HOCHSCHILD COHOMOLOGY OF CLUSTER-TILTED ALGEBRAS OF TYPES $\mathbb{A}_n$ AND $\mathbb{D}_n$

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Abstract. In this note, we study the Hochschild cohomology for cluster-tilted algebras of Dynkin types $\mathbb{A}_n$ and $\mathbb{D}_n$. We first show that all cluster-tilted algebras of type $\mathbb{A}_n$ are (D, A)-stacked monomial algebras (with $D = 2$ and $A = 1$), and then investigate their Hochschild cohomology rings modulo nilpotence. Also we describe the Hochschild cohomology rings modulo nilpotence for some cluster-tilted algebras of type $\mathbb{D}_n$ which are derived equivalent to a (D, A)-stacked monomial algebra. Finally we determine the structures of the Hochschild cohomology rings modulo nilpotence for algebras in a class of some special biserial algebras which contains a cluster-tilted algebra of type $\mathbb{D}_4$.

1. Introduction

The purpose in this note is to study the Hochschild cohomology for cluster-tilted algebras of Dynkin types $\mathbb{A}_n$ and $\mathbb{D}_n$.

Throughout this note, let K denote an algebraically closed field. Let A be a finite-dimensional K-algebra, and let A^{e} be the enveloping algebra $A^{\mathrm{op}} \otimes_K A$ of A (hence right A^{e}-modules correspond to A-A-bimodules). Then the *Hochschild cohomology ring* $\mathrm{HH}^*(A)$ of A is defined by the graded ring

$$\mathrm{HH}^*(A) := \mathrm{Ext}^*_{A^{\mathrm{e}}}(A, A) = \bigoplus_{i \geq 0} \mathrm{Ext}^i_{A^{\mathrm{e}}}(A, A),$$

where the product is given by the Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative K-algebra.

Let $\mathcal{N}_A$ be the ideal in $\mathrm{HH}^*(A)$ generated by all homogeneous nilpotent elements. The following question is important in the study of the Hochschild cohomology rings for finite-dimensional algebras:

Question ([23]). When is the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}_A$ finitely generated as an algebra?

It is shown that the Hochschild cohomology rings modulo nilpotence are finitely generated in the following cases: blocks of a group ring of a finite group [12, 25], monomial algebras [16], self-injective algebras of finite representation type [17], finite-dimensional hereditary algebras ([19]). On the other hand, Xu [26] gave an algebra whose Hochschild cohomology ring modulo nilpotence is infinitely generated (see also [23]).

In [7], Buan, Marsh and Reiten introduced cluster-tilted algebras, and since then they have been the subjects of many investigations (see for example [1, 3, 6, 7, 8, 9, 10, 11, 21]). We briefly recall their definition. Let $H = KQ$ be the path algebra of a finite acyclic

The detailed version of this paper will be submitted for publication elsewhere.

quiver Q over K, and let $D^b(H)$ the bounded derived category of H. Then the *cluster category* $\mathcal{C}_H$ *associated with* H is defined to be the orbit category $D^b(H)/\tau^{-1}[1]$, where τ denotes the Auslander-Reiten translation in $D^b(H)$, and [1] is the shift functor in $D^b(H)$ ([5, 10]). Note that, by [5], $\mathcal{C}_H$ is a Krull-Schmidt category, and by Keller [20] it is also a triangulated category. A basic object T in $\mathcal{C}_H$ is called a *cluster tilting object*, if it satisfies the following conditions ([5]):

(1) $\mathrm{Ext}^1_{\mathcal{C}_H}(T, T) = 0$; and
(2) the number of the indecomposable summands of T equals the number of vertices of Q.

Let Δ be the underlying graph of Q. Then the endomorphism ring $\mathrm{End}_{\mathcal{C}_H}(T)$ of a cluster tilting object T in $\mathcal{C}_H$ is called a cluster-tilted algebra of type Δ ([7]). In this note, we deal with cluster-tilted algebras of Dynkin types $\mathbb{A}_n$ and $\mathbb{D}_n$. Note that by [7] these algebras are of finite representation type.

