HOCHSCHILD COHOMOLOGY OF CLUSTER-TILTED ALGEBRAS OF TYPES $\mathbb{A}_n$ AND $\mathbb{D}_n$

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Abstract. In this note, we study the Hochschild cohomology for cluster-tilted algebras of Dynkin types $\mathbb{A}_n$ and $\mathbb{D}_n$. We first show that all cluster-tilted algebras of type $\mathbb{A}_n$ are $(D, A)$-stacked monomial algebras (with $D = 2$ and $A = 1$), and then investigate their Hochschild cohomology rings modulo nilpotence. Also we describe the Hochschild cohomology rings modulo nilpotence for some cluster-tilted algebras of type $\mathbb{D}_n$ which are derived equivalent to a $(D, A)$-stacked monomial algebra. Finally we determine the structures of the Hochschild cohomology rings modulo nilpotence for algebras in a class of some special biserial algebras which contains a cluster-tilted algebra of type $\mathbb{D}_4$.

1. Introduction

The purpose in this note is to study the Hochschild cohomology for cluster-tilted algebras of Dynkin types $\mathbb{A}_n$ and $\mathbb{D}_n$.

Throughout this note, let $K$ denote an algebraically closed field. Let $A$ be a finite-dimensional $K$-algebra, and let $A^e$ be the enveloping algebra $A^{\text{op}} \otimes_K A$ of $A$ (hence right $A^e$-modules correspond to $A$-$A$-bimodules). Then the Hochschild cohomology ring $\text{HH}^*(A)$ of $A$ is defined by the graded ring

$$\text{HH}^*(A) := \text{Ext}^*_A(A, A) = \bigoplus_{i \geq 0} \text{Ext}^i_A(A, A),$$

where the product is given by the Yoneda product. It is well-known that $\text{HH}^*(A)$ is a graded commutative $K$-algebra.

Let $N_A$ be the ideal in $\text{HH}^*(A)$ generated by all homogeneous nilpotent elements. The following question is important in the study of the Hochschild cohomology rings for finite-dimensional algebras:

**Question ([23]).** When is the Hochschild cohomology ring modulo nilpotence $\text{HH}^*(A)/N_A$ finitely generated as an algebra?

It is shown that the Hochschild cohomology rings modulo nilpotence are finitely generated in the following cases: blocks of a group ring of a finite group [12, 25], monomial algebras [16], self-injective algebras of finite representation type [17], finite-dimensional hereditary algebras ([19]). On the other hand, Xu [26] gave an algebra whose Hochschild cohomology ring modulo nilpotence is infinitely generated (see also [23]).

In [7], Buan, Marsh and Reiten introduced cluster-tilted algebras, and since then they have been the subjects of many investigations (see for example [1, 3, 6, 7, 8, 9, 10, 11, 21]). We briefly recall their definition. Let $H = KQ$ be the path algebra of a finite acyclic
quiver \( Q \) over \( K \), and let \( D^b(H) \) the bounded derived category of \( H \). Then the cluster category \( C_H \) associated with \( H \) is defined to be the orbit category \( D^b(H)/\tau^{-1}[1] \), where \( \tau \) denotes the Auslander-Reiten translation in \( D^b(H) \), and \( [1] \) is the shift functor in \( D^b(H) \) ([5, 10]). Note that, by [5], \( C_H \) is a Krull-Schmidt category, and by Keller [20] it is also a triangulated category. A basic object \( T \) in \( C_H \) is called a cluster tilting object, if it satisfies the following conditions ([5]):

1. \( \text{Ext}^1_{C_H}(T, T) = 0 \); and
2. the number of the indecomposable summands of \( T \) equals the number of vertices of \( Q \).

Let \( \Delta \) be the underlying graph of \( Q \). Then the endomorphism ring \( \text{End}_{C_H}(T) \) of a cluster tilting object \( T \) in \( C_H \) is called a cluster-tilted algebra of type \( A_n \) ([7]). In this note, we deal with cluster-tilted algebras of Dynkin types \( A_n \) and \( D_n \). Note that by [7] these algebras are of finite representation type.

In Section 2, we show that cluster-tilted algebras of type \( A_n \) are \((D, A)\)-stacked monomial algebras (with \( D = 2 \) and \( A = 1 \)) of [18] (Lemma 3), and then describe the structures of their Hochschild cohomology rings modulo nilpotence by using [18] (Theorem 4). In Section 3, we determine the Hochschild cohomology rings modulo nilpotence for some cluster-tilted algebras of type \( D_n \) which are derived equivalent to a \((D, A)\)-stacked monomial algebra (Proposition 7). We also describe the Hochschild cohomology rings modulo nilpotence for algebras in a class of some special biserial algebras which contains a cluster-tilted algebra of type \( D_4 \) (Theorem 9).

