QUIVER PRESENTATIONS OF GROTHENDIECK CONSTRUCTIONS

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ABSTRACT. We give quiver presentations of the Grothendieck constructions of functors from a small category to the 2-category of $\Bbbk$-categories for a commutative ring $\Bbbk$.

Key Words: Grothendieck construction, functors, quivers.

1. Introduction

Throughout this report I is a small category, $\Bbbk$ is a commutative ring, and $\Bbbk$-**Cat** denotes the 2-category of all $\Bbbk$-categories, $\Bbbk$-functors between them and natural transformations between $\Bbbk$-functors.

The Grothendieck construction is a way to form a single category $\mathrm{Gr}(X)$ from a diagram X of small categories indexed by a small category I, which first appeared in [4, §8 of Exposé VI]. As is exposed by Tamaki [7] this construction has been used as a useful tool in homotopy theory (e.g., [8]) or topological combinatorics (e.g., [9]). This can be also regarded as a generalization of orbit category construction from a category with a group action.

In [2] we defined a notion of derived equivalences of (oplax) functors from I to $\Bbbk$-**Cat**, and in [3] we have shown that if (oplax) functors $X, X' \colon I \to \Bbbk\text{-}\mathbf{Cat}$ are derived equivalent, then so are their Grothendieck constructions $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$. An easy example of a derived equivalent pair of functors is given by using diagonal functors: For a category $\mathcal{C}$ define the *diagonal* functor $\Delta(\mathcal{C}) \colon I \to \Bbbk\text{-}\mathbf{Cat}$ to be a functor sending all objects of I to $\mathcal{C}$ and all morphisms in I to the identity functor of $\mathcal{C}$. Then if categories $\mathcal{C}$ and $\mathcal{C}'$ are derived equivalent, then so are their diagonal functors $\Delta(\mathcal{C})$ and $\Delta(\mathcal{C}')$. Therefore, to compute examples of derived equivalent pairs using this result, it will be useful to present Grothendieck constructions of functors by quivers with relations. We already have computations in two special cases. First for a $\Bbbk$-algebra A, which we regard as a $\Bbbk$-category with a single object, we noted in [3] that if I is a semigroup G, a poset S, or the free category $\mathbb{P}Q$ of a quiver Q, then the Grothendieck construction $\mathrm{Gr}(\Delta(A))$ of the diagonal functor $\Delta(A)$ is isomorphic to the semigroup algebra AG, the incidence algebra AS, or the path-algebra AQ, respectively. Second in [1] we gave a quiver presentation of the orbit category $\mathcal{C}/G$ for each $\Bbbk$-category $\mathcal{C}$ with an action of a semigroup G in the case that $\Bbbk$ is a field, which can be seen as a computation of a quiver presentation of the Grothendieck construction $\mathrm{Gr}(X)$ of each functor $X \colon G \to \Bbbk\text{-}\mathbf{Cat}$.

In this report we generalize these two results as follows:

1. We compute the Grothendieck construction $\mathrm{Gr}(\Delta(A))$ of the diagonal functor $\Delta(A)$ for each $\Bbbk$-algebra A and each small category I.

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(2) We give a quiver presentation of the Grothendieck construction $\mathrm{Gr}(X)$ for each functor $X\colon I \to \Bbbk\text{-}\mathbf{Cat}$ and each small category I when $\Bbbk$ is a field.

2. Preliminaries

Throughout this report $Q = (Q_0, Q_1, t, h)$ is a quiver, where $t(\alpha) \in Q_0$ is the *tail* and $h(\alpha) \in Q_0$ is the *head* of each arrow α of Q. For each path μ of Q, the tail and the head of μ is denoted by $t(\mu)$ and $h(\mu)$, respectively. For each non-negative integer n the set of all paths of Q of length at least n is denoted by $Q_{\geq n}$. In particular $Q_{\geq 0}$ denotes the set of all paths of Q.

A category $\mathcal{C}$ is called a $\Bbbk$-*category* if for each $x, y \in \mathcal{C}$, $\mathcal{C}(x, y)$ is a $\Bbbk$-module and the compositions are $\Bbbk$-bilinear.

Definition 1. Let Q be a quiver.

(1) The *free* category $\mathbb{P}Q$ of Q is the category whose underlying quiver is $(Q_0, Q_{\geq 0}, t, h)$ with the usual composition of paths.