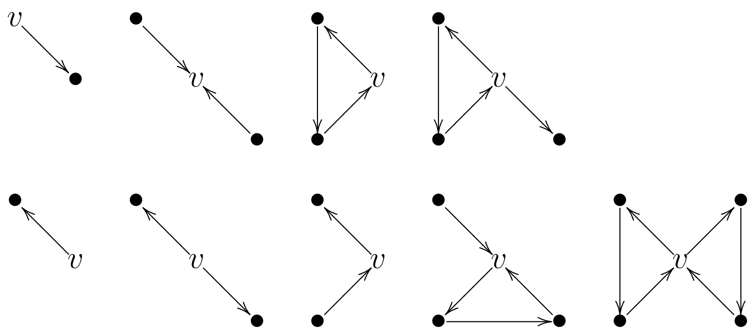
In Section 2, we show that cluster-tilted algebras of type $\mathbb{A}_n$ are (D, A)-stacked monomial algebras (with $D = 2$ and $A = 1$) of [18] (Lemma 3), and then describe the structures of their Hochschild cohomology rings modulo nilpotence by using [18] (Theorem 4). In Section 3, we determine the Hochschild cohomology rings modulo nilpotence for some cluster-tilted algebras of type $\mathbb{D}_n$ which are derived equivalent to a (D, A)-stacked monomial algebra (Proposition 7). We also describe the Hochschild cohomology rings modulo nilpotence for algebras in a class of some special biserial algebras which contains a cluster-tilted algebra of type $\mathbb{D}_4$ (Theorem 9).

2. Cluster-tilted algebras of type $\mathbb{A}_n$ and the Hochschild cohomology rings modulo nilpotence

In this section we describe the structure of the Hochschild cohomology rings modulo nilpotence for cluster-tilted algebras of type $\mathbb{A}_n$ ($n \geq 1$).

First we recall the presentation by the quiver and relations of cluster-tilted algebras of type $\mathbb{A}_n$ given in [3, 9]. For a vertex x in a quiver Γ, the *neighborhood* of x is the full subquiver of Γ consisting of x and the vertices which are end-points of arrows starting at x or start-points of arrows ending with x. Let $n \geq 2$ be an integer, and let Q_n be the class of quivers Q satisfying the following:

(1) Q has n vertices.
(2) The neighborhood of each vertex v of Q is one of the following forms:



(3) There is no cycles in the underlying graph of Q apart from those induced by oriented cycles contained in neighborhoods of vertices of Q.

Let $Q_1 = \{Q'\}$, where Q' is the quiver which has a single vertex and no arrows. It is shown in [9, Proposition 2.4] that a quiver Γ is mutation equivalent $\mathbb{A}_n$ if and only if $\Gamma \in Q_n$.

In [9], Buan and Vatne proved the following (see also [3]):

Proposition 1 ([9, Proposition 3.1])**.** *The cluster-tilted algebras of type $\mathbb{A}_n$ are exactly the algebras KQ/I, where $Q \in Q_n$, and*

$$I = \langle p \mid p \text{ is a path of length } 2\text{, and on an oriented } 3\text{-cycle in } Q \rangle \tag{2.1}$$

As a consequence we see that cluster-tilted algebras of type $\mathbb{A}_n$ are gentle algebras of [2]:

Corollary 2 ([9, Corollary 3.2])**.** *The cluster-tilted algebras of type $\mathbb{A}_n$ are gentle algebras.*

Green and Snashall [18] introduced (D, A)-*stacked monomial algebras* by using the notion of overlaps of paths, where D and A are positive integers with $D \geq 2$ and $A \geq 1$, and gave generators and relations of the Hochschild cohomology rings modulo nilpotence for (D, A)-stacked monomial algebras completely. (In this note, we do not state the definition of (D, A)-stacked algebras and the result of [18]; see for their details [13, Section 1], [18, Section 3], or [23, Section 3].)

It is known that $(2, 1)$-stacked monomial algebras are precisely Koszul monomial algebras (equivalently, quadratic monomial algebras), and also $(D, 1)$-stacked monomial algebras are exactly D-Koszul monomial algebras (see [4]). By the definition, we directly see that all gentle algebras are $(2, 1)$-stacked monomial algebras (see [13]). Hence, by Corollary 2, we have the following:

Lemma 3. *All cluster-tilted algebras of type $\mathbb{A}_n$ are $(2, 1)$-stacked monomial algebras, and so are Koszul monomial algebras.*