2. Cluster-tilted algebras of type \( A_n \) and the Hochschild cohomology rings modulo nilpotence

In this section we describe the structure of the Hochschild cohomology rings modulo nilpotence for cluster-tilted algebras of type \( A_n \) (\( n \geq 1 \)).

First we recall the presentation by the quiver and relations of cluster-tilted algebras of type \( A_n \) given in [3, 9]. For a vertex \( x \) in a quiver \( \Gamma \), the neighborhood of \( x \) is the full subquiver of \( \Gamma \) consisting of \( x \) and the vertices which are end-points of arrows starting at \( x \) or start-points of arrows ending with \( x \). Let \( n \geq 2 \) be an integer, and let \( Q_n \) be the class of quivers \( Q \) satisfying the following:

1. \( Q \) has \( n \) vertices.
2. The neighborhood of each vertex \( v \) of \( Q \) is one of the following forms:

\begin{center}
\begin{tikzpicture}
\node (v) at (0,0) {$v$};
\node (w) at (1,0) {$w$};
\draw (v) -- (w);
\end{tikzpicture}
\end{center}
(3) There is no cycles in the underlying graph of $Q$ apart from those induced by oriented cycles contained in neighborhoods of vertices of $Q$.

Let $Q_1 = \{Q'\}$, where $Q'$ is the quiver which has a single vertex and no arrows. It is shown in [9, Proposition 2.4] that a quiver $\Gamma$ is mutation equivalent $A_n$ if and only if $\Gamma \in Q_n$.

In [9], Buan and Vatne proved the following (see also [3]):

**Proposition 1** ([9, Proposition 3.1]). The cluster-tilted algebras of type $A_n$ are exactly the algebras $KQ/I$, where $Q \in Q_n$, and

$$I = \langle p \mid p \text{ is a path of length } 2, \text{ and on an oriented } 3\text{-cycle in } Q \rangle$$

As a consequence we see that cluster-tilted algebras of type $A_n$ are gentle algebras of [2]:

**Corollary 2** ([9, Corollary 3.2]). The cluster-tilted algebras of type $A_n$ are gentle algebras.

Green and Snashall [18] introduced $(D,A)$-stacked monomial algebras by using the notion of overlaps of paths, where $D$ and $A$ are positive integers with $D \geq 2$ and $A \geq 1$, and gave generators and relations of the Hochschild cohomology rings modulo nilpotence for $(D,A)$-stacked monomial algebras completely. (In this note, we do not state the definition of $(D,A)$-stacked algebras and the result of [18]; see for their details [13, Section 1], [18, Section 3], or [23, Section 3].)

It is known that $(2,1)$-stacked monomial algebras are precisely Koszul monomial algebras (equivalently, quadratic monomial algebras), and also $(D,1)$-stacked monomial algebras are exactly $D$-Koszul monomial algebras (see [4]). By the definition, we directly see that all gentle algebras are $(2,1)$-stacked monomial algebras (see [13]). Hence, by Corollary 2, we have the following:

**Lemma 3.** All cluster-tilted algebras of type $A_n$ are $(2,1)$-stacked monomial algebras, and so are Koszul monomial algebras.

By Lemma 3, we can apply the result of [18] to describe the Hochshild cohomology rings of cluster-tilted algebras of type $A_n$. Applying [18, Theorem 3.4] with Proposition 1, we have the following theorem:

**Theorem 4.** Let $n$ be a positive integer, and let $\Lambda = KQ/I$ be a cluster-tilted algebra of type $A_n$, where $Q \in Q_n$ and $I$ is the ideal given by (2.1). Suppose that $\text{char } K \neq 2$. Moreover, let $r$ be the number of oriented $3$-cycles in $Q$. Then

$$\text{HH}^*(\Lambda)/N_A \cong \begin{cases} K[x_1, \ldots, x_r]/\langle x_ix_j \mid i \neq j \rangle & \text{if } r > 0 \cr K & \text{if } r = 0, \end{cases}$$

where $\text{deg } x_i = 6$ for $i = 1, \ldots, r$.

