EXAMPLE OF CATEGORIFICATION OF A CLUSTER ALGEBRA

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Abstract. We present here two detailed examples of additive categorifications of the cluster algebra structure of a coordinate ring of a maximal unipotent subgroup of a simple Lie group. The first one is of simply-laced type (A_3) and relies on an article by Geiß, Leclerc and Schröer. The second is of non simply-laced type (C_2) and relies on an article by the author of this note. This is aimed to be accessible, specially for people who are not familiar with this subject.

1. Introduction: the total positivity problem

Let N be the subgroup of $\mathrm{SL}_4(\mathbb{C})$ consisting of upper triangular matrices with diagonal 1. We say that $X \in N$ is *totally positive* if its 12 non-trivial minors are positive real numbers (a minor is non-trivial if it is not constant on N and not product of other minors). As a consequence of various results of Fomin and Zelevinsky [3] (see also [1]), in a (very) special case, we get

Proposition 1 (Fomin-Zelevinsky). *$X \in N$ is totally positive if and only if the minors $\Delta_4^1(X)$, $\Delta_{34}^{12}(X)$, $\Delta_{234}^{123}(X)$, $\Delta_{24}^{12}(X)$, $\Delta_4^2(X)$, $\Delta_4^3(X)$ are positive.*

where $\Delta_{c_1 \dots c_k}^{\ell_1 \dots \ell_k}(X)$ is the minor of X with rows $\ell_1, \dots, \ell_k$ and columns $c_1, \dots, c_k$.

Remark that, as the algebraic variety N has dimension 6, we can not expect to find a criterion with less than 6 inequalities to check the total positivity of a matrix.

To prove this, just remark that we have the following equality:

$$\Delta_{24}^{12}\Delta_{34}^{23} = \Delta_{234}^{123}\Delta_4^2 + \Delta_4^3\Delta_{34}^{12}$$

which immediately implies that $\Delta_4^1(X)$, $\Delta_{34}^{12}(X)$, $\Delta_{234}^{123}(X)$, $\Delta_{24}^{12}(X)$, $\Delta_4^2(X)$, $\Delta_4^3(X)$ are positive if and only if $\Delta_4^1(X)$, $\Delta_{34}^{12}(X)$, $\Delta_{234}^{123}(X)$, $\Delta_{34}^{23}(X)$, $\Delta_4^2(X)$, $\Delta_4^3(X)$ are positive. Such an equality is called an *exchange identity*. In Figure 1, we wrote 14 sets of minors which are related by exchange identities whenever they are linked by an edge. As every minor appears in this graph, it induces the previous proposition.

These observations lead to the definition of a *cluster algebra* [4]. A cluster algebra is an algebra endowed with an additional combinatorial structure. Namely, a (generally infinite) set of distinguished elements called *cluster variables* grouped into subsets of the same cardinality n, called *clusters* and a finite set $\{x_{n+1}, x_{n+2}, \dots, x_m\}$ called the set of *coefficients*. For each cluster $\{x_1, x_2, \dots, x_n\}$, the *extended cluster* $\{x_1, \dots, x_n, x_{n+1}, \dots, x_m\}$ is a transcendence basis of the algebra. Moreover, each cluster $\{x_1, x_2, \dots, x_n\}$ has n

The paper is in a final form and no version of it will be submitted for publication elsewhere.