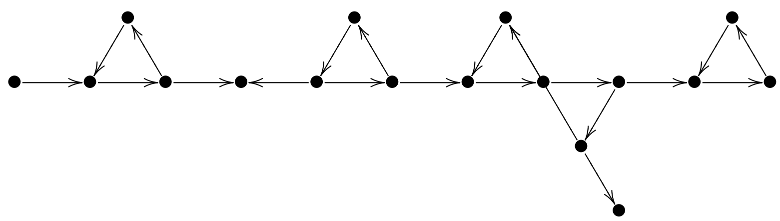
By Lemma 3, we can apply the result of [18] to describe the Hochshild cohomology rings of cluster-tilted algebras of type $\mathbb{A}_n$. Applying [18, Theorem 3.4] with Proposition 1, we have the following theorem:

Theorem 4. *Let n be a positive integer, and let $\Lambda = KQ/I$ be a cluster-tilted algebra of type $\mathbb{A}_n$, where $Q \in Q_n$ and I is the ideal given by (2.1). Suppose that $\operatorname{char} K \neq 2$. Moreover, let r be the number of oriented 3-cycles in Q. Then*

$$\mathrm{HH}^*(\Lambda)/\mathcal{N}_\Lambda \simeq \begin{cases} K[x_1, \dots, x_r]/\langle x_i x_j \mid i \neq j \rangle & \text{if } r > 0 \\ K & \text{if } r = 0, \end{cases}$$

where $\deg x_i = 6$ for $i = 1, \dots, r$.

Example 5. Let Q be the following quiver with 17 vertices and five oriented 3-cycles:



Then $Q \in Q_{17}$. Suppose $\operatorname{char} K \neq 2$, and let $\Lambda := KQ/I$, where I is the ideal generated by all possible paths of length 2 on oriented 3-cycles. Then Λ is a cluster-tilted algebra of type $\mathbb{A}_{17}$, and by Theorem 4 we have $\mathrm{HH}^*(\Lambda)/\mathcal{N}_\Lambda \simeq K[x_1, \dots, x_5]/\langle x_i x_j \mid i \neq j \rangle$, where $\deg x_i = 6$ $(1 \leq i \leq 5)$.

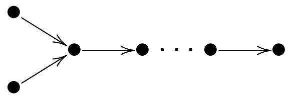
3. Cluster-tilted algebras of type $\mathbb{D}_n$ and the Hochschild cohomology rings modulo nilpotence

The purpose in this section is to describe the Hochschild cohomology rings modulo nilpotence for some cluster-tilted algebras of type $\mathbb{D}_n$ $(n \geq 4)$ which are derived equivalent to a (D, A)-stacked monomial algebra.

In [3, Theorem 2.3], Bastian, Holm and Ladkani introduced specific quivers, called "standard forms" for derived equivalences, and proved that any cluster-tilted algebra of type $\mathbb{D}_n$ is derived equivalent to one of cluster-tilted algebras of type $\mathbb{D}_n$ whose quiver is a standard form.

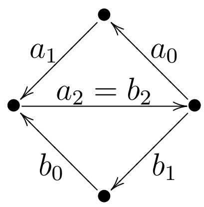
It is known that Hochschild cohomology ring is invariant under derived equivalence, so that it suffices to deal with cluster-tilted algebras of type $\mathbb{D}_n$ whose quivers are standard forms. In this note, we consider the following quivers Γ_i $(1 \leq i \leq 4)$. Clearly these quivers are standard forms of [3, Theorem 2.3].

Γ_1 :

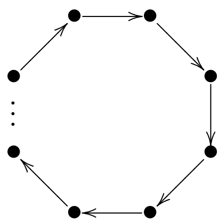


with $m\,(\geq 4)$ vertices,

Γ_2 :

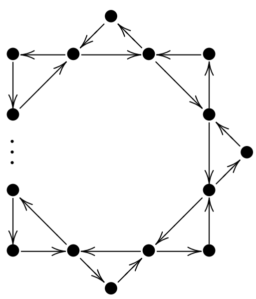


Γ_3 :



with m vertices, where $m\,(\geq 5)$ is odd, or $m = 4$,

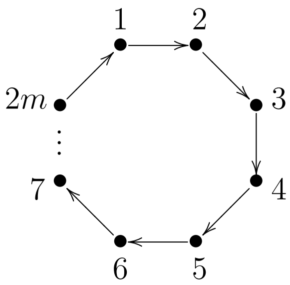
Γ_4 :



with $2m$ vertices, where $m \geq 3$.

