n-REPRESENTATION INFINITE ALGEBRAS

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Abstract. We introduce the class of n-representation infinite algebras and discuss some of their homological properties. We also present the family of n-representation infinite algebras of type $\widetilde{A}$.

1. Introduction

This brief survey contains the results from my presentation at the *44th Symposium on Ring Theory and Representation Theory* in Okayama. It is based on joint work Osamu Iyama and Steffen Oppermann. A detailed final version will be published elsewhere.

The class of hereditary finite dimensional algebras is one of the best understood in terms of representation theory, especially in the context of Auslander-Reiten theory. This applies in particular to representation finite hereditary algebras. In higher dimensional Auslander-Reiten theory an analogue of these algebras is given by the class of n-representation finite algebras [1, 2]. Recall that a finite dimensional algebra is called n-representation finite if it has global dimension at most n and admits an n-cluster tilting module. Since a 1-cluster tilting module is the same as an additive generator of the module category, 1-representation finite means precisely hereditary and representation finite.

The aim of this report is to define the class of n-representation infinite algebras, that will in a similar way be a higher dimensional analogue of representation infinite hereditary algebras. To do this we begin by recalling some properties of n-representation finite algebras.

Let K be a field and Λ a finite dimensional K-algebra with $\operatorname{gl.dim}\Lambda \leq n$. We always assume that Λ is ring indecomposable. Denote by $\operatorname{mod}\Lambda$ the category of finite dimensional left Λ-modules and by $\mathcal{D}^{\mathrm{b}}(\Lambda)$ the bounded derived category of $\operatorname{mod}\Lambda$. Combining the K-dual $D := \operatorname{Hom}_K(-, K)$ with the Λ-dual we obtain the Nakayama functor

$$\nu := D\mathbf{R}\mathrm{Hom}(-, \Lambda) : \mathcal{D}^{\mathrm{b}}(\Lambda) \to \mathcal{D}^{\mathrm{b}}(\Lambda).$$

It is a Serre functor in the sense that there is a functorial ismorphism

$$\operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(\Lambda)}(X, Y) \simeq D \operatorname{Hom}_{\mathcal{D}^{\mathrm{b}}(\Lambda)}(Y, \nu(X)).$$

We combine ν with the shift functor on $\mathcal{D}^{\mathrm{b}}(\Lambda)$ to obtain the autoequivalence

$$\nu_n := \nu \circ [-n] : \mathcal{D}^{\mathrm{b}}(\Lambda) \to \mathcal{D}^{\mathrm{b}}(\Lambda).$$

It plays the role of the higher Auslander-Reiten translation in $\mathcal{D}^{\mathrm{b}}(\Lambda)$. More precisely, define

$$\tau_n := D \operatorname{Ext}^n_\Lambda(-, \Lambda) : \operatorname{mod}\Lambda \to \operatorname{mod}\Lambda$$

The detailed version of this paper will be submitted for publication elsewhere.

and

$$\tau_n^- := \mathrm{Ext}^n_\Lambda(D\Lambda, -) : \mathrm{mod}\,\Lambda \to \mathrm{mod}\,\Lambda.$$

Then $\tau_n = H^0(\nu_n -)$ and $\tau_n^- = H^0(\nu_n^{-1} -)$. Using these functors we can capture the notion of n-representation finiteness in the following way.

Proposition 1. [3] *Let Λ be a finite dimensional K-algebra with* $\mathrm{gl.dim}\,\Lambda \leq n$. *Then the following conditions are equivalent.*

(a) Λ *is n-representation finite.*
(b) *For every indecomposable projective Λ-module P, there is a non-negative integer ℓ_P such that $\nu_n^{-\ell_P} P$ is an indecomposable injective Λ-module.*

We remark that if condition (b) is satisfied then $\nu_n^{-i}P \simeq \tau_n^{-i}P$ for all $0 \leq i \leq \ell_P$ and

$$\bigoplus_P \bigoplus_{i=0}^{\ell_P} \tau_n^{-i} P = \bigoplus_P \bigoplus_{i=0}^{\ell_P} \nu_n^{-i} P$$

is an n-cluster tilting Λ-module [1]. Furthermore, since ν^{-1} sends injectives to projectives we have

$$\nu_n^{-(\ell_P+1)} P = \nu^{-1}(\nu_n^{-\ell_P} P)[n] = P'[n] \in \mathrm{mod}\,\Lambda[n]$$

for some indecomposable projective P'. We conclude that knowing the τ_n^--orbits of the indecomposable projectives in $\mathrm{mod}\,\Lambda$ is enough to determine their ν_n^{-1}-orbits. Comparing this to the classical case $n = 1$ gives us a hint how to define n-representation infinite algebras.

2. n-representation infinite algebras

Recall that if $n = 1$ and Λ is representation infinite, then $\tau^{-i}P$ is never injective for an indecomposable projective Λ-module P. In fact $\nu_1^{-i} P = \tau^{-i} P \in \mathrm{mod}\,\Lambda$ for all $i \geq 0$. Inspired by this we make the following definition.

Definition 2. Let Λ be a finite dimensional K-algebra with $\mathrm{gl.dim}\,\Lambda \leq n$. We say that Λ is n-representation infinite if

$$\nu_n^{-i}\Lambda \in \mathrm{mod}\,\Lambda$$

for all $i \geq 0$.

We remark that this condition is equivalent to $\nu_n^i(D\Lambda) \in \mathrm{mod}\,\Lambda$ for all $i \geq 0$. In the classical setting of $n = 1$ every indecomposable module is either preprojective, preinjective or regular. We define higher analogues of these classes of modules as follows.