**Example 5.** Let $Q$ be the following quiver with 17 vertices and five oriented 3-cycles:
Then \( Q \in Q_{17} \). Suppose \( \text{char } K \neq 2 \), and let \( A := KQ/I \), where \( I \) is the ideal generated by all possible paths of length 2 on oriented 3-cycles. Then \( A \) is a cluster-tilted algebra of type \( A_{17} \), and by Theorem 4 we have \( HH^*(A)/\mathcal{N}_A \cong K[x_1, \ldots, x_5]/\langle x_i x_j \mid i \neq j \rangle \), where \( \deg x_i = 6 \) (1 \( \leq i \leq 5 \)).

3. CLUSTER-TILTED ALGEBRAS OF TYPE \( \mathbb{D}_n \) AND THE HOCHSCHILD COHOMOLOGY RINGS MODULO NILPOTENCE

The purpose in this section is to describe the Hochschild cohomology rings modulo nilpotence for some cluster-tilted algebras of type \( \mathbb{D}_n \) (\( n \geq 4 \)) which are derived equivalent to a \((D,A)\)-stacked monomial algebra.

In [3, Theorem 2.3], Bastian, Holm and Ladkani introduced specific quivers, called “standard forms” for derived equivalences, and proved that any cluster-tilted algebra of type \( \mathbb{D}_n \) is derived equivalent to one of cluster-tilted algebras of type \( \mathbb{D}_n \) whose quiver is a standard form.

It is known that Hochschild cohomology ring is invariant under derived equivalence, so that it suffices to deal with cluster-tilted algebras of type \( \mathbb{D}_n \) whose quivers are standard forms. In this note, we consider the following quivers \( \Gamma_i \) (1 \( \leq i \leq 4 \)). Clearly these quivers are standard forms of [3, Theorem 2.3].

\[
\Gamma_1 : \begin{array}{c}
\bullet & \xrightarrow{a_0} & \bullet & \xrightarrow{a_1} & \bullet & \xrightarrow{a_2} & \bullet & \xrightarrow{a_0} & \bullet \\
\downarrow & & & & & & & & \\
\bullet & & & & & & & & \bullet
\end{array}
\quad \text{with } m \geq 4 \text{ vertices},
\]

\[
\Gamma_2 : \begin{array}{c}
\bullet & \xrightarrow{b_0} & \bullet & \xrightarrow{a_0} & \bullet & \xrightarrow{a_2 = b_2} & \bullet & \xrightarrow{b_1} & \bullet \\
\downarrow & & & & & & & & \\
\bullet & & & & & & & & \bullet
\end{array}
\]

\[
\Gamma_3 : \begin{array}{c}
\bullet & \xrightarrow{a_0} & \bullet & \xrightarrow{a_1} & \bullet & \xrightarrow{a_2} & \bullet & \xrightarrow{a_0} & \bullet \\
\downarrow & & & & & & & & \\
\bullet & & & & & & & & \bullet
\end{array}
\quad \text{with } m \text{ vertices, where } m \geq 5 \text{ is odd, or } m = 4,
\]

\[
\Gamma_4 : \begin{array}{c}
\bullet & \xrightarrow{a_0} & \bullet & \xrightarrow{a_1} & \bullet & \xrightarrow{a_2} & \bullet & \xrightarrow{a_0} & \bullet \\
\downarrow & & & & & & & & \\
\bullet & & & & & & & & \bullet
\end{array}
\quad \text{with } 2m \text{ vertices, where } m \geq 3.
\]
Proposition 7. For the algebras $A_1$, $A_3$ and $A_4$ above, we have

\[ \dim_K \text{HH}^i(A_k) = \begin{cases} 
  k + 1 & \text{if } i \equiv 0 \pmod{6} \\
  k + 1 & \text{if } i \equiv 1 \pmod{6} \\
  k & \text{if } i \equiv 2 \pmod{6} \\
  k + 1 & \text{if } i \equiv 3 \pmod{6} \text{ and char } K \mid 3k + 2 \\
  k & \text{if } i \equiv 3 \pmod{6} \text{ and char } K \not\mid 3k + 2 \\
  k + 1 & \text{if } i \equiv 4 \pmod{6} \text{ and char } K \mid 3k + 2 \\
  k & \text{if } i \equiv 4 \pmod{6} \text{ and char } K \not\mid 3k + 2 \\
  k & \text{if } i \equiv 5 \pmod{6}.
\end{cases} \]