(2) The *path* $\Bbbk$-category of Q is the $\Bbbk$-linearization of $\mathbb{P}Q$ and is denoted by $\Bbbk Q$.

Definition 2. Let $\mathcal{C}$ be a category. We set

$$\mathrm{Rel}(\mathcal{C}) := \bigcup_{(i,j)\in\mathcal{C}_0\times\mathcal{C}_0} \mathcal{C}(i,j) \times \mathcal{C}(i,j),$$

elements of which are called *relations* of $\mathcal{C}$. Let $R \subseteq \mathrm{Rel}(\mathcal{C})$. For each $i, j \in \mathcal{C}_0$ we set

$$R(i,j) := R \cap (\mathcal{C}(i,j) \times \mathcal{C}(i,j)).$$

(1) The smallest congruence relation

$$R^c := \bigcup_{(i,j)\in\mathcal{C}_0\times\mathcal{C}_0} \{(dac, dbc) \mid c \in \mathcal{C}(-,i), d \in \mathcal{C}(j,-), (a,b) \in R(i,j)\}$$

containing R is called the *congruence relation* generated by R.

(2) For each $i, j \in \mathcal{C}_0$ we set

$$\begin{aligned}
R^{-1}(i,j) &:= \{(g,f) \in \mathcal{C}(i,j) \times \mathcal{C}(i,j) \mid (f,g) \in R(i,j)\}\\
1_{\mathcal{C}(i,j)} &:= \{(f,f) \mid f \in \mathcal{C}(i,j)\}\\
S(i,j) &:= R(i,j) \cup R^{-1}(i,j) \cup 1_{\mathcal{C}(i,j)}\\
S(i,j)^1 &:= S(i,j)\\
S(i,j)^n &:= \{(h,f) \mid \exists g \in \mathcal{C}(i,j), (g,f) \in S(i,j), (h,g) \in S(i,j)^{n-1}\} \quad (\text{for all } n \geq 2)\\
S(i,j)^\infty &:= \bigcup_{n\geq 1} S(i,j)^n, \text{ and set}
\end{aligned}$$

$$R^e := \bigcup_{(i,j)\in\mathcal{C}_0\times\mathcal{C}_0} S(i,j)^\infty.$$

R^e is called the *equivalence relation* generated by R.

(3) We set $R^{\#} := (R^c)^e$ (cf. [5]).

The following is well known (cf. [6]).

Proposition 3. *Let $\mathcal{C}$ be a category, and $R \subseteq \mathrm{Rel}(\mathcal{C})$. Then the category $\mathcal{C}/R^{\#}$ and the functor $F : \mathcal{C} \to \mathcal{C}/R^{\#}$ defined above satisfy the following conditions.*

(i) *For each $i, j \in \mathcal{C}_0$ and each $(f, f') \in R(i, j)$ we have $Ff = Ff'$.*

(ii) *If a functor $G : \mathcal{C} \to \mathcal{D}$ satisfies $Gf = Gf'$ for all $f, f' \in \mathcal{C}(i, j)$ and all $i, j \in \mathcal{C}_0$ with $(f, f') \in R(i, j)$, then there exists a unique functor $G' : \mathcal{C}/R^{\#} \to \mathcal{D}$ such that $G' \circ F = G$.*

Definition 4. Let Q be a quiver and $R \subseteq \mathrm{Rel}(\mathbb{P}Q)$. We set

$$\langle Q \mid R \rangle := \mathbb{P}Q/R^{\#}.$$

The following is straightforward.

Proposition 5. *Let $\mathcal{C}$ be a category, Q the underlying quiver of $\mathcal{C}$, and set*

$$R := \{(e_i, \mathbb{1}_i), (\mu, [\mu]) \mid i \in Q_0, \mu \in Q_{\geq 2}\} \subseteq \mathrm{Rel}(\mathbb{P}Q),$$

where e_i is the path of length 0 at each vertex $i \in Q_0$, and $[\mu] := \alpha_n \circ \cdots \circ \alpha_1$ (the composite in $\mathcal{C}$) for all paths $\mu = \alpha_n \ldots \alpha_1 \in Q_{\geq 2}$ with $\alpha_1, \ldots, \alpha_n \in Q_1$. Then

$$\mathcal{C} \cong \langle Q \mid R \rangle.$$

By this statement, an arbitrary category is presented by a quiver and relations. Throughout the rest of this report I is a small category with a presentation $I = \langle Q \mid R \rangle$.