Remark 6. For $i = 1, \ldots, 4$, let $\Lambda_i = K\Gamma_i/I_i$ be the cluster-tilted algebra of type $\mathbb{D}_n$ corresponding to Γ_i. Then we see from [3, 24] that

(1) Λ_1 is the path algebra of a Dynkin quiver of type $\mathbb{D}_m$.
(2) Λ_2 is of type $\mathbb{D}_4$, and $I_2 = \langle a_1a_2, b_1b_2, a_2a_0, b_2b_0, a_0a_1 - b_0b_1\rangle$. We immediately see that Λ_2 is a special biserial algebra of [22], but not a self-injective algebra.
(3) Λ_3 is of type $\mathbb{D}_m$, and $I_3 = \langle p \mid p \text{ is a path of length } m-1\rangle$. Hence Λ_3 is a $(m-1, 1)$-stacked monomial algebra, and is also a self-injective Nakayama algebra.
(4) Λ_4 is of type $\mathbb{D}_{2m}$, and it follows by [3, Lemma 4.5] that Λ_4 is derived equivalent to the $(2m-1, 1)$-stacked monomial algebra $\Lambda' = KQ'/I'$, where Q' is the cyclic quiver with $2m$ vertices



and I' is generated by all paths of length $2m-1$. Note that Λ' is a self-injective Nakayama algebra, and moreover is a cluster-tilted algebra of type $\mathbb{D}_{2m}$ ([21, 24]).

In [19], Happel described the Hochschild cohomology for path algebras. Using this result and [18, Theorem 3.4], we have the following proposition:

Proposition 7. *For the algebras Λ_1, Λ_3 and Λ_4 above, we have*

$$\mathrm{HH}^*(\Lambda_1) \simeq \mathrm{HH}^*(\Lambda_1)/\mathcal{N}_{\Lambda_1} \simeq K$$
$$\mathrm{HH}^*(\Lambda_3)/\mathcal{N}_{\Lambda_3} \simeq \mathrm{HH}^*(\Lambda_4)/\mathcal{N}_{\Lambda_4} \simeq K[x].$$

Finally we describe the Hochschild cohomology ring modulo nilpotence of the algebra $A_k := \Gamma_2/J_k$, where $k \geq 0$ and J_k is the ideal generated by the following elements:

$$(a_1a_2a_0)^k a_1a_2, \quad b_1b_2, \quad (a_2a_0a_1)^k a_2a_0, \quad b_2b_0, \quad (a_0a_1a_2)^k a_0a_1 - b_0b_1.$$

If $k = 0$, then $J_0 = I_2$, and so $A_0 = \Gamma_2/J_0$ coincides with the algebra Λ_2. Note that, for all $k \geq 0$, A_k is a special biserial algebra and not a self-injective algebra.

Now the dimensions of the Hochschild cohomology groups of A_k are given as follows:

Theorem 8 ([14])**.** *For $k \geq 0$ and $i \geq 0$ we have*

$$\dim_K \mathrm{HH}^i(A_k) = \begin{cases} k+1 & \text{if } i \equiv 0 \pmod 6 \\ k+1 & \text{if } i \equiv 1 \pmod 6 \\ k & \text{if } i \equiv 2 \pmod 6 \\ k+1 & \text{if } i \equiv 3 \pmod 6 \text{ and } \operatorname{char} K \mid 3k+2 \\ k & \text{if } i \equiv 3 \pmod 6 \text{ and } \operatorname{char} K \nmid 3k+2 \\ k+1 & \text{if } i \equiv 4 \pmod 6 \text{ and } \operatorname{char} K \mid 3k+2 \\ k & \text{if } i \equiv 4 \pmod 6 \text{ and } \operatorname{char} K \nmid 3k+2 \\ k & \text{if } i \equiv 5 \pmod 6. \end{cases}$$

Moreover the Hochschild cohomology ring modulo nilpotence of A_k is given as follows:

Theorem 9 ([15])**.** *For $k \geq 0$, we have*

$$\mathrm{HH}^*(A_k)/\mathcal{N}_{A_k} \simeq K[x], \quad \textit{where} \quad \deg x = \begin{cases} 3 & \textit{if } k = 0 \textit{ and } \mathrm{char}\, K = 2 \\ 6 & \textit{otherwise.} \end{cases}$$

Hence $\mathrm{HH}^*(A_k)/\mathcal{N}_{A_k}$ $(k \geq 0)$ *is finitely generated as an algebra.*

Remark 10. It seems that most of computations of the Hochschild cohomology rings modulo nilpotence for cluster-tilted algebras of type $\mathbb{D}_n$ except those in the derived equivalence classes of Λ_i $(1 \leq i \leq 4)$ are open questions.

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