Remark 6. For $i = 1, \ldots, 4$, let $A_i := K\Gamma_i/I_i$ be the cluster-tilted algebra of type $\mathbb{D}_n$ corresponding to $\Gamma_i$. Then we see from [3, 24] that

1. $A_1$ is the path algebra of a Dynkin quiver of type $\mathbb{D}_m$.
2. $A_2$ is of type $\mathbb{D}_4$, and $I_2 = \langle a_1a_2, b_1b_2, a_2a_0, b_2b_0, a_0a_1 - b_0b_1 \rangle$. We immediately see that $A_2$ is a special biserial algebra of [22], but not a self-injective algebra.
3. $A_3$ is of type $\mathbb{D}_m$, and $I_3 = \langle p \mid p \text{ is a path of length } m - 1 \rangle$. Hence $A_3$ is a $(m-1,1)$-stacked monomial algebra, and is also a self-injective Nakayama algebra.
4. $A_4$ is of type $\mathbb{D}_2m$, and it follows by [3, Lemma 4.5] that $A_4$ is derived equivalent to the $(2m - 1,1)$-stacked monomial algebra $A' = KQ'/I'$, where $Q'$ is the cyclic quiver with $2m$ vertices.

and $I'$ is generated by all paths of length $2m - 1$. Note that $A'$ is a self-injective Nakayama algebra, and moreover is a cluster-tilted algebra of type $\mathbb{D}_2m$ ([21, 24]).

In [19], Happel described the Hochschild cohomology for path algebras. Using this result and [18, Theorem 3.4], we have the following proposition:

Proposition 8. For the algebras $A_1$, $A_3$ and $A_4$ above, we have

\[ \text{HH}^*(A_1) \simeq \text{HH}^*(A_1)/\mathcal{N}_{A_1} \simeq K \]

\[ \text{HH}^*(A_3)/\mathcal{N}_{A_3} \simeq \text{HH}^*(A_4)/\mathcal{N}_{A_4} \simeq K[x]. \]

Finally we describe the Hochschild cohomology ring modulo nilpotence of the algebra $A_k := \Gamma_2/J_k$, where $k \geq 0$ and $J_k$ is the ideal generated by the following elements:

\[ (a_1a_2a_0)^ka_1a_2, b_1b_2, (a_2a_0a_1)^ka_2a_0, b_2b_0, (a_0a_1a_2)^ka_0a_1 - b_0b_1. \]

If $k = 0$, then $J_0 = I_2$, and so $A_0 = \Gamma_2/J_0$ coincides with the algebra $A_2$. Note that, for all $k \geq 0$, $A_k$ is a special biserial algebra and not a self-injective algebra.

Now the dimensions of the Hochschild cohomology groups of $A_k$ are given as follows:

Theorem 9 ([14]). For $k \geq 0$ and $i \geq 0$ we have

\[ \dim_K \text{HH}^i(A_k) = \begin{cases} 
  k + 1 & \text{if } i \equiv 0 \pmod{6} \\
  k + 1 & \text{if } i \equiv 1 \pmod{6} \\
  k & \text{if } i \equiv 2 \pmod{6} \\
  k + 1 & \text{if } i \equiv 3 \pmod{6} \text{ and char } K \mid 3k + 2 \\
  k & \text{if } i \equiv 3 \pmod{6} \text{ and char } K \not\mid 3k + 2 \\
  k + 1 & \text{if } i \equiv 4 \pmod{6} \text{ and char } K \mid 3k + 2 \\
  k & \text{if } i \equiv 4 \pmod{6} \text{ and char } K \not\mid 3k + 2 \\
  k & \text{if } i \equiv 5 \pmod{6}.
\end{cases} \]
Moreover the Hochschild cohomology ring modulo nilpotence of $A_k$ is given as follows:

**Theorem 9** ([15]). For $k \geq 0$, we have

$$\text{HH}^*(A_k)/\mathcal{N}_{A_k} \simeq K[x],$$

where

$$\deg x = \begin{cases} 3 & \text{if } k = 0 \text{ and } \text{char } K = 2 \\ 6 & \text{otherwise.} \end{cases}$$

Hence $\text{HH}^*(A_k)/\mathcal{N}_{A_k}$ $(k \geq 0)$ is finitely generated as an algebra.

**Remark 10.** It seems that most of computations of the Hochschild cohomology rings modulo nilpotence for cluster-tilted algebras of type $D_n$ except those in the derived equivalence classes of $A_i$ $(1 \leq i \leq 4)$ are open questions.

**References**


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