-module Q' such that $Q' \otimes_{R[t]} (R \otimes_k V) \cong Q$. For a fixed integer i we now consider the Fitting ideal $J := \mathcal{F}_i^{R[t]}(Q') \subseteqq R[t]$. Apply Lemma 19 to the ring homomorphism $R[t] \to R = R[t]/tR[t]$, and noting that $Q' \otimes_{R[t]} R \cong N$, we have

$$\mathcal{F}_i^R(N) = J + tR[t]/tR[t] \qquad (2.1)$$

as an ideal of $R = R[t]/tR[t]$. On the other hand, applying Lemma 19 to $R[t] \to R \otimes_k K = T^{-1}R[t]$ where $T = k[t] \backslash \{0\}$, we have $\mathcal{F}_i^R(M)T^{-1}R[t] = JT^{-1}R[t]$. Therefore there is an element $f(t) \in T$ such that $f(t)J \subseteq \mathcal{F}_i^R(M)R[t]$.

Now to prove the inclusion $\mathcal{F}_i^R(N) \subseteqq \mathcal{F}_i^R(M)$, take an arbitrary element $a \in \mathcal{F}_i^R(N)$. It follows from (2.1) that there is a polynomial of the form $a + b_1 t + b_2 t^2 + \cdots + b_r t^r$ ($b_i \in R$) that belongs to J. Then, we have $f(t)(a + b_1 t + b_2 t^2 + \cdots + b_r t^r) \in \mathcal{F}_i^R(M)R[t]$. Since $f(t)$ is a non-zero polynomial whose coefficients are all in k, looking at the coefficient of the non-zero term of the least degree in the polynomial $f(t)(a + b_1 t + \cdots + b_r t^r)$, we have that $a \in \mathcal{F}_i^R(M)$. $\square$

Example 21. Let $R = k[[x, y]]/(x^2, y^2)$. Note that R is an artinian Gorenstein local ring. Now consider the modules $M_\lambda = R/(x - \lambda y)R$ for all $\lambda \in k$. We denote by k the unique simple module $R/(x, y)R$ over R.

(1) R degenerates to $M_\lambda \oplus M_{-\lambda}$ for $\forall \lambda \in k$, since there is an exact sequence $0 \to M_{-\lambda} \to R \to M_\lambda \to 0$.

(2) There is a sequence of degenerations from $R \oplus k^2$ to $M_\lambda \oplus M_\mu \oplus k^2$ for any choice of $\lambda, \mu \in k$. ([9, Example 3.1])

(Proof) There are exact sequences; $0 \to \mathfrak{m} \to R \oplus \mathfrak{m}/(xy) \to R/(xy) \to 0$, $0 \to M_\lambda \to \mathfrak{m} \to k \to 0$ and $0 \to k \xrightarrow{x - \mu y} R/(xy) \to M_\mu \to 0$ for any $\lambda, \mu \in k$. Noting $\mathfrak{m}/(xy) \cong k^2$, we have a sequence of degenerations $R \oplus k^2 \Rightarrow \mathfrak{m} \oplus R/(xy) \Rightarrow (M_\mu \oplus k) \oplus (M_\lambda \oplus k) = M_\lambda \oplus M_\mu \oplus k^2$. □

(3) There is no sequence of degenerations from R to $M_\lambda \oplus M_\mu$ if $\lambda + \mu \neq 0$.

(Proof) If there are such degenerations, then we have an inclusion of Fitting ideals; $\mathcal{F}_n^R(M_\lambda \oplus M_\mu) \subseteq \mathcal{F}_n^R(R)$ for all n. Note that $\mathcal{F}_0^R(R) = 0$, and

$$\mathcal{F}_0^R(M_\lambda \oplus M_\mu) = \mathcal{F}_0^R(M_\lambda)\mathcal{F}_0^R(M_\mu) = (x - \lambda y)(x - \mu y)R = (\lambda + \mu)xyR.$$

Hence we must have $\lambda + \mu = 0$. □

This example shows the cancellation law does not hold for degeneration.

