SUBCATEGORIES OF EXTENSION MODULES RELATED TO SERRE SUBCATEGORIES

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Abstract. We consider subcategories consisting of the extensions of modules in two given Serre subcategories to find a method of constructing Serre subcategories of the module category. We shall give a criterion for this subcategory to be a Serre subcategory.

1. Introduction

Let R be a commutative Noetherian ring. We denote by R-Mod the category of R-modules and by R-mod the full subcategory consisting of finitely generated R-modules.

In [2], P. Gabriel showed that one has lattice isomorphisms between the set of Serre subcategories of R-mod, the set of Serre subcategories of R-Mod which are closed under arbitrary direct sums and the set of specialization closed subsets of $\mathrm{Spec}\,(R)$. By this result, Serre subcategories of R-mod are classified. However, it has not yet classified Serre subcategories of R-Mod. In this paper, we shall give a way of constructing Serre subcategories of R-Mod by considering subcategories of extension modules related to Serre subcategories.

2. The definition of a subcategory of extension modules by Serre subcategories

We assume that all full subcategories of R-Mod are closed under isomorphisms. We recall that a subcategory $\mathcal{S}$ of R-Mod is said to be Serre subcategory if the following condition is satisfied: For any short exact sequence

$$0 \to L \to M \to N \to 0$$

of R-modules, it holds that M is in $\mathcal{S}$ if and only if L and N are in $\mathcal{S}$. In other words, $\mathcal{S}$ is called a Serre subcategory if it is closed under submodules, quotient modules and extensions.

We give the definition of a subcategory of extension modules by Serre subcategories.

Definition 1. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. We denote by $(\mathcal{S}_1, \mathcal{S}_2)$ a subcategory consisting of R-modules M with a short exact sequence

$$0 \to X \to M \to Y \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$, that is

$$(\mathcal{S}_1, \mathcal{S}_2) = \left\{ M \in R\text{-Mod} \;\middle|\; \begin{array}{c} \text{there are } X \in \mathcal{S}_1 \text{ and } Y \in \mathcal{S}_2 \text{ such that} \\ 0 \to X \to M \to Y \to 0 \\ \text{is a short exact sequence.} \end{array} \right\}.$$

The detailed version of this paper has been submitted for publication elsewhere.

Remark 2. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod.

(1) Since the zero module belongs to any Serre subcategory, one has $\mathcal{S}_1 \subseteq (\mathcal{S}_1, \mathcal{S}_2)$ and $\mathcal{S}_2 \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.

(2) It holds $\mathcal{S}_1 \supseteq \mathcal{S}_2$ if and only if $(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{S}_1$.

(3) It holds $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $(\mathcal{S}_1, \mathcal{S}_2) = \mathcal{S}_2$.

(4) A subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under finite direct sums.

Example 3. We denote by $\mathcal{S}_{f.g.}$ the subcategory consisting of finitely generated R-modules and by $\mathcal{S}_{Artin}$ the subcategory consisting of Artinian R-modules. If R is a complete local ring, then a subcategory $(\mathcal{S}_{f.g.}, \mathcal{S}_{Artin})$ is known as the subcategory consisting of Matlis reflexive R-modules. Therefore, $(\mathcal{S}_{f.g.}, \mathcal{S}_{Artin})$ is a Serre subcategory of R-Mod.

The following example shows that a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ needs not be a Serre subcategory for Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$.

Example 4. We shall see that the subcategory $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ needs not be closed under extensions.

Let R be a one dimensional Gorenstein local ring with a maximal ideal $\mathfrak{m}$. Then one has a minimal injective resolution

$$0 \to R \to \bigoplus_{\substack{\mathfrak{p} \in \mathrm{Spec}(R) \\ \mathrm{ht}\mathfrak{p} = 0}} E_R(R/\mathfrak{p}) \to E_R(R/\mathfrak{m}) \to 0$$

of R. ($E_R(M)$ denotes the injective hull of an R-module M.) We note that R and $E_R(R/\mathfrak{m})$ are in $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$.

Now, we assume that a subcategory $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ is closed under extensions. Then $E_R(R) = \oplus_{\mathrm{ht}\mathfrak{p}=0} E_R(R/\mathfrak{p})$ is in $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$. It follows from the definition of $(\mathcal{S}_{Artin}, \mathcal{S}_{f.g.})$ that there exists an Artinian R-submodule X of $E_R(R)$ such that $E_R(R)/X$ is a finitely generated R-module.

If $X = 0$, then $E_R(R)$ is a finitely generated injective R-module. It follows from the Bass formula that one has $\dim R = \operatorname{depth} R = \operatorname{inj\,dim} E_R(R) = 0$. However, this equality contradicts $\dim R = 1$. On the other hand, if $X \neq 0$, then X is a non-zero Artinian R-module. Therefore, one has $\mathrm{Ass}_R(X) = \{\mathfrak{m}\}$. Since X is an R-submodule of $E_R(R)$, one has

$$\mathrm{Ass}_R(X) \subseteq \mathrm{Ass}_R(E_R(R)) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mathrm{ht}\,\mathfrak{p} = 0\}.$$

This is contradiction as well.