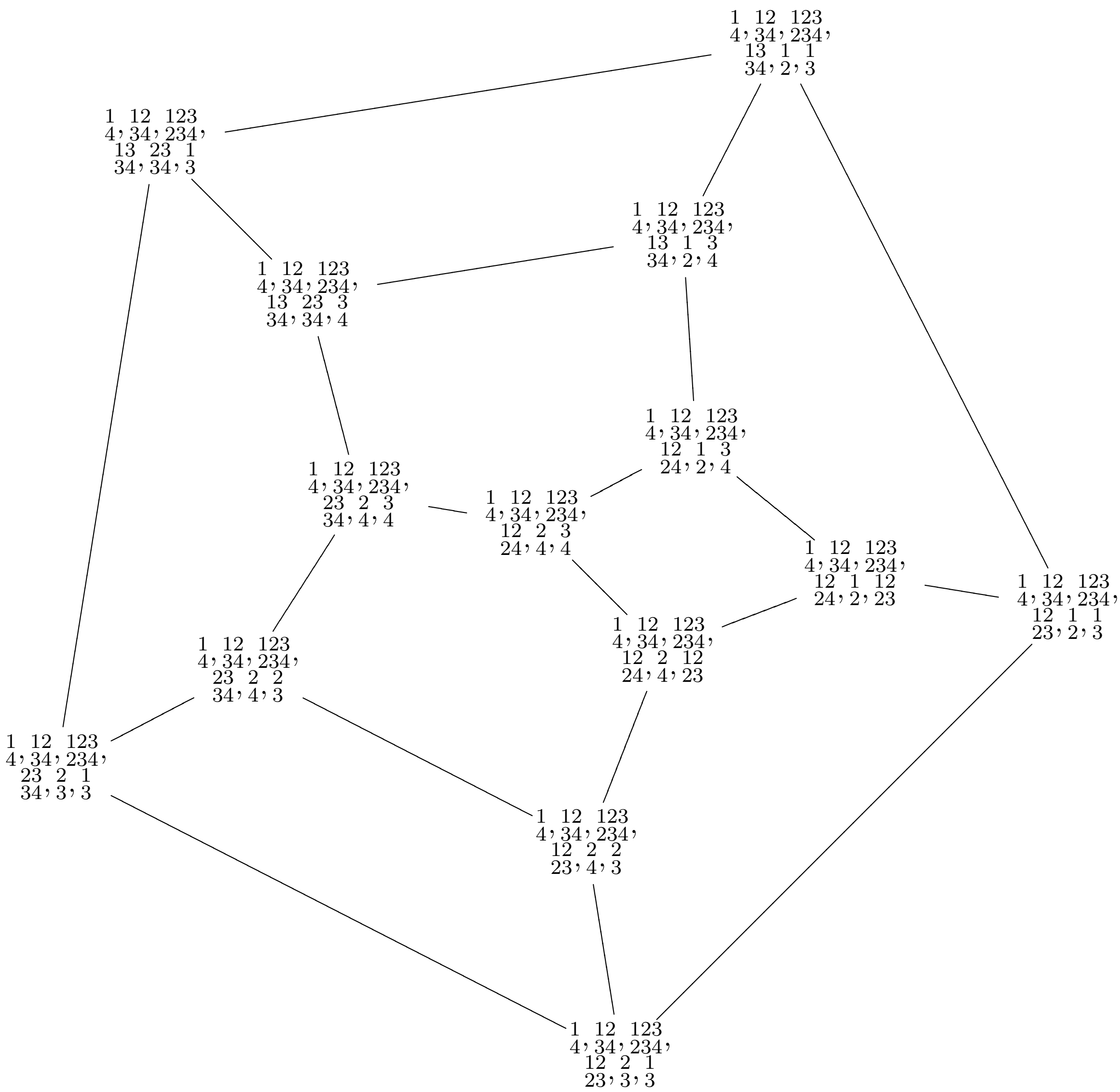


Figure 1. Exchange graph of minors

neighbours obtained by replacing one of its elements x_k by a new one x'_k related by a relation

$$x_k x'_k = M_1 + M_2$$

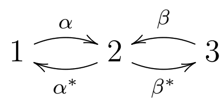
where M_1 and M_2 are mutually prime monomials in $\{x_1, \dots, x_{k-1}, x_{k+1}, \dots, x_m\}$, given by precise combinatorial rules. These replacements, called *mutations* and denoted by μ_k are involutive. For precise definitions and details about these constructions, we refer to [4].

In the previous example, the coefficients are Δ_4^1, Δ_{34}^{12} and Δ_{234}^{123} and the cluster variables are all the other non-trivial minors. The extended clusters are the sets appearing at the vertices of Figure 1.

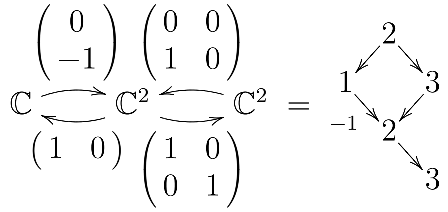
The aim of the following sections is to describe examples of *additive categorifications* of cluster algebras. It consists of enhancing the cluster algebra structure with an additive category, some objects of which reflect the combinatorial structure of the cluster algebra; moreover, there is an explicit formula, the *cluster character* associating to these particular objects elements of the algebra, in a way which is compatible with the combinatorial structure. The examples we develop here rely on (abelian) module categories. They are particular cases of categorifications by exact categories appearing in [6] (simply-laced case) and [2] (non simply-laced case). The study of cluster algebras and their categorifications has been particularly successful these last years. For a survey on categorification by triangulated categories and a much more complete bibliography, see [7].

2. The preprojective algebra and the cluster character

Let Q be the following quiver (oriented graph):



As usual, denote by $\mathbb{C}Q$ the $\mathbb{C}$-algebra, a basis of which is formed by the paths (including 0-length paths supported by each of the three vertices) and the multiplication of which is defined by concatenation of paths when it is possible and vanishes when paths can not be composed (we write here the composition from left to right, on the contrary to the usual composition of maps). Thus, a (right) $\mathbb{C}Q$-module is naturally graded by idempotents (0-length paths) corresponding to vertices and the action of arrows seen as elements of the algebra can naturally be identified with linear maps between the corresponding homogeneous subspaces of the representation. We shall use the following right-hand side convenient notation:



where each of the digits represents a basis vector of the representation and each arrow a non-zero scalar (1 when not specified) in the corresponding matrix entry.

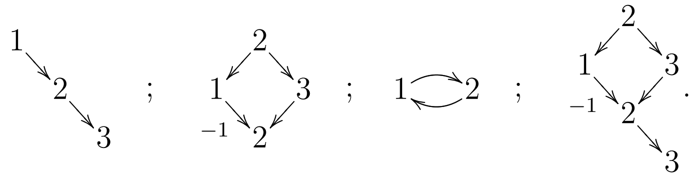
Let us now introduce the preprojective algebra of Q:

Definition 2. The *preprojective algebra* of Q is defined by

$$\Pi_Q = \frac{\mathbb{C}Q}{(\alpha\alpha^*, \alpha^*\alpha + \beta^*\beta, \beta\beta^*)}$$

the representations of which are seen as particular representations of $\mathbb{C}Q$ (in other words, $\operatorname{mod}\Pi_Q$ is a full subcategory of $\operatorname{mod}\mathbb{C}Q$).