Example 22. Let $R = k[[t]]$ be a formal power series ring over a field k with one variable t and let M be an R-module of length n. It is easy to see that there is an isomorphism

$$M \cong R/(t^{p_1}) \oplus \cdots \oplus R/(t^{p_n}), \tag{2.2}$$

where

$$p_1 \geq p_2 \geq \cdots \geq p_n \geq 0 \quad \text{and} \quad \sum_{i=1}^{n} p_i = n. \tag{2.3}$$

In this case the ith Fitting ideal of M is given as

$$\mathcal{F}_i^R(M) = (t^{p_{i+1}+\cdots+p_n}) \quad (i \geq 0).$$

We denote by p_M the sequence $(p_1, p_2, \cdots, p_n)$ of non-negative integers. Recall that such a sequence satisfying (2.3) is called a partition of n.

Conversely, given a partition $p = (p_1, p_2, \cdots, p_n)$ of n, we can associate an R-module of length n by (2.2), which we denote by $M(p)$. In such a way there is a one-one correspondence between the set of partitions of n and the set of isomorphism classes of R-modules of length n.

Let $p = (p_1, p_2, \cdots, p_n)$ and $q = (q_1, q_2, \cdots, q_n)$ be partitions of n. Then we denote $p \succeq q$ if it satisfies $\sum_{i=1}^{j} p_i \geq \sum_{i=1}^{j} q_i$ for all $1 \leq j \leq n$. This $\succeq$ is known to be a partial order on the set of partitions of n and called the dominance order.

Then we can show that there is a degeneration from M to N if and only if $p_M \succeq p_N$.

3. Stable degenerations of CM modules

In this section we are interested in the stable analogue of degenerations of Cohen-Macaulay modules over a commutative Gorenstein local ring. For this purpose, $(R, \mathfrak{m}, k)$ always denotes a Gorenstein local ring which is a k-algebra, and $V = k[t]_{(t)}$ and $K = k(t)$ where t is a variable. We note that $R \otimes_k V$ and $R \otimes_k K$ are Gorenstein as well as R and we have the equality of Krull dimension;

$$\dim R \otimes_k V = \dim R + 1, \quad \dim R \otimes_k K = \dim R.$$

If $\dim R = 0$ (i.e. R is artinian), then the rings $R \otimes_k V$ and $R \otimes_k K$ are local. However we should note that $R \otimes_k V$ and $R \otimes_k K$ will never be local rings if $\dim R > 0$. Since $R \otimes_k K$ is non-local, there may be a lot of projective modules which are not free.

Example 23. Let $R = k[[x, y]]/(x^3 - y^2)$. It is known that the maximal ideal $\mathfrak{m} = (x, y)$ is a unique non-free indecomposable Cohen-Macaulay module over R. See [10, Proposition 5.11]. In fact it is given by a matrix factorization of the polynomial $x^3 - y^2$;

$$(\varphi, \psi) = \left(\begin{pmatrix} y & x \\ x^2 & y \end{pmatrix}, \quad \begin{pmatrix} y & -x \\ -x^2 & y \end{pmatrix} \right).$$

Therefore there is an exact sequence

$$\cdots \longrightarrow R^2 \xrightarrow{\varphi} R^2 \xrightarrow{\psi} R^2 \xrightarrow{\varphi} R^2 \longrightarrow \mathfrak{m} \longrightarrow 0.$$

Now we deform these matrices and consider the pair of matrices over $R \otimes_k K$;

$$(\Phi, \Psi) = \left(\begin{pmatrix} y - xt & x - t^2 \\ x^2 & y + xt \end{pmatrix}, \quad \begin{pmatrix} y + xt & -x + t^2 \\ -x^2 & y - xt \end{pmatrix} \right).$$

Define the $R \otimes_k K$-module P by the following exact sequence;

$$\cdots \longrightarrow (R \otimes_k K)^2 \xrightarrow{\Psi} (R \otimes_k K)^2 \xrightarrow{\Phi} (R \otimes_k K)^2 \longrightarrow P \longrightarrow 0.$$

In this case we can prove that P is a projective module of rank one over $R \otimes_k K$ but non-free. (Hence the Picard group of $R \otimes_k K$ is non-trivial.)