3. The main result

In this section, we shall give a criterion for a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ to be a Serre subcategory for Serre subcategories $\mathcal{S}_1$ and $\mathcal{S}_2$.

First of all, it is easy to see that the following assertion holds.

Proposition 5. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. Then a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under submodules and quotient modules.*

Lemma 6. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. We suppose that a sequence $0 \to L \to M \to N \to 0$ of R-modules is exact. Then the following assertions hold.*

(1) *If $L \in \mathcal{S}_1$ and $N \in (\mathcal{S}_1, \mathcal{S}_2)$, then $M \in (\mathcal{S}_1, \mathcal{S}_2)$.*

(2) *If $L \in (\mathcal{S}_1, \mathcal{S}_2)$ and $N \in \mathcal{S}_2$, then $M \in (\mathcal{S}_1, \mathcal{S}_2)$.*

Proof. (1) We assume that L is in $\mathcal{S}_1$ and N is in $(\mathcal{S}_1, \mathcal{S}_2)$. Since N belongs to $(\mathcal{S}_1, \mathcal{S}_2)$, there exists a short exact sequence

$$0 \to X \to N \to Y \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$. Then we consider the following pull buck diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & X' & \longrightarrow & X & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & Y & = & Y & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

of R-modules with exact rows and columns. Since $\mathcal{S}_1$ is a Serre subcategory, it follows from the first row in the diagram that X' belongs to $\mathcal{S}_1$. Consequently, we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$ by the middle column in the diagram.

(2) We can show that the assertion holds by the similar argument in the proof of (1). □

Now, we can show the main purpose of this paper.

Theorem 7. *Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be Serre subcategories of R-Mod. Then the following conditions are equivalent:*

(1) *A subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is a Serre subcategory;*

(2) *One has $(\mathcal{S}_2, \mathcal{S}_1) \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.*

Proof. (1) $\Rightarrow$ (2) We assume that M is in $(\mathcal{S}_2, \mathcal{S}_1)$. By the definition of a subcategory $(\mathcal{S}_2, \mathcal{S}_1)$, there exists a short exact sequence

$$0 \to Y \to M \to X \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$. We note that X and Y are also in $(\mathcal{S}_1, \mathcal{S}_2)$. Since a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under extensions by the assumption (1), we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$.

(2) ⇒ (1) We only have to prove that a subcategory $(\mathcal{S}_1, \mathcal{S}_2)$ is closed under extensions by Proposition 5. Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules such that L and N are in $(\mathcal{S}_1, \mathcal{S}_2)$. We shall show that M is also in $(\mathcal{S}_1, \mathcal{S}_2)$.

Since L is in $(\mathcal{S}_1, \mathcal{S}_2)$, there exists a short exact sequence

$$0 \to S \to L \to L/S \to 0$$

of R-modules where S is in $\mathcal{S}_1$ such that L/S is in $\mathcal{S}_2$. We consider the following push out diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & S & = & S & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & L/S & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

of R-modules with exact rows and columns. Next, since N is in $(\mathcal{S}_1, \mathcal{S}_2)$, we have a short exact sequence

$$0 \to T \to N \to N/T \to 0$$

of R-modules where T is in S_1 such that N/T is in $\mathcal{S}_2$. We consider the following pull back diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L/S & \longrightarrow & P' & \longrightarrow & T & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & L/S & \longrightarrow & P & \longrightarrow & N & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & N/T & = & N/T & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

of R-modules with exact rows and columns.

In the first row of the second diagram, since L/S is in $\mathcal{S}_2$ and T is in $\mathcal{S}_1$, P' is in $(\mathcal{S}_2, \mathcal{S}_1)$. Now here, it follows from the assumption (2) that P' is in $(\mathcal{S}_1, \mathcal{S}_2)$. Next, in the middle column of the second diagram, we have the short exact sequence such that P' is in $(\mathcal{S}_1, \mathcal{S}_2)$ and N/T is in $\mathcal{S}_2$. Therefore, it follows from Lemma 6 that P is in $(\mathcal{S}_1, \mathcal{S}_2)$. Finally, in the middle column of the first diagram, there exists the short exact sequence such that S is in $\mathcal{S}_1$ and P is in $(\mathcal{S}_1, \mathcal{S}_2)$. Consequently, we see that M is in $(\mathcal{S}_1, \mathcal{S}_2)$ by Lemma 6.

The proof is completed. □

Corollary 8. *A subcategory* $(\mathcal{S}_{f.g.}, \mathcal{S})$ *is a Serre subcategory for a Serre subcategory* $\mathcal{S}$ *of* R-Mod.