Example 3. Among the following representations of $\mathbb{C}Q$, the first one and the second one are representations of Π_Q:



One of the property, which is discussed in many places (for example in [6]), of the preprojective algebra of Q, fundamental for this categorification, is

Proposition 4. *The category* $\operatorname{mod}\Pi_Q$ *is* stably 2-Calabi-Yau. *In other words, for every* $X, Y \in \operatorname{mod}\Pi_Q$,

$$\operatorname{Ext}^1(X,Y) \simeq \operatorname{Ext}^1(Y,X)^*$$

functorially in X *and* Y*, where* $\operatorname{Ext}^1(Y,X)^*$ *is the* $\mathbb{C}$*-dual of* $\operatorname{Ext}^1(Y,X)$*. In particular, it is a Frobenius category (is has enough projective objects and enough injective objects and they coincide).*

Let us now define the three following one-parameter subgroups of N:

$$x_1(t) = \begin{pmatrix} 1 & t & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_2(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & t & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad x_3(t) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

For $X \in \operatorname{mod}\Pi_Q$ and any sequence of vertices $a_1, a_2, \ldots, a_n$ of Q, we denote by

$$\Phi_{X,a_1a_2\ldots a_n} = \left\{ 0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X \mid \forall i \in \{1,2,\ldots,n\}, \frac{X_i}{X_{i-1}} \simeq S_{a_i} \right\}$$

the *variety of composition series* of X of type $a_1a_2\ldots a_n$ (S_{a_i} is the simple module, of dimension 1, supported at vertex a_i). This is a closed algebraic subvariety of the product of Grassmannians

$$\operatorname{Gr}_1(X) \times \operatorname{Gr}_2(X) \times \cdots \times \operatorname{Gr}_n(X).$$

We denote by χ the Euler characteristic. Using results of Lusztig and Kashiwara-Saito, Geiß-Leclerc-Schroër proved the following result:

Theorem 5 ([6])**.** *Let* $X \in \operatorname{mod}\Pi_Q$*. There is a unique* $\varphi_X \in \mathbb{C}[N]$ *such that*

$$\varphi_X\left(x_{a_1}(t_1)x_{a_2}(t_2)\ldots x_{a_6}(t_6)\right) = \sum_{i_1,i_2,\ldots,i_6 \in \mathbb{N}} \chi\left(\Phi_{X,a_1^{i_1}a_2^{i_2}\ldots a_6^{i_6}}\right) \frac{t_1^{i_1}t_2^{i_2}\ldots t_6^{i_6}}{i_1!i_2!\ldots i_6!}$$

for every word $a_1a_2a_3a_4a_5a_6$ *representing the longest element of* $\mathfrak{S}_4$ *(*$a_k^{i_k}$ *is the repetition* i_k *times of* a_k*).*

The map $\varphi : \operatorname{mod}\Pi_Q \to \mathbb{C}[N]$ is called a *cluster character.*

Remark 6. (1) The uniqueness in the previous theorem is easy because it is well known that

$$x_{a_1}(t_1)x_{a_2}(t_2)\ldots x_{a_6}(t_6)$$

runs over a dense subset of N ;

$X \in \operatorname{mod} \Pi_Q$	S_1	S_2	S_3	$\begin{smallmatrix} 1 & & \\ & \searrow & \\ & & 2 \end{smallmatrix}$	$\begin{smallmatrix} & & 2 \\ & \swarrow & \\ 1 & & \end{smallmatrix}$	$\begin{smallmatrix} 2 & & \\ & \searrow & \\ & & 3 \end{smallmatrix}$	$\begin{smallmatrix} & & 3 \\ & \swarrow & \\ 2 & & \end{smallmatrix}$
$\varphi_X \in \mathbb{C}[N]$	Δ_2^1	Δ_3^2	Δ_4^3	Δ_{23}^{12}	Δ_3^1	Δ_{34}^{23}	Δ_4^2

$X \in \operatorname{mod} \Pi_Q$	$\begin{smallmatrix} & & 2 & & \\ & \swarrow & & \searrow & \\ 1 & & & & 3 \end{smallmatrix}$	$\begin{smallmatrix} 1 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 2 & & \end{smallmatrix}$	$\begin{smallmatrix} 1[dr] & & \\ 2 & & \\ & \searrow & \\ & & 3 \end{smallmatrix}$	$\begin{smallmatrix} & & 2 & & \\ & \swarrow & & \searrow & \\ 1 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 2 & & \end{smallmatrix}$	$\begin{smallmatrix} & & & & 3 \\ & & & \swarrow & \\ & & 2 & & \\ & \swarrow & & & \\ 1 & & & & \end{smallmatrix}$
$\varphi_X \in \mathbb{C}[N]$	Δ_{34}^{13}	Δ_{24}^{12}	Δ_{234}^{123}	Δ_{34}^{12}	Δ_4^1

FIGURE 2. Cluster character

(2) the existence is much harder and strongly relies on the construction of semi-canonical bases by Lusztig [8]. In particular, the fact that it does not depend on the choice of $a_1a_2a_3a_4a_5a_6$ is not clear *a priori* (see the following examples).

Example 7. We suppose that $a_1a_2a_3a_4a_5a_6 = 213213$. Then

$$x_{a_1}(t_1)x_{a_2}(t_2)x_{a_3}(t_3)x_{a_4}(t_4)x_{a_5}(t_5)x_{a_6}(t_6) = \begin{pmatrix} 1 & t_2+t_5 & t_2t_4 & t_2t_4t_6 \\ 0 & 1 & t_1+t_4 & t_1t_3+t_1t_6+t_4t_6 \\ 0 & 0 & 1 & t_3+t_6 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- The module S_1 has only one composition series, of type 1. Therefore $\Phi_1(S_1)$ is one point and $\Phi_{\mathbf{a}}(S_1) = \emptyset$ for any other $\mathbf{a}$. Identifying the two members in the formula of the previous theorem,

$$\varphi_{S_1}\left(x_{a_1}(t_1)x_{a_2}(t_2)x_{a_3}(t_3)x_{a_4}(t_4)x_{a_5}(t_5)x_{a_6}(t_6)\right) = t_2 + t_5 = \Delta_2^1.$$

- The module

$$P_2 = \begin{smallmatrix} & & 2 & & \\ & \swarrow & & \searrow & \\ 1 & & & & 3 \\ & \searrow & & \swarrow & \\ & & 2 & & \end{smallmatrix}$$

has two composition series, of type 2312 and 2132. Therefore,

$$\varphi_{P_2}\left(x_{a_1}(t_1)x_{a_2}(t_2)x_{a_3}(t_3)x_{a_4}(t_4)x_{a_5}(t_5)x_{a_6}(t_6)\right) = t_1t_2t_3t_4 = \Delta_{34}^{12}.$$

Remark that, in this case, the only composition series which is playing a role is 2132, even if the situation is symmetric. This justify the second part of the previous remark.