3. Grothendieck constructions of Diagonal functors

Definition 6. Let $X : I \to \Bbbk\text{-}\mathbf{Cat}$ be a functor. Then a category $\mathrm{Gr}(X)$, called the *Grothendiek construction* of X, is defined as follows:

(i) $(\mathrm{Gr}(X))_0 := \bigcup_{i \in I_0} \{(i, x) \mid x \in X(i)_0\}$

(ii) For $(i, x), (j, y) \in (\mathrm{Gr}(X))_0$

$$\mathrm{Gr}(X)((i, x), (j, y)) := \bigoplus_{a \in I(i,j)} X(j)(X(a)x, y)$$

(iii) For $f = (f_a)_{a \in I(i,j)} \in \mathrm{Gr}(X)((i, x), (j, y))$ and $g = (g_b)_{b \in I(j,k)} \in \mathrm{Gr}(X)((j, y), (k, z))$

$$g \circ f := \Big(\sum_{\substack{c = ba \\ a \in I(i,j) \\ b \in I(j,k)}} g_b X(b) f_a \Big)_{c \in I(i,k)}$$

Definition 7. Let $\mathcal{C} \in \Bbbk\text{-}\mathbf{Cat}_0$. Then the *diagonal functor* $\Delta(\mathcal{C})$ of $\mathcal{C}$ is a functor from I to $\Bbbk\text{-}\mathbf{Cat}$ sending each arrow $a\colon i \to j$ in I to $\mathbb{1}_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}$ in $\Bbbk\text{-}\mathbf{Cat}$.

In this section, we fix a $\Bbbk$-algebra A which we regard as a $\Bbbk$-category with a single object $*$ and with $A(*, *) = A$. The *quiver algebra* AQ of Q over A is the A-linearization of $\mathbb{P}Q$, namely $AQ := A \otimes_{\Bbbk} \Bbbk Q$.

Theorem 8. *We have an isomorphism $\mathrm{Gr}(\Delta(A)) \cong AQ/\langle R \rangle_A$, where $\langle R \rangle_A$ is the ideal of AQ generated by the elements $g - h$ with $(g, h) \in R$.*

Remark 9. Theorem 8 can be written in the form

$$\mathrm{Gr}(\Delta(A)) \cong A \otimes_{\Bbbk} (\Bbbk Q/\langle R \rangle_{\Bbbk}).$$

4. The quiver presentation of Grothendieck constructions

In this section we give a quiver presentation of the Grothendieck construction of an arbitrary functor $I \to \Bbbk\text{-}\mathbf{Cat}$. Throughout this section we assume that $\Bbbk$ is a field.

Theorem 10. *Let $X : I \to \Bbbk\text{-}\mathbf{Cat}$ be a functor, and for each $i \in I$ set $X(i) = \Bbbk Q^{(i)}/\langle R^{(i)}\rangle$ with $\Phi^{(i)} \colon \Bbbk Q^{(i)} \to X(i)$ the canonical morphism, where $R^{(i)} \subseteq \Bbbk Q^{(i)}$, $\langle R^{(i)}\rangle \cap \{e_x \mid x \in Q(i)_0\} = \emptyset$. Then Grothendieck construction is presented by the quiver with relations (Q, R') defined as follows.*

Quiver: $Q' = (Q'_0, Q'_1, t', h')$, where

(i) $Q'_0 := \bigcup_{i\in I}\{{}_ix \mid x \in Q_0^{(i)}\}$.

(ii) $Q'_1 := \bigcup_{i\in I}\{\{{}_i\alpha \mid \alpha \in Q_1^{(i)}\} \cup \{(a, {}_ix) : {}_ix \to {}_j(ax) \mid a : i \to j \in Q_1, x \in Q_0^{(i)}, ax \neq 0\}\}$,
where we set $ax := X(\overline{a})(x)$.