Let A be a commutative Gorenstein ring which is not necessarily local. We say that a finitely generated A-module M is CM if $\mathrm{Ext}^i_A(M, A) = 0$ for all $i > 0$. We consider the category of all CM modules over A with all A-module homomorphisms:

$$\mathrm{CM}(A) := \{M \in \mathrm{mod}(A) \mid M \text{ is a Cohen-Macaulay module over } A\}.$$

We can then consider the stable category of $\mathrm{CM}(A)$, which we denote by $\underline{\mathrm{CM}}(A)$. This is similarly defined as in local cases, but the morphisms of $\underline{\mathrm{CM}}(A)$ are elements of $\underline{\mathrm{Hom}}_A(M, N) := \mathrm{Hom}_A(M, N)/P(M, N)$ for $M, N \in \underline{\mathrm{CM}}(A)$, where $P(M, N)$ denotes the set of morphisms from M to N factoring through projective A-modules (not necessarily free).

Note that $\underline{M} \cong \underline{N}$ in $\underline{\mathrm{CM}}(A)$ if and only if there are projective A-modules P_1 and P_2 such that $M \oplus P_1 \cong N \oplus P_2$ in $\mathrm{CM}(A)$.

Under such circumstances it is known that $\underline{\mathrm{CM}}(A)$ has a structure of triangulated category as well as in local cases.

Let $x \in A$ be a non-zero divisor on A. Note that x is a non-zero divisor on every CM module over A. Thus the functor $- \otimes_A A/xA$ sends a CM module over A to that over A/xA. Therefore it yields a functor $\mathrm{CM}(A) \to \mathrm{CM}(A/xA)$. Since this functor maps projective A-modules to projective A/xA-modules, it induces the functor $\mathcal{R} : \underline{\mathrm{CM}}(A) \to \underline{\mathrm{CM}}(A/xA)$. It is easy to verify that $\mathcal{R}$ is a triangle functor.

Now let $S \subset A$ be a multiplicative subset of A. Then, by a similar reason to the above, we have a triangle functor $\mathcal{L} : \underline{\mathrm{CM}}(A) \to \underline{\mathrm{CM}}(S^{-1}A)$ which maps $\underline{M}$ to $\underline{S^{-1}M}$.

As before, let $(R, \mathfrak{m}, k)$ be a Gorenstein local ring that is a k-algebra and let $V = k[t]_{(t)}$ and $K = k(t)$. Since $R \otimes_k V$ and $R \otimes_k K$ are Gorenstein rings, we can apply the observation above. Actually, $t \in R \otimes_k V$ is a non-zero divisor on $R \otimes_k V$ and there are isomorphisms of k-algebras; $(R \otimes_k V)/t(R \otimes_k V) \cong R$ and $(R \otimes_k V)_t \cong R \otimes_k K$. Thus there are triangle functors $\mathcal{L} : \underline{\mathrm{CM}}(R \otimes_k V) \to \underline{\mathrm{CM}}(R \otimes_k K)$ defined by the localization by t, and

$\mathcal{R} : \underline{\mathrm{CM}}(R \otimes_k V) \to \underline{\mathrm{CM}}(R)$ defined by taking $- \otimes_{R \otimes_k V} (R \otimes_k V)/t(R \otimes_k V) = - \otimes_V V/tV$. Now we define the stable degeneration of CM modules.

Definition 24. Let $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R)$. We say that $\underline{M}$ **stably degenerates to** $\underline{N}$ if there is a Cohen-Macaulay module $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ such that $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$ in $\underline{\mathrm{CM}}(R \otimes_k K)$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$ in $\underline{\mathrm{CM}}(R)$.

Lemma 25. [15, Lemma 4.2, Proposition 4.3]

(1) *Let $M, N \in \mathrm{CM}(R)$. If M degenerates to N, then $\underline{M}$ stably degenerates to $\underline{N}$.*
(2) *Suppose that there is a triangle in $\underline{\mathrm{CM}}(R)$;*

$$\underline{L} \xrightarrow{\ \alpha\ } \underline{M} \xrightarrow{\ \beta\ } \underline{N} \xrightarrow{\ \gamma\ } \underline{L}[1].$$

Then $\underline{M}$ stably degenerates to $\underline{L} \oplus \underline{N}$.