The other indecomposable representations of Π_Q and their cluster character values are collected in Figure 2.

Two important properties of this cluster character were proved by Geiß-Leclerc-Schroër (see for example [6]):

Proposition 8. *Let $X, Y \in \operatorname{mod} \Pi_Q$.*

(1) $\varphi_{X \oplus Y} = \varphi_X \varphi_Y$.

(2) *Suppose that* $\dim \mathrm{Ext}^1(X,Y) = 1$ *(and therefore* $\dim \mathrm{Ext}^1(Y,X) = 1$*) and let*

$$0 \to X \to T_a \to Y \to 0 \quad \text{and} \quad 0 \to Y \to T_b \to X \to 0$$

be two (unique up to isomorphism) non-split short exact sequences. Then

$$\varphi_X \varphi_Y = \varphi_{T_a} + \varphi_{T_b}.$$

3. Minimal approximations

This section recall the definition and elementary properties of approximations. It is there for the sake of ease. In what follows, $\mathrm{mod}\,\Pi_Q$ can be replaced by any additive Hom-finite category over a field.

Definition 9. Let X and T be two objects of $\mathrm{mod}\,\Pi_Q$. A *left* $\mathrm{add}(T)$*-approximation* of X is a morphism $f : X \to T'$ such that

- $T' \in \mathrm{add}(T)$ (which means that every indecomposable summand of T' is an indecomposable summand of T) ;
- every morphism $g : X \to T$ factors through f.

If, moreover, there is no strict direct summand T'' of T' and left *add*(T)-approximation $f' : X \to T''$, then f is said to be a *minimal left* $\mathrm{add}(T)$*-approximation.*

In the same way, we can define

Definition 10. Let X and T be two objects in $\mathrm{mod}\,\Pi_Q$. A *right* $\mathrm{add}(T)$*-approximation* of X is a morphism $f : T' \to X$ such that

- $T' \in \mathrm{add}(T)$;
- every morphism $g : T \to X$ factors through f.

If, moreover, there is no strict direct summand T'' of T' and right *add*(T)-approximation $f' : T'' \to X$, then f is said to be a *minimal right* $\mathrm{add}(T)$*-approximation.*

Now, a classical proposition which permits to explicitly compute approximations:

Proposition 11. *Let* X *and* $T \simeq T_1^{i_1} \oplus T_2^{i_2} \oplus \cdots \oplus T_n^{i_n}$ *be two objects in* $\mathrm{mod}\,\Pi_Q$ *(the* T_i*'s are non-isomorphic indecomposable). For* $i, j \in \{1, \ldots, n\}$*, we denote by* I_{ij} *the subvector space of* $\mathrm{Hom}(T_i, T_j)$ *consisting of the non-invertible morphisms (*$I_{ij} = \mathrm{Hom}(T_i, T_j)$ *if* $i \neq j$*). Thus, for* $j \in \{1, \ldots, n\}$*, we obtain a linear map*

$$\bigoplus_{i \in \{1,\ldots,n\}} I_{ij} \otimes \mathrm{Hom}(X, T_i) \xrightarrow{\varphi_j} \mathrm{Hom}(X, T_j)$$

$$(g, f) \mapsto g \circ f.$$

Let $\mathcal{B}_j$ *be a basis of* $\mathrm{coker}\,\varphi_j$ *lifted to* $\mathrm{Hom}(X, T_j)$*. Then the morphism*

$$X \xrightarrow{(f)_{j \in \{1,\ldots,n\}, f \in \mathcal{B}_j}} \bigoplus_{j \in \{1,\ldots,n\}} T_j^{\#\mathcal{B}_j}$$

is a minimal left $\mathrm{add}(T)$*-approximation of* X*. Moreover, any minimal left* $\mathrm{add}(T)$*-approximation of* X *is isomorphic to it.*

The previous proposition has a dual version which permits to compute minimal right approximations. In practice, this computation relies on searching morphisms up to factorization through other objects. There is an explicit example of computation in Example 19.

4. Maximal rigid objects and their mutations

Let us introduce the objects the combinatorics of which will play the role of the cluster algebra structure.

Definition 12. Let $X \in \operatorname{mod} \Pi_Q$.

- The module X is said to be *rigid* if it has no self-extension, (*i.e.*, $\operatorname{Ext}^1(X, X) = 0$).
- The module X is said to be *basic maximal rigid* if it is basic (*i.e.*, it does not have two isomorphic indecomposable summands), rigid, and maximal for these two properties.

Remark 13. A basic maximal rigid Π_Q-module contains Π_Q as a direct summand (because Π_Q is both projective and injective and therefore has no extension with any module).

Example 14. The object

$$\begin{smallmatrix}1 & & 3\\ & \searrow\ \swarrow & \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} & 3\\ \swarrow & \\ 2 & \end{smallmatrix} \oplus \begin{smallmatrix}1 & \\ & \searrow \\ & 2\end{smallmatrix} \oplus \begin{smallmatrix} 1[dr] & \\ 2 & \\ \searrow & \\ & 3\end{smallmatrix} \oplus \begin{smallmatrix} & 2 & \\ \swarrow & & \searrow \\ 1 & & 3 \\ \searrow & & \swarrow \\ & 2 & \end{smallmatrix} \oplus \begin{smallmatrix} & & 3\\ & \swarrow & \\ 2 & & \\ \swarrow & & \\ 1 & & \end{smallmatrix},$$

the last three summands of which are the indecomposable projective-injective Π_Q-modules, is basic maximal rigid. It is easy to check that it is basic and rigid, but more difficult to prove that it is maximal for these properties (see [6] for more details).