(iii) *For $\alpha \in Q_1^{(i)}$, $t'({}_i\alpha) = t^{(i)}(\alpha)$ and $h'({}_i\alpha) = h^{(i)}(\alpha)$.*

(iv) *For $a : i \to j \in Q_1, x \in Q_0^{(i)}$, $t'(a, {}_ix) = {}_ix$ and $h'(a, {}_ix) = {}_j(ax)$.*

Relations: $R' := R'_1 \cup R'_2 \cup R'_3$, where

(i) $R'_1 := \{\sigma^{(i)}(\mu) \mid i \in Q_0, \mu \in R^{(i)}\}$,
where we set $\sigma^{(i)} : \Bbbk Q^{(i)} \hookrightarrow \Bbbk Q'$.

(ii) $R'_2 := \{\pi(g, {}_ix) - \pi(h, {}_ix) \mid i, j \in Q_0, (g, h) \in R(i, j), x \in Q_0^{(i)}\}$, *where for each path a in Q we set*

$$\pi(a, {}_ix) := (a_n, {}_{i_{n-1}}(a_{n-1}a_{n-2}\dots a_1x))\dots(a_2, {}_{i_1}(a_1x))(a_1, {}_ix)$$

if $a = a_n \dots a_2a_1$ for some $a_1, \dots, a_n$ arrows in Q, and

$$\pi(a, {}_ix) := e_{{}_ix}$$

if $a = e_i$ for some $i \in Q_0$.

(iii) $R'_3 := \{(a, {}_iy){}_i\alpha - {}_j(a\alpha)(a, {}_ix) \mid a : i \to j \in Q_1, \alpha : x \to y \in Q_1^{(i)}\}$, *where we take $a\alpha : ax \to ay$ so that $\Phi^{(j)}(a\alpha) \in X(\overline{a})\Phi^{(i)}(\alpha)$:*

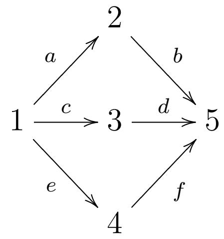
$$\begin{array}{ccc} \alpha \in \Bbbk Q^{(i)} & \xrightarrow{\Phi^{(i)}} & X(i) \\ & & \downarrow{\scriptstyle X(\overline{a})} \\ a\alpha \in \Bbbk Q^{(j)} & \xrightarrow{\Phi^{(j)}} & X(j). \end{array}$$

Note that the ideal $\langle R'\rangle$ is independent of the choice of $a\alpha$ because $R'_1 \subseteq R'$.

5. Examples

In this section, we illustrate Theorems 8 and 10 by some examples.

