DERIVED AUTOEQUIVALENCES AND BRAID RELATIONS

JOSEPH GRANT

Abstract. We will consider braid relations between autoequivalences of derived categories of symmetric algebras. We first recall the construction of spherical twists for symmetric algebras and the braid relations that they satisfy, as illustrated by Brauer tree algebras. Then we explain the construction of periodic twists, which generalise spherical twists for symmetric algebras. Finally, we explain a lifting theorem for periodic twists, and show how this gives a new interpretation of the action on the derived Picard group of lifts of longest elements of the symmetric group to the braid group.

1. Preliminaries

Let k be an algebraically closed field. All algebras we consider will be finite-dimensional k-algebras, and for simplicity we will also assume that A is basic. We will denote the category of finite-dimensional left A-modules by A-mod, and of finite-dimensional right A-modules by mod-A.

Given an algebra A and a left (or right) A-module M, we have a right (or left, respectively) A-module $M^* = \mathrm{Hom}_k(M, k)$ with A-action $fa(m) = f(am)$ for $m \in M$, $f \in M^*$, and $a \in A$. This gives a duality

$$(-)^* : A\text{-mod} \xrightarrow{\sim} \text{mod-}A.$$

Similarly, if M is an A-B-bimodule for algebras A and B, then M^* is a B-A-bimodule.

There is another way to construct a right module from $M \in A$-mod: we set $M^\vee = \mathrm{Hom}_A(M, A)$, where the action is given here by $fa(m) = f(m)a$ for $m \in M$, $f \in M^\vee$, and $a \in A$. This defines a functor

$$(-)^\vee : A\text{-mod} \to \text{mod-}A.$$

but in general this is not an equivalence. However, in the cases we consider below this will be an equivalence.

Any algebra A has a natural structure of an A-A-bimodule given by the multiplication. We say that A is a *symmetric* algebra if there exists an isomorphism of A-A-bimodules $A \xrightarrow{\sim} A^*$. Symmetric algebras have various equivalent definitions: one is that $(-)^*$ and $(-)^\vee$ are natually isomorphic functors, and another is a Calabi-Yau type condition on the derived category. For more information on this, we refer the reader to [Ric2, Section 3].

We will be interested in bounded derived categories of module categories over algebras A, which we will denote $\mathrm{D^b}(A)$. We refer the reader to [Wei, Chapter 10] for their definition and basic properties. In partuicular, we will study autoequivalences of $\mathrm{D^b}(A)$. Clearly the autoequivalences form a group, but in fact we can restrict ourselves to a particular subset. One way to define an endofunctor of $\mathrm{D^b}(A)$ is to take the derived tensor product with

The detailed version of this paper will be submitted for publication elsewhere.

a cochain complex X of A-A-bimodules. If this gives us an equivalence of triangulated categories, we call X a *two-sided tilting complex* [Ric1]. Rickard showed that tensoring with two-sided tilting complexes does give a subgroup of the group of autoequivalences [Ric1]. We call this subgroup the *derived Picard group of* A, and denote it $\mathrm{DPic}(A)$. Here we can work with ordinary tensor products, and will not need to consider derived tensor products, as all our two-sided tilting complexes will be presented as cochain complexes of A-A-bimodules which are projective on both sides.

2. Spherical Twists and Braid Relations

Let A be a symmetric algebra and let P be a projective A-module. Following [ST], we say that P is *spherical* if $\mathrm{End}_A(P) \cong k[x]/\langle x^2 \rangle$. In this case, consider the cochain complex of A-A-bimodules

$$P \otimes_k P^\vee \to A$$

concentrated in degrees 1 and 0, where the nonzero map is given by evaluation. We will denote this complex by X_P. It defines an object in the bounded derived category $\mathrm{D^b}(A)$, which we will also denote by X_P. Then tensoring with X_P defines an endofunctor

$$X_P \otimes_A - : \mathrm{D^b}(A) \to \mathrm{D^b}(A)$$

which we denote by F_P.

Theorem 1 ([RZ] for Brauer tree algebras, [ST] in general)**.** *If the projective A-module P is spherical then F_P is an autoequivalence.*

Now let $P_1, \ldots, P_n$ be a collection of n spherical projective A-modules. Following [ST], we say that $\{P_1, \ldots, P_n\}$ is an A_n-collection if

$$\dim_k \mathrm{Hom}_A(P_i, P_j) = \begin{cases} 0 & \text{if } |i-j| > 1; \\ 1 & \text{if } |i-j| = 1 \end{cases}$$

for all $1 \leq i, j \leq n$.

Theorem 2 ([RZ] for Brauer tree algebras, [ST] in general)**.** *If $\{P_1, \ldots, P_n\}$ is an A_n-collection then the spherical twists $F_i = F_{P_i}$ satisfy the braid relations*

- $F_iF_j \cong F_jF_i$ *if* $|i-j| > 1$*;*
- $F_iF_jF_i \cong F_jF_iF_j$ *if* $|i-j| = 1$

for all $1 \leq i, j \leq n$.