Remark 15. We can prove that all basic maximal rigid objects have the same number of indecomposable summands (six in the example we are talking about).

The following result permits to define a mutation on basic maximal rigid objects. Considered as an operation on isomorphism classes of basic maximal rigid objects, the induced combinatorial structure will correspond to the one of a cluster algebra.

Theorem 16 ([6]). *Let $T \simeq T_1 \oplus T_2 \oplus T_3 \oplus P_1 \oplus P_2 \oplus P_3 \in \operatorname{mod} \Pi_Q$ be basic maximal rigid such that P_1, P_2 and P_3 are the indecomposable projective Π_Q-modules and T_1, T_2 and T_3 are indecomposable non-projective Π_Q-modules. Then, for $i \in \{1, 2, 3\}$, there are two (unique) short exact sequences*

$$0 \to T_i \xrightarrow{f} T_a \xrightarrow{f'} T_i^* \to 0 \quad \textit{and} \quad 0 \to T_i^* \xrightarrow{g} T_b \xrightarrow{g'} T_i \to 0$$

such that

1. *f and g are minimal left $\operatorname{add}(T/T_i)$-approximations ;*
2. *f' and g' are minimal right $\operatorname{add}(T/T_i)$-approximations ;*
3. *T_i^* is indecomposable and non-projective ;*
4. *$\dim \operatorname{Ext}^1(T_i, T_i^*) = \dim \operatorname{Ext}^1(T_i^*, T_i) = 1$ and the two short exact sequences do not split ;*
5. *$\mu_i(T) = T/T_i \oplus T_i^*$ is basic maximal rigid ;*

(6) *T_a and T_b do not have common summands.*

Remark 17. In the previous theorem, the existence and uniqueness, regarding the first two conditions, are automatic, except the fact that the extremities of the two short exact sequences coincide up to order. This fact strongly relies on the stably 2-Calabi-Yau property. It implies that μ_i is involutive.

Definition 18. In the previous theorem, μ_i is called the *mutation in direction i*. The short exact sequences appearing are called *exchange sequences.*

Example 19. Let

$$T = {}^{1}\searrow_{2}\swarrow^{3} \oplus {}_{2}\swarrow^{3} \oplus {}^{1}\searrow_{2} \oplus \begin{matrix}1[dr] & & \\ & 2 & \\ & & \searrow 3\end{matrix} \oplus \begin{matrix} & 2 & \\ \swarrow & & \searrow \\ 1 & & 3 \\ \searrow & & \swarrow \\ & 2 & \end{matrix} \oplus \begin{matrix} & & 3 \\ & 2 \swarrow & \\ 1 \swarrow & & \end{matrix}.$$

Using Proposition 11, we get a left add $\left(T/{}_{2}\swarrow^{3}\right)$-approximation of ${}_{2}\swarrow^{3}$:

$${}_{2}\swarrow^{3} \to {}^{1}\searrow_{2}\swarrow^{3}$$

and computing the cokernel, we get the exchange sequence:

$$0 \to {}_{2}\swarrow^{3} \to {}^{1}\searrow_{2}\swarrow^{3} \to S_1 \to 0$$

so that

$$\mu_2(T) = {}^{1}\searrow_{2}\swarrow^{3} \oplus S_1 \oplus {}^{1}\searrow_{2} \oplus \begin{matrix}1[dr] & & \\ & 2 & \\ & & \searrow 3\end{matrix} \oplus \begin{matrix} & 2 & \\ \swarrow & & \searrow \\ 1 & & 3 \\ \searrow & & \swarrow \\ & 2 & \end{matrix} \oplus \begin{matrix} & & 3 \\ & 2 \swarrow & \\ 1 \swarrow & & \end{matrix}.$$

Doing mutation in the reverse direction:

$$0 \to S_1 \to \begin{matrix} & & 3 \\ & 2 \swarrow & \\ 1 \swarrow & & \end{matrix} \to {}_{2}\swarrow^{3} \to 0.$$

Let us now compute $\mu_1\mu_2(T)$ with its two exchange sequences:

$$0 \to {}^{1}\searrow_{2}\swarrow^{3} \to S_1 \oplus \begin{matrix} & 2 & \\ \swarrow & & \searrow \\ 1 & & 3 \\ \searrow & & \swarrow \\ & 2 & \end{matrix} \to {}_{1}\swarrow^{2} \to 0$$

$$0 \to {}_{1}\swarrow^{2} \to {}^{1}\searrow_{2} \oplus \begin{matrix} & & 3 \\ & 2 \swarrow & \\ 1 \swarrow & & \end{matrix} \to {}^{1}\searrow_{2}\swarrow^{3} \to 0$$

$$\mu_1\mu_2(T) = {}_{1}\swarrow^{2} \oplus S_1 \oplus {}^{1}\searrow_{2} \oplus \begin{matrix}1[dr] & & \\ & 2 & \\ & & \searrow 3\end{matrix} \oplus \begin{matrix} & 2 & \\ \swarrow & & \searrow \\ 1 & & 3 \\ \searrow & & \swarrow \\ & 2 & \end{matrix} \oplus \begin{matrix} & & 3 \\ & 2 \swarrow & \\ 1 \swarrow & & \end{matrix}.$$

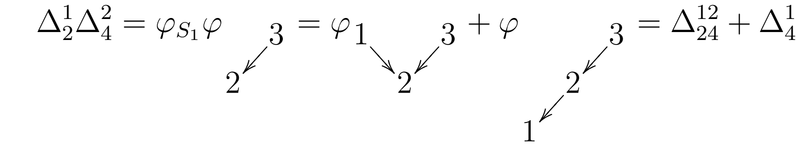
Computing inductively all the mutations, we obtain the *exchange graph of maximal rigid objects of* Π_Q (Figure 3).