Example 11. Let Q be the quiver

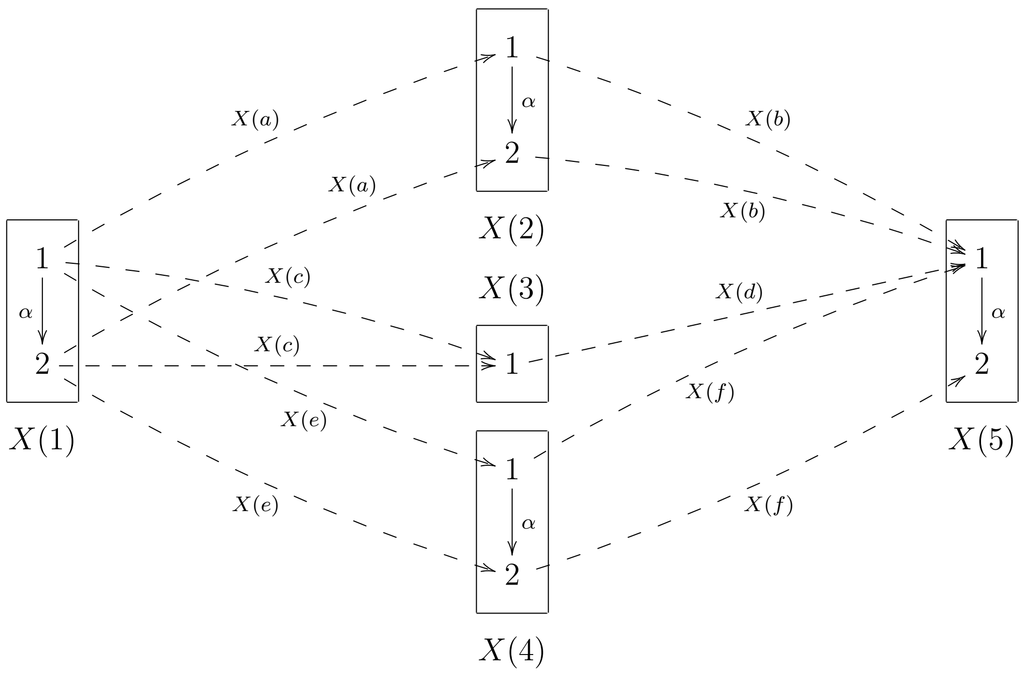


and let $R = \{(ba, dc)\}$. Then the category $I := \langle Q \mid R\rangle$ is not given as a semigroup, as a poset or as the free category of a quiver. For any algebra A consider the diagonal functor $\Delta(A)\colon I \to \Bbbk\text{-}\mathbf{Cat}$. Then by Theorem 8 the category $\mathrm{Gr}(\Delta(A))$ is given by

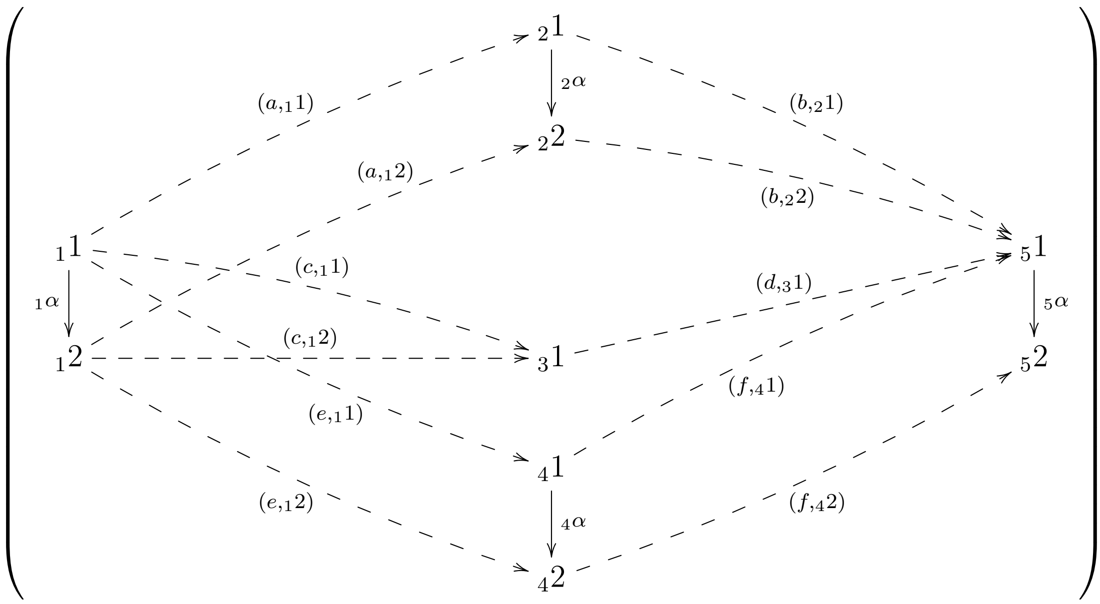
$$AQ/\langle ba - dc\rangle.$$

Remark 12. Let Q and Q' be quivers having neither double arrows nor loops, and let $f\colon Q_0 \to Q'_0$ be a map (a *vertex map* between Q and Q'). If $Q(x, y) \neq \emptyset$ $(x, y \in Q_0)$ implies $Q'(f(x), f(y)) \neq \emptyset$ or $f(x) = f(y)$, then f induces a $\Bbbk$-functor $\hat{f}\colon \Bbbk P \to \Bbbk P'$ defined by the following correspondence: For each $x \in Q_0$, $\hat{f}(e_x) := e_{f(x)}$, and for each arrow $a\colon x \to y$ in Q, $f(a)$ is the unique arrow $f(x) \to f(y)$ (resp. $e_{f(x)}$) if $f(x) \neq f(y)$ (resp. if $f(x) = f(y)$).