Proof. Let $\mathcal{S}$ be a Serre subcategory of R-Mod. To prove our assertion, it is enough to show that one has $(\mathcal{S}, \mathcal{S}_{f.g.}) \subseteq (\mathcal{S}_{f.g.}, \mathcal{S})$ by Theorem 7. Let M be in $(\mathcal{S}, \mathcal{S}_{f.g.})$. Then there exists a short exact sequence $0 \to Y \to M \to M/Y \to 0$ of R-modules where Y is in $\mathcal{S}$ such that M/Y is in $\mathcal{S}_{f.g.}$. It is easy to see that there exists a finitely generated R-submodule X of M such that $M = X + Y$. Since $X \oplus Y$ is in $(\mathcal{S}_{f.g.}, \mathcal{S})$ and M is a homomorphic image of $X \oplus Y$, M is in $(\mathcal{S}_{f.g.}, \mathcal{S})$ by Proposition 5. □

We note that a subcategory $\mathcal{S}_{Artin}$ consisting of Artinian R-modules is a Serre subcategory which is closed under injective hulls. (Also see [1, Example 2.4].) Therefore we can see that a subcategory $(\mathcal{S}, \mathcal{S}_{Artin})$ is also Serre subcategory for a Serre subcategory of R-Mod by the following assertion.

Corollary 9. *Let* $\mathcal{S}_2$ *be a Serre subcategory of* R-Mod *which is closed under injective hulls. Then a subcategory* $(\mathcal{S}_1, \mathcal{S}_2)$ *is a Serre subcategory for a Serre subcategory* $\mathcal{S}_1$ *of* R-Mod.

Proof. By Theorem 7, it is enough to show that one has $(\mathcal{S}_2, \mathcal{S}_1) \subseteq (\mathcal{S}_1, \mathcal{S}_2)$.

We assume that M is in $(\mathcal{S}_2, \mathcal{S}_1)$ and shall show that M is in $(\mathcal{S}_1, \mathcal{S}_2)$. Then there exists a short exact sequence

$$0 \to Y \to M \to X \to 0$$

of R-modules where X is in $\mathcal{S}_1$ and Y is in $\mathcal{S}_2$. Since $\mathcal{S}_2$ is closed under injective hulls, we note that the injective hull $E_R(Y)$ of Y is also in $\mathcal{S}_2$. We consider a push out diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \longrightarrow & M & \longrightarrow & X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & E_R(Y) & \longrightarrow & T & \longrightarrow & X & \longrightarrow & 0
\end{array}$$

of R-modules with exact rows and injective vertical maps. The second exact sequence splits, and we have an injective homomorphism $M \to X \oplus E_R(Y)$. Since there is a short exact sequence

$$0 \to X \to X \oplus E_R(Y) \to E_R(Y) \to 0$$

of R-modules, the R-module $X \oplus E_R(Y)$ is in $(\mathcal{S}_1, \mathcal{S}_2)$. Consequently, we see that M is also in $(\mathcal{S}_1, \mathcal{S}_2)$ by Proposition 5.

The proof is completed. □

Example 10. Let R be a domain but not a field and let Q be a field of fractions of R. We denote by $\mathcal{S}_{Tor}$ a subcategory consisting of torsion R-modules, that is

$$\mathcal{S}_{Tor} = \{M \in R\text{-Mod} \mid M \otimes_R Q = 0\}.$$

Then we shall see that one has

$$(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subsetneqq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor}) = \{M \in R\text{-Mod} \mid \dim_Q M \otimes_R Q < \infty\}.$$

Therefore, a subcategory $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$ is a Serre subcategory by Corollary 8, but a subcategory $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$ is not closed under extensions by Theorem 7.

First of all, we shall show that the above equality holds. We suppose that M is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. Then there exists a short exact sequence

$$0 \to X \to M \to Y \to 0$$

of R-modules where X is in $\mathcal{S}_{f.g.}$ and Y is in $\mathcal{S}_{Tor}$. We apply an exact functor $- \otimes_R Q$ to this sequence. Then we see that one has $M \otimes_R Q \cong X \otimes_R Q$ and this module is a finite dimensional Q-vector space.

Conversely, let M be an R-module with $\dim_Q M \otimes_R Q < \infty$. Then we can denote $M \otimes_R Q = \sum_{i=1}^{n} Q(m_i \otimes 1_Q)$ with $m_i \in M$ and the unit element 1_Q of Q. We consider a short exact sequence

$$0 \to \sum_{i=1}^{n} Rm_i \to M \to M/\sum_{i=1}^{n} Rm_i \to 0$$

of R-modules. It is clear that $\sum_{i=1}^{n} Rm_i$ is in $\mathcal{S}_{f.g.}$ and $M/\sum_{i=1}^{n} Rm_i$ is in $\mathcal{S}_{Tor}$. So M is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. Consequently, the above equality holds.

Next, it is clear that $M \otimes_R Q$ has finite dimension as Q-vector space for an R-module M of $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$. Thus, one has $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subseteq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$.

Finally, we shall see that a field of fractions Q of R is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$ but not in $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$, so one has $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.}) \subsetneqq (\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. Indeed, it follows from $\dim_Q Q \otimes_R Q = 1$ that Q is in $(\mathcal{S}_{f.g.}, \mathcal{S}_{Tor})$. On the other hand, we assume that Q is in $(\mathcal{S}_{Tor}, \mathcal{S}_{f.g.})$. Since R is a domain, a torsion R-submodule of Q is only the zero module. It means that Q must be a finitely generated R-module. But, this is a contradiction.

References

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