Then, using Proposition 8 and Theorem 16 together with other technical results, we get the following proposition:

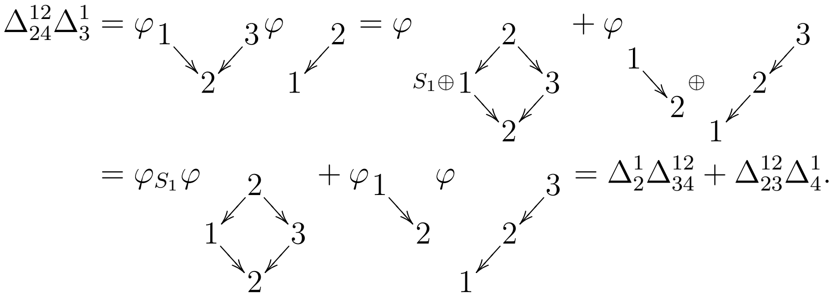
Proposition 20 ([6])**.** *If we project the mutation of maximal rigid objects to* $\mathbb{C}[N]$ *through the cluster character* φ*, we get a cluster algebra structure on* $\mathbb{C}[N]$ *(in the sense of* [4]*). Moreover, this structure is the one proposed combinatorially in* [1]*. Under this projection, we get the correspondence:*

$$\{\textit{non projective indecomposable objects}\} \leftrightarrow \{\textit{cluster variables}\}$$
$$\{\textit{projective indecomposable objects}\} \leftrightarrow \{\textit{coefficients}\}$$
$$\{\textit{basic maximal rigid objects}\} \leftrightarrow \{\textit{extended clusters}\}$$

Example 21. Taking the notation of Example 19 and looking at Figure 2, we get:



and



which can be easily checked by hand. These are part of the equalities which appear in the proof of Proposition 1.

5. From simply-laced case to general one

Define the following symplectic form:

$$\Psi = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

and the subgroup

$$N' = \{M \in N | {}^t M \Psi M = \Psi\} \quad \text{or, equivalently} \quad N' = N^{\mathbb{Z}/2\mathbb{Z}}$$

where $\mathbb{Z}/2\mathbb{Z} = \langle g \rangle$ acts on N by $M \mapsto \Psi^{-1} ({}^t M^{-1}) \Psi$. The group N' is a maximal unipotent subgroup of a symplectic group of type C_2.

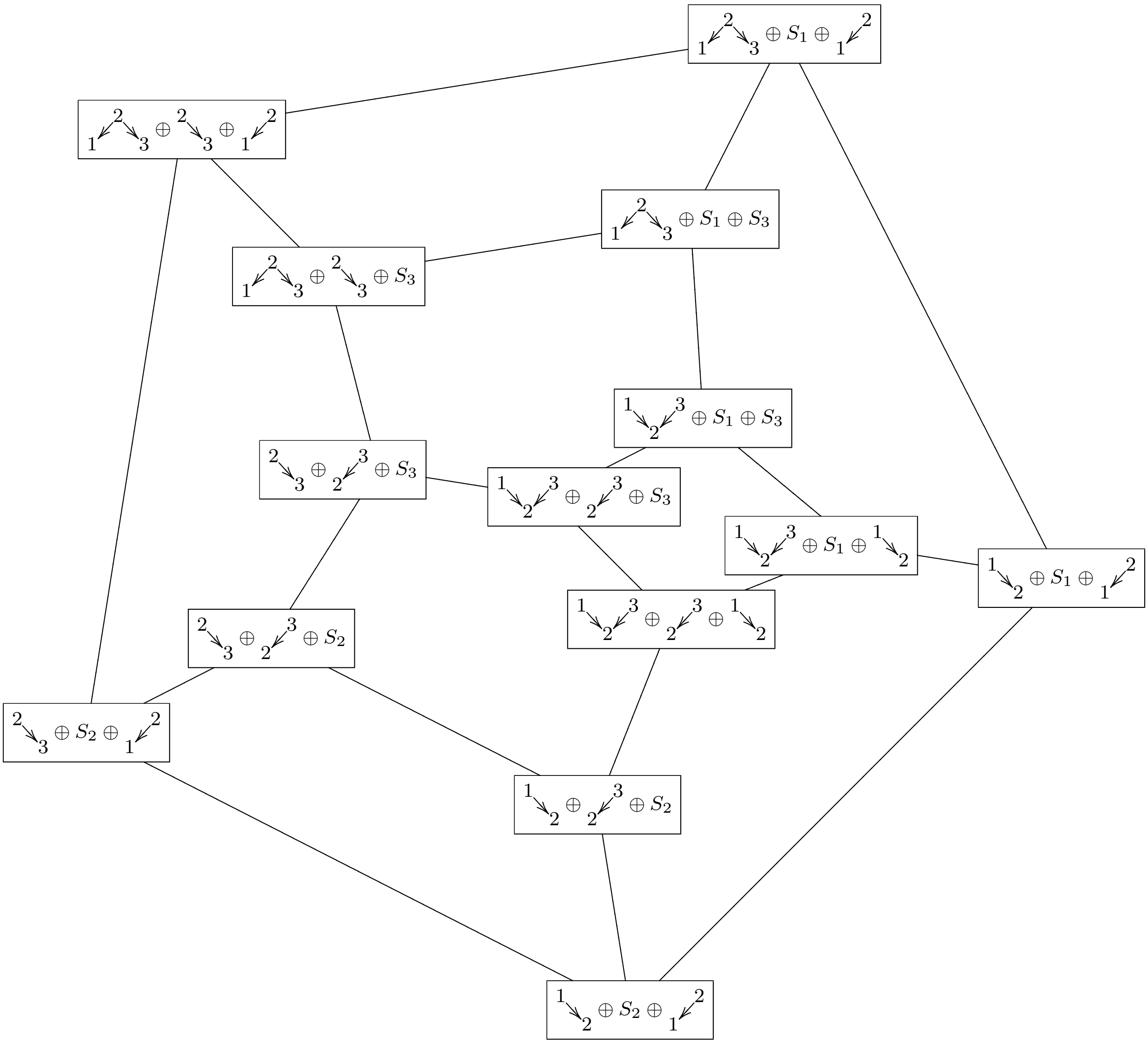


FIGURE 3. Exchange graph of maximal rigid objects (up to projective summands)