Another way to say this is as follows: let B_{n+1} be the braid group on the letters $\{1, \ldots, n, n+1\}$. This is generated by elements $s_1, \ldots, s_n$ and has relations

- $s_is_j = s_js_i$ if $|i-j| > 1$;
- $s_is_js_i = s_js_is_j$ if $|i-j| = 1$.

If A has an A_n-collection then we have a group homomorphism

$$B_{n+1} \to \mathrm{DPic}(A)$$

which sends s_i to the spherical twist F_i.

Let S_{n+1} be the symmetric group on the letters $\{1, \ldots, n, n+1\}$. We also denote the generators of S_i by $s_1, \ldots, s_n$, and there is an obvious group epimorphism $B_{n+1} \twoheadrightarrow S_{n+1}$.

3. Periodic Twists

We now describe a generalization of the spherical twists described above.

An algebra E is called *twisted periodic* if there is an algebra automorphism $\sigma : E \xrightarrow{\sim} E$ and an exact sequence of E-E-bimodules

$$0 \to E_\sigma \to Y_{n-1} \to Y_{n-2} \to \cdots \to Y_1 \to Y_0 \to E \to 0$$

where each Y_i is a projective E-E-bimodule. This just says that the E-E-bimodule E has a periodic resolution which is projective up to some automorphism (twist). We say that E has a period n.

Let A be a symmetric algebra and P a projective A-module. Let $E = \mathrm{End}_A(P)^{\mathrm{op}}$, so P is an A-E-bimodule, and suppose that E is a periodic algebra. We denote the cochain complex

$$Y_{n-1} \to Y_{n-2} \to \cdots \to Y_1 \to Y_0$$

concentrated in degrees $n-1$ to 0 by Y. Then we have a natural map $f : Y \to E$ of cochain complexes of E-E-bimodules. We use this to construct a map $g : P \otimes_E Y \otimes_E P^\vee \to A$ of cochain complexes of A-A-bimodules defined as the following composition

$$P \otimes_E Y \otimes_E P^\vee \to P \otimes_E E \otimes_E P^\vee \xrightarrow{\sim} P \otimes_E P^\vee \to A$$

where the first map is given by $P \otimes_E f \otimes_E P^\vee$ and the last is given by an evaluation map.

We take the cone of the map g to obtain a cochain complex

$$P \otimes_E Y_{n-1} \otimes_E P^\vee \to P \otimes_E Y_{n-2} \otimes_E P^\vee \to \cdots \to P \otimes_E Y_0 \otimes_E P^\vee \to A$$

concentrated in degrees n to 0, which we denote X. By tensoring over A we obtain an endofunctor

$$X \otimes_A - : \mathrm{D}^{\mathrm{b}}(A) \to \mathrm{D}^{\mathrm{b}}(A)$$

which we denote by Ψ_P.

Theorem 3 ([Gra])**.** *If the algebra E is twisted periodic then Ψ_P is an autoequivalence.*

Note that the functor Ψ_P depends on the resolution Y that we choose.

If $E \cong k[x]/\langle x^2 \rangle$ then we recover the spherical twists described above by using the exact sequence

$$0 \to E_\sigma \to E \otimes_k E \to E \to 0$$

where σ is the algebra automorphism which sends x to $-x$.

4. Brauer Tree Algebras of Lines

We define a collection of algebras Γ_n, $n \geq 1$, which are isomorphic to the Brauer tree algebras of lines without multiplicity. Let $\Gamma_1 = k[x]/\langle x^2 \rangle$ and let $\Gamma_2 = kQ_2/I_2$, where Q_2 is the quiver

$$Q_2 \quad = \quad 1 \underset{\beta}{\overset{\alpha}{\rightleftarrows}} 2$$

and I_2 is the ideal generated by $\alpha\beta\alpha$ and $\beta\alpha\beta$. For $n \geq 3$, let $\Gamma_n = kQ_n/I_n$ where Q_n is the quiver

$$Q_n \quad = \quad 1 \underset{\beta_2}{\overset{\alpha_1}{\rightleftarrows}} 2 \underset{\beta_3}{\overset{\alpha_2}{\rightleftarrows}} \cdots \underset{\beta_n}{\overset{\alpha_{n-1}}{\rightleftarrows}} n$$

and I_n is the ideal generated by $\alpha_{i-1}\alpha_i$, $\beta_{i+1}\beta_i$, and $\alpha_i\beta_{i+1} - \beta_i\alpha_{i-1}$ for $2 \leq i \leq n-1$. Note that if we take the indecomposable projective Γ_n-modules $P_i, P_{i+1}, \ldots, P_j$ for $1 \leq i < j \leq n$, we have $\mathrm{End}_A(P_i \oplus P_{i+1} \oplus \ldots \oplus P_j)^{\mathrm{op}} \cong \Gamma_{j-i}$.

One can check that for all $n \geq 1$, each indecomposable projective Γ_n-module is spherical. Spherical twists for these algebras were studied in detail in [RZ].