Lemma 26. [15, Proposition 4.4] *Let $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R)$ and suppose that $\underline{M}$ stably degenerates to $\underline{N}$. Then the following hold.*

(1) *$\underline{M}[1]$ (resp. $\underline{M}[-1]$) stably degenerates to $\underline{N}[1]$ (resp. $\underline{N}[-1]$).*
(2) *$\underline{M^*}$ stably degenerates to $\underline{N^*}$, where M^* denotes the R-dual $\mathrm{Hom}_R(M, R)$.*

Lemma 27. [15, Proposition 4.5] *Let $\underline{M}$, $\underline{N}$, $\underline{X} \in \underline{\mathrm{CM}}(R)$. If $\underline{M} \oplus \underline{X}$ stably degenerates to $\underline{N}$, then $\underline{M}$ stably degenerates to $\underline{N} \oplus \underline{X}[1]$.*

Remark 28. The zero object in $\underline{\mathrm{CM}}(R)$ can stably degenerate to a non-zero object. In fact, in Example 13 the free module R degenerates to an ideal N. Hence it follows from Proposition 25(1) that $\underline{0} = \underline{R}$ stably degenerates to $\underline{N}$.

For another example, note that there is a triangle

$$\underline{X} \longrightarrow \underline{0} \longrightarrow \underline{X}[1] \xrightarrow{\ 1\ } \underline{X}[1],$$

for any $\underline{X} \in \underline{\mathrm{CM}}(R)$. Hence $\underline{0}$ stably degenerates to $\underline{X} \oplus \underline{X}[1]$ by Proposition 25(2).

Let $(R, \mathfrak{m}, k)$ be a Gorenstein complete local k-algebra and assume for simplicity that k is an infinite field. For Cohen-Macaulay R-modules M and N we consider the following four conditions:

(1) $R^m \oplus M$ degenerates to $R^n \oplus N$ for some $m, n \in \mathbb{N}$.
(2) There is a triangle $\underline{Z} \xrightarrow{\binom{\phi}{\psi}} \underline{M} \oplus \underline{Z} \to \underline{N} \to \underline{Z}[1]$ in $\underline{\mathrm{CM}}(R)$, where $\underline{\psi}$ is a nilpotent element of $\underline{\mathrm{End}}_R(Z)$.
(3) $\underline{M}$ stably degenerates to $\underline{N}$.
(4) There exists an $X \in \mathrm{CM}(R)$ such that $M \oplus R^m \oplus X$ degenerates to $N \oplus R^n \oplus X$ for some $m, n \in \mathbb{N}$.

In [15] we proved the following implications and equivalences of these conditions:

Theorem 29. (*i*) *In general,* (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) *holds.*

(*ii*) *If* $\dim R = 0$*, then* (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) *holds.*
(*iii*) *If R is an isolated singularity of any dimension, then* (2) $\Leftrightarrow$ (3) *holds.*
(*iv*) *There is an example of isolated singularity of* $\dim R = 1$ *for which* (2) $\Rightarrow$ (1) *fails.*
(*v*) *There is an example of* $\dim R = 0$ *for which* (4) $\Rightarrow$ (3) *fails.*

We give here an outline of some of the proofs.

Proof of (1) ⇒ (2) : By Theorem 14, there exists an exact sequence

$$0 \to Z \xrightarrow{\binom{\phi}{\psi}} (R^m \oplus M) \oplus Z \to (R^n \oplus N) \to 0,$$

where ψ is nilpotent. In such a case Z ia a Cohen-Macaulay module as well. Then converting this into a triangle in $\underline{\mathrm{CM}}(R)$, and noting that the nilpotency of $\psi \in \mathrm{End}_R(Z)$ forces the nilpotency of $\underline{\psi} \in \underline{\mathrm{End}}_R(Z)$, we can see that (2) holds. □

Proof of (2) ⇒ (3): Suppose that there exists a triangle $\underline{Z} \xrightarrow{\binom{\phi}{\psi}} \underline{M} \oplus \underline{Z} \to \underline{N} \to \underline{Z}[1]$, where $\underline{\psi}$ is nilpotent. Then we have a triangle of the form;

$$\underline{Z \otimes_k V} \xrightarrow{\binom{\phi}{t+\psi}} \underline{M \otimes_k V} \oplus \underline{Z \otimes_k V} \longrightarrow \underline{Q} \longrightarrow \underline{Z \otimes_k V}[1],$$

for a $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$. Note $\mathcal{L}(t + \underline{\psi})$ is an isomorphism in $\underline{\mathrm{CM}}(R \otimes_k K)$. Thus $\mathcal{L}(\underline{Q}) \cong \mathcal{L}(\underline{M \otimes_k V}) = \underline{M \otimes_k K}$. On the other hand, since $\mathcal{R}(t + \underline{\psi}) = \underline{\psi}$, $\mathcal{R}(\underline{Q}) \cong N$. Thus $\underline{M}$ stably degenerates to $\underline{N}$. □