The only non-trivial action of $\mathbb{Z}/2\mathbb{Z}$ on Q induces an action on Π_Q and therefore on $\operatorname{mod}\Pi_Q$. Denote by $\pi : \mathbb{C}[N] \to \mathbb{C}[N']$ the canonical projection. We can now formulate the following result:

Theorem 22 ([2])**.** (1) *If T is a $\mathbb{Z}/2\mathbb{Z}$-stable basic maximal rigid Π_Q-module, then $\mu_1\mu_3(T) = \mu_3\mu_1(T)$. Moreover, $\mu_1\mu_3(T)$ and $\mu_2(T)$ are also $\mathbb{Z}/2\mathbb{Z}$-stable.*

(2) *If $X \in \operatorname{mod}\Pi_Q$, then $\pi(\varphi_X) = \pi(\varphi_{gX})$.*

(3) *If we denote $\bar{\mu}_2 = \mu_2$ and $\bar{\mu}_1 = \mu_1\mu_3 = \mu_3\mu_1$, acting on the set of $\mathbb{Z}/2\mathbb{Z}$-stable maximal rigid Π_Q-modules, $\bar{\mu}$ induces through $\pi \circ \varphi$ the structure of a cluster algebra on $\mathbb{C}[N']$, the clusters of which are projections of the $\mathbb{Z}/2\mathbb{Z}$-stable ones of $\mathbb{C}[N]$.*

Example 23. We have

$$\Delta_{23}^{12}\begin{pmatrix}1 & a_{12} & a_{13} & a_{14}\\ 0 & 1 & a_{23} & a_{24}\\ 0 & 0 & 1 & a_{34}\\ 0 & 0 & 0 & 1\end{pmatrix} = a_{12}a_{23} - a_{13} \quad \text{and} \quad \Delta_{4}^{2}\begin{pmatrix}1 & a_{12} & a_{13} & a_{14}\\ 0 & 1 & a_{23} & a_{24}\\ 0 & 0 & 1 & a_{34}\\ 0 & 0 & 0 & 1\end{pmatrix} = a_{24}.$$

Moreover,

$$\Psi^{-1}\ {}^{t}\begin{pmatrix}1 & a_{12} & a_{13} & a_{14}\\ 0 & 1 & a_{23} & a_{24}\\ 0 & 0 & 1 & a_{34}\\ 0 & 0 & 0 & 1\end{pmatrix}^{-1}\Psi$$

$$= \begin{pmatrix}1 & a_{34} & a_{23}a_{34} - a_{24} & a_{12}a_{23}a_{34} - a_{12}a_{24} - a_{13}a_{34} + a_{14}\\ 0 & 1 & a_{23} & a_{12}a_{23} - a_{13}\\ 0 & 0 & 1 & a_{12}\\ 0 & 0 & 0 & 1\end{pmatrix}$$

which implies that, as expected,

$$\pi\left(\Delta_{23}^{12}\right) = \pi\varphi_{\substack{1\\ \searrow\\ 2}} = \pi\varphi_{\substack{3\\ \swarrow\\ 2}} = \pi\left(\Delta_{4}^{2}\right).$$

The exchange graph of the $\mathbb{Z}/2\mathbb{Z}$-stable basic maximal rigid objects of $\operatorname{mod}\Pi_Q$ is presented on Figure 4, in relation to the exchange graph of the basic maximal rigid objects. It permits, in view of Figure 1 to describe the clusters of $\mathbb{C}[N']$:

$$\begin{array}{ccccc}
 & & \Delta_4^1 = \Delta_{234}^{123}, \Delta_{34}^{12}, \Delta_{24}^{12}, \Delta_2^1 = \Delta_4^3 & & \\
 & \diagup & & \diagdown & \\
\Delta_4^1 = \Delta_{234}^{123}, \Delta_{34}^{12}, \Delta_{24}^{12}, \Delta_4^2 = \Delta_{23}^{12} & & & & \Delta_4^1 = \Delta_{234}^{123}, \Delta_{34}^{12}, \Delta_{34}^{13}, \Delta_2^1 = \Delta_4^3 \\
| & & & & | \\
\Delta_4^1 = \Delta_{234}^{123}, \Delta_{34}^{12}, \Delta_3^2, \Delta_4^2 = \Delta_{23}^{12} & & & & \Delta_4^1 = \Delta_{234}^{123}, \Delta_{34}^{12}, \Delta_{34}^{13}, \Delta_{34}^{23} = \Delta_3^1 \\
 & \diagdown & & \diagup & \\
 & & \Delta_4^1 = \Delta_{234}^{123}, \Delta_{34}^{12}, \Delta_3^2, \Delta_{34}^{23} = \Delta_3^1 & &
\end{array}.$$

6. Scope of these results and consequences

The example presented here can be generalized to the coordinate rings of:

- The groups of the form

$$N(w) = N \cap \left(w^{-1}N_{-}w\right) \quad \text{and} \quad N^{w} = N \cap \left(B_{-}wB_{-}\right)$$