We have the following observation:

Lemma 4. *Let A be a symmetric algebra. A collection $\{P_1, \ldots, P_n\}$ of projective A-modules is an A_n-collection if and only if*

$$\mathrm{End}_A(\bigoplus_{i=1}^{n} P_i)^{\mathrm{op}} \cong \Gamma_n.$$

The algebras Γ_n are of finite representation type, and hence are twisted periodic, but in fact we can say more.

Theorem 5 ([BBK]). *The algebra n is twisted periodic with period n and automorphism σ_n induced by the quiver automorphism which sends the vertex i to $n - i + 1$.*

A natural question is: what do the associated periodic twists look like? It was noted in [Gra] that periodic twists associated to Γ_2 are isomorphic to the composition $F_1F_2F_1$ of spherical twists. We will show that this pattern continues.

5. A Lifting Theorem

Let A be a symmetric algebra and let $P_1, \ldots, P_n$ be a collection of indecomposable projective A-modules. We will use the following notation:

- $P = \bigoplus_{i=1}^{n} P_i$;
- $E = \mathrm{End}_A(P)^{\mathrm{op}}$;
- $E_i = \mathrm{End}_A(P_i)^{\mathrm{op}}$;
- $Q_i = \mathrm{Hom}_A(P, P_i)$,
- $Q = \bigoplus_{i=1}^{n} Q_i$;

so $\{Q_i | 1 \leq i \leq n\}$ is a complete set of representatives of the isoclasses of indecomposable projective E-modules. Note that $\mathrm{End}_E(Q_i)^{\mathrm{op}} \cong E_i$. We will explain a connection between compositions of periodic twists for E and compositions of corresponding periodic twists for A.

Theorem 6 (Lifting Theorem). *Suppose that E and each E_i are twisted periodic with fixed truncated resolutions Y and Y_i. Let $\Psi_i = \Psi_{P_i} : \mathrm{D^b}(A) \xrightarrow{\sim} \mathrm{D^b}(A)$ and $\Psi'_i = \Psi_{Q_i} : \mathrm{D^b}(E) \xrightarrow{\sim} \mathrm{D^b}(E)$. If*

$$\Psi_Q \cong \Psi'_{i_\ell} \ldots \Psi'_{i_2}\Psi'_{i_1}$$

for some $1 \leq i_1, \ldots, i_\ell \leq n$ then

$$\Psi_P \cong \Psi_{i_\ell} \ldots \Psi_{i_2}\Psi_{i_1}.$$

We now specialise to the case where $\{P_1, \ldots, P_n\}$ is an A_n-collection, so $E \cong \Gamma_n$, and $F_i = \Psi_i$ and $F_i' = \Psi_i'$ are spherical twists.

Recall that the symmetric group S_{n+1} has a unique longest element, often denoted w_0. We choose a particular presentation

$$w_0 = s_1(s_2 s_1) \ldots (s_n \ldots s_2 s_1) \in S_{n+1}$$

and define an element w_0 of the braid group by the same presentation. Rouquier and Zimmermann showed how this element acts on the derived Picard group of an algebra Γ_n:

Theorem 7 ([RZ, Theorem 4.5]). *The image of the element w_0 under the group morphism $B_{n+1} \to \mathrm{DPic}(\Gamma_n)$ is the functor $-_{\sigma_n}[n]$ which twists on the right by the automorphism σ_n and shifts cochain complexes n places to the left.*

By Theorem 5 we see that $\Psi_{\Gamma_n} : \mathrm{D^b}(\Gamma_n) \to \mathrm{D^b}(\Gamma_n)$ is the same functor, and hence by applying the lifting theorem we obtain the following:

Corollary 8. *Suppose the symmetric algebra A has an A_n-collection $\{P_1, \ldots, P_n\}$. Then the image of w_0 in the group morphism $B_{n+1} \to \mathrm{DPic}(A)$ is Ψ_P, where $P = \bigoplus_{i=1}^n P_i$.*

We also obtain a new proof of the braid relations by using Theorem 7 in the case $n = 2$, or alternatively by performing a straightforward calculation with Γ_2, and then applying the lifting theorem.

References

[BBK] S. Brenner, M. Butler, and A. King, *Periodic algebras which are almost Koszul*, Algebr. Represent. Theory **5** (2002), no. 4, 331-367

[Gra] J. Grant, *Derived autoequivalences from periodic algebras*, arXiv:1106.2733v1 [math.RT]

[Ric1] J. Rickard, *Derived equivalences as derived functors*, J. London Math. Soc. (2) **43** (1991) 37-48

[Ric2] J. Rickard, *Equivalences of derived categories for symmetric algebras*, J. Algebra **257** (2002), no. 2, 460-481

[RZ] R. Rouquier and A. Zimmermann, *Picard groups for derived module categories*, Proc. London Math. Soc. (3) **87** (2003), no. 1, 197-225

[ST] P. Seidel and R. Thomas, *Braid group actions on derived categories of coherent sheaves*, Duke Math. J. **108** (2001), no. 1, 37-108

[Wei] C. A. Weibel, *An introduction to homological algebra*, Cambridge studies in advanced mathematics **38**, 1994

Graduate School of Mathematics
Nagoya University
Chikusa-ku, Nagoya, 464-8602, Japan
E-mail address: joseph.grant@gmail.com