Proof of (3) ⇒ (1) when $\dim R = 0$: In this proof we assume $\dim R = 0$. Suppose that $\underline{M}$ stably degenerates to $\underline{N}$. Then there is a $\underline{Q} \in \underline{\mathrm{CM}}(R \otimes_k V)$ with $\mathcal{L}(\underline{Q}) \cong \underline{M \otimes_k K}$ and $\mathcal{R}(\underline{Q}) \cong \underline{N}$. By definition, we have isomorphisms $Q_t \oplus P_1 \cong (M \otimes_k K) \oplus P_2$ in $\mathrm{CM}(R \otimes_k K)$ for some projective $R \otimes_k K$-modules P_1, P_2, and $Q/tQ \oplus R^a \cong N \oplus R^b$ in $\mathrm{CM}(R)$ for some $a, b \in \mathbb{N}$. Since $R \otimes_k K$ is a local ring, P_1 and P_2 are free. Thus $Q_t \oplus (R \otimes_k K)^c \cong (M \otimes_k K) \oplus (R \otimes_k K)^d$ for some $c, d \in \mathbb{N}$. Setting $\tilde{Q} = Q \oplus (R \otimes_k V)^{a+c}$, we have isomorphisms

$$\tilde{Q}_t \cong (M \oplus R^{a+d}) \otimes_k K, \quad \tilde{Q}/t\tilde{Q} \cong N \oplus R^{b+c}.$$

Since $\tilde{Q}$ is V-flat, $M \oplus R^{a+d}$ degenerates to $N \oplus R^{b+c}$. □

The difficult part of the proof is to show the implications (3) ⇒ (4) and (3) ⇒ (2). Actually it is technically difficult to show the existence of a Cohen-Macaulay module Z and X in each case. To get over this difficulty, we use the following lemma called Swan's Lemma in Algebraic K-Theory.

Lemma 30. [8, Lemma 5.1] *Let R be a noetherian ring and t a variable. Assume that an $R[t]$-module L is a submodule of $W \otimes_R R[t]$ with W being a finitely generated R-module. Then there is an exact sequence of $R[t]$-modules;*

$$0 \longrightarrow X \otimes_R R[t] \longrightarrow Y \otimes_R R[t] \longrightarrow L \longrightarrow 0,$$

where X and Y are finitely generated R-modules.

By virtue of Swan's lemma we can prove the following proposition that will play an essential role in the proof of Theorem 29.

Proposition 31. *Let R be a Gorenstein local k-algebra, where k is an infinite field. Suppose we are given a Cohen-Macaulay $R \otimes_k V$-module P' satisfying that the localization*

$P = P'_t$ *by* t *is a projective* $R \otimes_k K$*-module. Then there is a Cohen-Macaulay* R*-module* X *with a triangle in* $\underline{\mathrm{CM}}(R \otimes_k V)$ *of the following form:*

$$\underline{X \otimes_k V} \longrightarrow \underline{X \otimes_k V} \longrightarrow \underline{P'} \longrightarrow \underline{X \otimes_k V}[1]. \tag{3.1}$$

As a direct consequence of Theorem 29, we have the following corollary.

Corollary 32. *Let* $(R_1, \mathfrak{m}_1, k)$ *and* $(R_2, \mathfrak{m}_2, k)$ *be Gorenstein complete local* k*-algebras. Assume that the both* R_1 *and* R_2 *are isolated singularities, and that* k *is an infinite field. Suppose there is a* k*-linear equivalence* $F : \underline{\mathrm{CM}}(R_1) \to \underline{\mathrm{CM}}(R_2)$ *of triangulated categories. Then, for* $\underline{M}$, $\underline{N} \in \underline{\mathrm{CM}}(R_1)$, $\underline{M}$ *stably degenerates to* $\underline{N}$ *if and only if* $F(\underline{M})$ *stably degenerates to* $F(\underline{N})$.

Remark 33. Let $(R_1, \mathfrak{m}_1, k)$ and $(R_2, \mathfrak{m}_2, k)$ be Gorenstein complete local k-algebras as above. Then it hardly occurs that there is a k-linear equivalence of categories between $\mathrm{CM}(R_1)$ and $\mathrm{CM}(R_2)$. In fact, if it occurs, then R_1 is isomorphic to R_2 as a k-algebra. (See [4, Proposition 5.1].)