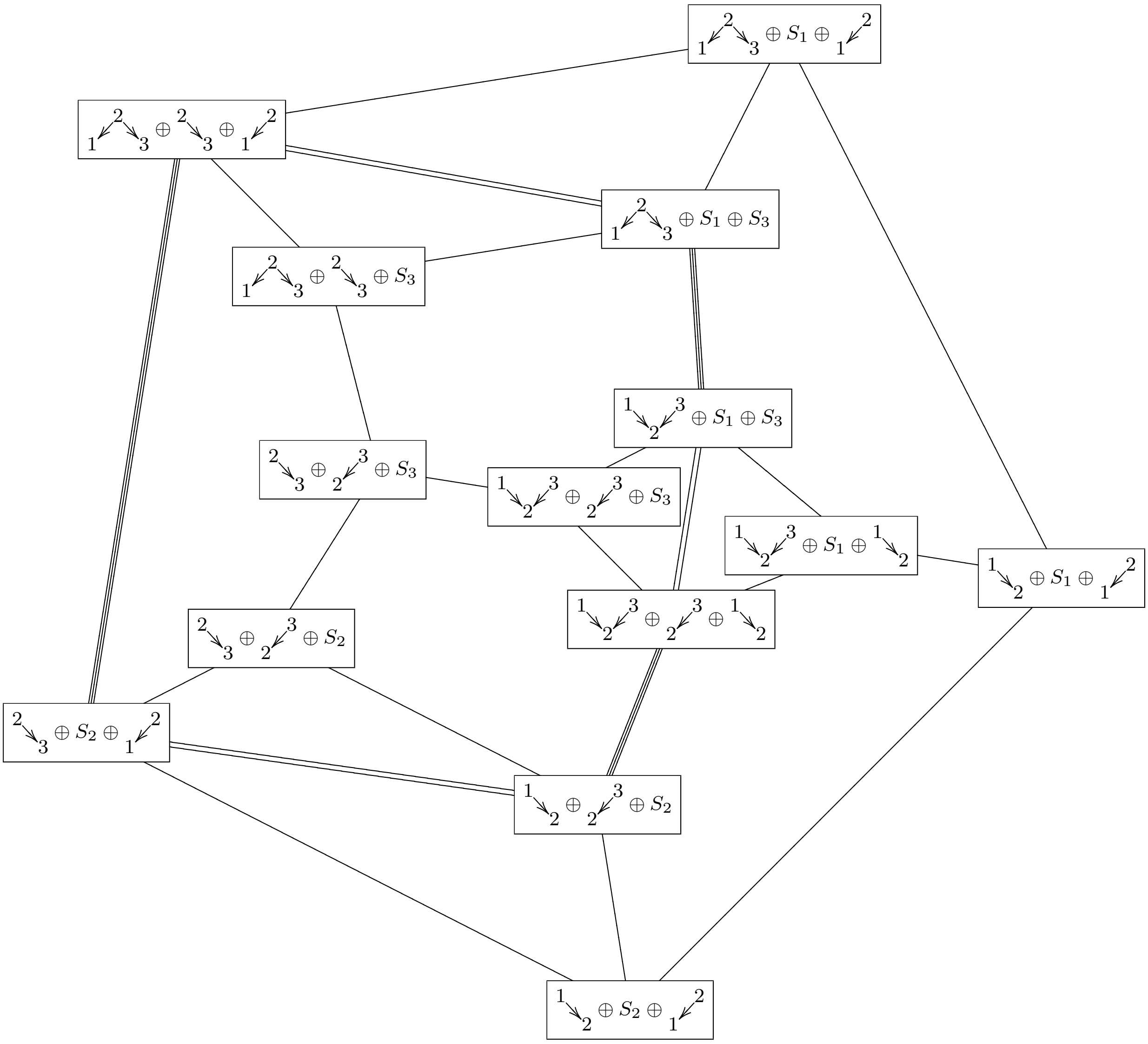


FIGURE 4. Exchange graph of $\mathbb{Z}/2\mathbb{Z}$-stable maximal rigid objects

where N is a maximal unipotent subgroup of a Kac-Moody group, N_- its opposite unipotent group, B_- the corresponding Borel subgroup, and w is an element of the corresponding Weyl group. In particular, if N is of Lie type and w is the longest element, then $N(w) = N$.

- Partial flag varieties corresponding to classical Lie groups.

These results were obtained in [5] and [6] for the simply-laced cases and in [2] for the non simply-laced cases.

It permits for example to prove in these cases that all the cluster monomials (products of elements of a same extended cluster) are linearly independent (result which is now generalized but was new at that time) and other more specific results (for example the classification of partial flag varieties the coordinate rings of which have finite cluster type, that is a finite number of clusters).

References

[1] A. Berenstein, S. Fomin, A. Zelevinsky, *Cluster algebras. III. Upper bounds and double Bruhat cells*, Duke Math. J. **126** (2005), no. 1, 1–52.

[2] L. Demonet, *Categorification of skew-symmetrizable cluster algebras*, Algebr. Represent. Theory **14** (2011), no. 6, 1087–1162.

[3] S. Fomin, A. Zelevinsky, *Double Bruhat cells and total positivity*, J. Amer. Math. Soc. **12** (1999), no. 2, 335–380.

[4] ———, *Cluster algebras. I. Foundations*, J. Amer. Math. Soc. **15** (2002), no. 2, 497–529.

[5] C. Geiß, B. Leclerc, J. Schröer, *Partial flag varieties and preprojective algebras*, Ann. Inst. Fourier (Grenoble) **58** (2008), no. 3, 825–876.

[6] ———, *Kac-Moody groups and cluster algebras*, Adv. Math. **228** (2011), no. 1, 329–433.

[7] B. Keller, *Cluster algebras and derived categories*, arXiv:1202.4161 [math.RT].

[8] G. Lusztig, *Semicanonical bases arising from enveloping algebras*, Adv. Math. **151** (2000), no. 2, 129–139.

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