Example 13. Let $I = \langle Q \mid R\rangle$ be as in the previous example. Define a functor X : $I \to \Bbbk\text{-}\mathbf{Cat}$ by the $\Bbbk$-linearizations of the following quivers in frames and the $\Bbbk$-functors induced by the vertex maps expressed by broken arrows between them:



Then by Theorem 10 Gr(X) is presented by the quiver

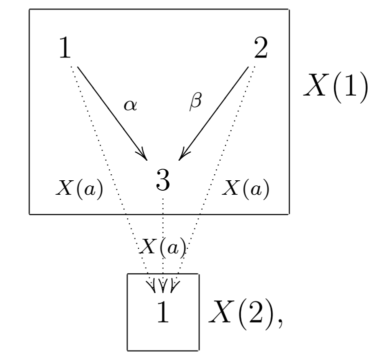
$Q' =$ 

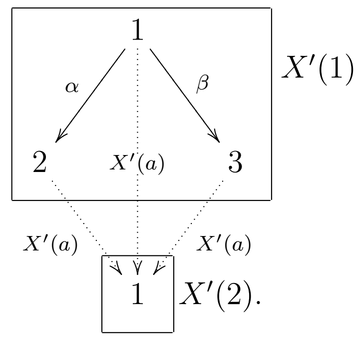
with relations

$$\begin{aligned} R' &= \{\pi(ba,{}_11) - \pi(dc,{}_11), \pi(ba,{}_12) - \pi(dc,{}_12)\} \\ &\quad \cup \{(a,{}_iy)_i\alpha - {}_j(a\alpha)(a,{}_ix) \mid a : i \to j \in Q_1, \alpha : x \to y \in Q_1^{(i)}\}, \end{aligned}$$

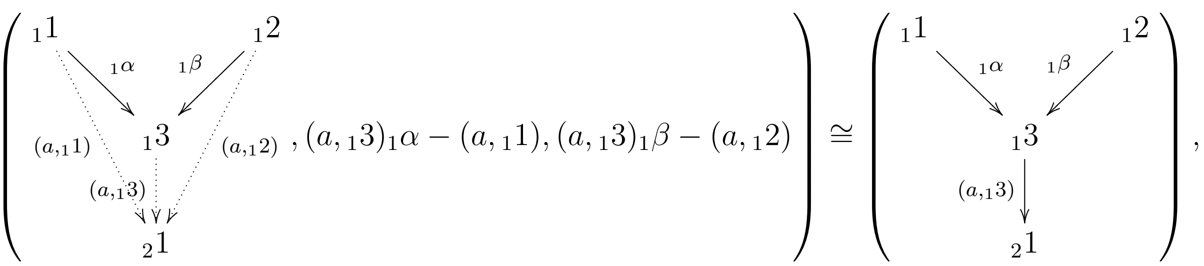
where the new arrows are presented by broken arrows.

Example 14. Let $Q = (\ 1 \xrightarrow{a} 2\ )$ and $I := \langle Q \rangle$. Define functors $X, X' \colon I \to \Bbbk\text{-}\mathbf{Cat}$ by the $\Bbbk$-linearizations of the following quivers in frames and the $\Bbbk$-functors induced by the vertex maps expressed by dotted arrows between them:

X : 

X' : 

Then by Theorem 10 Gr(X) is given by the following quiver with no relations



and $\mathrm{Gr}(X')$ is given by the following quiver with a commutativity relation

$$\left(\begin{array}{ccc} & {}_1 1 & \\ {}_1\alpha \swarrow & \vdots & \searrow {}_1\beta \\ {}_1 2 & (a,{}_1 1) & {}_1 3 \\ (a,{}_1 2) \searrow & \downarrow & \swarrow (a,{}_1 3) \\ & {}_2 1 & \end{array} , (a,{}_1 2){}_1\alpha - (a,{}_1 1), (a,{}_1 3){}_1\beta - (a,{}_1 1) \right) \cong \left(\begin{array}{ccc} & {}_1 1 & \\ {}_1\alpha \swarrow & & \searrow {}_1\beta \\ {}_1 2 & \circlearrowright & {}_1 3 \\ (a,{}_1 2) \searrow & & \swarrow (a,{}_1 3) \\ & {}_2 1 & \end{array} \right).$$

By using the main theorem in [3] derived equivalences between $X(1)$ and $X'(1)$ and between $X(2)$ and $X'(2)$ are glued together to have a derived equivalence between $\mathrm{Gr}(X)$ and $\mathrm{Gr}(X')$.

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