On the other hand, an equivalence between $\underline{\mathrm{CM}}(R_1)$ and $\underline{\mathrm{CM}}(R_2)$ may happen for non-isomorphic k-algebras. For example, let $R_1 = k[[x,y,z]]/(x^n+y^2+z^2)$ and $R_2 = k[[x]]/(x^n)$ with characteristic of k not being 2 and $n \in \mathbb{N}$. Then, by Knoerrer's periodicity ([10, Theorem 12.10]), we have an equivalence $\underline{\mathrm{CM}}(k[[x,y,z]]/(x^n + y^2 + z^2)) \cong \underline{\mathrm{CM}}(k[[x]]/(x^n))$. Since $k[[x]]/(x^n)$ is an artinian Gorenstein ring, the stable degeneration of modules over $k[[x]]/(x^n)$ is equivalent to a degeneration up to free summands by Theorem 29(ii). Moreover the degeneration problem for modules over $k[[x]]/(x^n)$ is known to be equivalent to the degeneration problem for Jordan canonical forms of square matrices of size n. (See Example 22.) Thus by virtue of Corollary 32, it is easy to describe the stable degenerations of Cohen-Macaulay modules over $k[[x,y,z]]/(x^n + y^2 + z^2)$.

References

1. K. Bongartz, *A generalization of a theorem of M. Auslander*, Bull. London Math. Soc. **21** (1989), no. 3, 255-256.
2. K. Bongartz, *On degenerations and extensions of finite-dimensional modules*, Adv. Math. **121** (1996), no. 2, 245–287.
3. D. Happel, *Triangulated categories in the representation theory of finite-dimensional algebras*, London Mathematical Society Lecture Note Series, vol.119. Cambridge University Press, Cambridge, 1988. x+208 pp.
4. N. Hiramatsu and Y. Yoshino, *Automorphism groups and Picard groups of additive full subcategories*, Math. Scand. **107** (2010), 5–29.
5. ———, *Examples of degenerations of Cohen-Macaulay modules*, Preprint (2010). [arXiv:1012.5346]
6. C.Huneke and G.Leuschke, *Two theorems about maximal Cohen-Macaulay modules*, Math. Ann. **324** (2002), no. 2, 391-404.
7. C.Huneke and G.Leuschke, *Local rings of countable Cohen-Macaulay type*, Proc. Amer. Math. Soc. **131** (2003), no. 10, 3003-3007.
8. T. Y. Lam, *Serre's conjecture*, Lecture Notes in Mathematics, Vol. 635. Springer-Verlag, Berlin-New York, 1978. xv+227 pp. ISBN: 3-540-08657-9
9. Ch. Riedtmann, *Degenerations for representations of quivers with relations*, Ann. Scient. École Normale Sup. 4^e sèrie **19** (1986), 275–301.
10. Y. Yoshino, *Cohen-Macaulay modules over Cohen-Macaulay rings*, London Mathematical Society, Lecture Notes Series vol. 146, Cambridge University Press, 1990. viii+177 pp.

11. ______, *On degenerations of Cohen-Macaulay modules*, Journal of Algebra **248** (2002), 272–290.
12. ______, *On degenerations of modules*, Journal of Algebra **278** (2004), 217–226.
13. ______, *Degeneration and G-dimension of modules*, Lecture Notes in Pure and Applied Mathematics vol. 244, 'Commutative Algebra' Chapman and Hall/CRC (2006), 259 – 265.
14. ______, *Universal lifts of chain complexes over non-commutative parameter algebras*, J. Math. Kyoto Univ. **48** (2008), no. 4, 793–845.
15. ______, *Stable degenerations of Cohen-Macaulay modules*, J. Algebra **332** (2011), 500–521.
16. G. Zwara, *A degeneration-like order for modules*, Arch. Math. **71** (1998), 437–444.
17. ______, *Degenerations of finite dimensional modules are given by extensions*, Compositio Math. **121** (2000), 205–218.

Department of Mathematics
Graduate School of Natural Science and Techonology
Okayama University
Okayama, 700-8530 JAPAN

E-mail address: yoshino@math.okayama-u.ac.jp