Definition 3. Let Λ be an n-representation infinite algebra. The full subcategories of n-preprojective, n-preinjective and n-regular modules are defined as

$$\begin{aligned} \mathcal{P} &:= \mathrm{add}\{\nu_n^{-i}\Lambda \mid i \geq 0\}, \\ \mathcal{I} &:= \mathrm{add}\{\nu_n^{i}(D\Lambda) \mid i \geq 0\}, \\ \mathcal{R} &:= \{X \in \mathrm{mod}\,\Lambda \mid \mathrm{Ext}^i_\Lambda(\mathcal{P}, X) = 0 = \mathrm{Ext}^i_\Lambda(X, \mathcal{I}) \text{ for all } i \geq 0\}, \end{aligned}$$

respectively.

Note that $\mathcal{P}$ and $\mathcal{I}$ are well-defined as subcategories of $\operatorname{mod}\Lambda$ since Λ is n-representation infinite. Many properties of representation infinite hereditary algebras generalize to n-representation infinite algebras. For instance n-regular modules can be characterized by $\mathcal{R} = \{X \in \operatorname{mod}\Lambda \mid \nu_n^i(X) \in \operatorname{mod}\Lambda \text{ for all } i \in \mathbb{Z}\}$. Moreover, one has the following result about vanishing of homomorphisms and extensions.

Theorem 4. *Let Λ be an n-representation infinite algebra. Then the following holds:*

$$\begin{array}{llllll} \operatorname{Hom}_\Lambda(\mathcal{R},\mathcal{P}) & = 0, & \operatorname{Hom}_\Lambda(\mathcal{I},\mathcal{P}) & = 0, & \operatorname{Hom}_\Lambda(\mathcal{I},\mathcal{R}) & = 0, \\ \operatorname{Ext}^n_\Lambda(\mathcal{P},\mathcal{R}) & = 0, & \operatorname{Ext}^n_\Lambda(\mathcal{P},\mathcal{I}) & = 0, & \operatorname{Ext}^n_\Lambda(\mathcal{R},\mathcal{I}) & = 0. \end{array}$$

3. n-representation infinite algebras of type $\widetilde{A}$

In this section we assume that K is an algebraically closed field of characteristic zero. We shall present a family of n-representation infinite algebras by generalizing one of the simplest classes of representation infinite hereditary algebras, namely path algebras of extended Dynkin quivers of type $\widetilde{A}$.

On can construct extended Dynkin quivers of type $\widetilde{A}$ by taking the following steps. Start with the double quiver of A_∞^∞:

$$\cdots \; -2 \rightleftarrows -1 \rightleftarrows 0 \rightleftarrows 1 \rightleftarrows 2 \; \cdots$$

Identify vertices and arrows modulo m for some $m \geq 1$ and remove one arrow from each 2-cycle. For instance, choosing $m = 2$ and removing the arrows starting in the odd vertex gives the Kronecker quiver:

$$0 \rightrightarrows 1.$$

We shall construct the n-representation infinite algebras of type $\widetilde{A}$ similarly. First we define the covering quiver Q. As vertices in Q we take the lattice

$$Q_0 = G := \left\{ v \in \mathbb{Z}^{n+1} \;\middle|\; \sum_{i=1}^{n+1} v_i = 0 \right\}.$$

It is freely generated as an abelian group by the elements $f_i := e_{i+1} - e_i$ for $1 \leq i \leq n$. We also define $f_{n+1} := e_1 - e_{n+1}$, so that $\sum_{i=1}^{n+1} f_i = 0$. As arrows in Q we take

$$Q_1 := \{a_i : v \to v + f_i \mid v \in G,\ 1 \leq i \leq n+1\}.$$

Then Q is the double of A_∞^∞ for $n = 1$. For $n \geq 2$ we need to introduce certain relations. Let $v \in Q_0$ and $i, j \in \{1, \ldots, n+1\}$. We consider the relation $r_{ij}^v := a_i a_j - a_j a_i$ from v to $v + f_i + f_j$ and let I be the two-sided ideal in KQ generated by

$$\{r_{ij}^v \mid v \in Q_0,\ 1 \leq i, j \leq n+1\}.$$

Since G is an abelian group it acts on itself by translations. This extends to a unique G-action on the quiver Q. We say that a subgroup $B \leq G$ is cofinite if G/B is finite. In that case we define $\Gamma(B)$ as the orbit algebra of KQ/I. More explicitly we define $Q/B := (Q_0/B, Q_1/B)$ and set

$$\Gamma(B) := K(Q/B)/\langle \overline{r}_{ij}^{\overline{v}} \mid \overline{v} \in Q_0/B,\ 1 \leq i, j \leq n+1 \rangle$$

where $\overline{r}_{ij}^{\overline{v}} := \overline{a}_i\overline{a}_j - \overline{a}_j\overline{a}_i$ and $\overline{a}$ denotes the B-orbit of a. As motivation for this construction we remark that $\Gamma(B)$ is isomorphic to a skew group algebra $K[x_1, \ldots, x_{n+1}] * H$ for some finite abelian subgroup $H < SL_{n+1}(K)$.

Next we consider the analogue of 2-cycles. For every $v \in Q_0$ and permutation σ of $1, \ldots, n+1$, there is a cyclic path $a_{\sigma(1)} \cdots a_{\sigma(n+1)}$ from v to v. We call such cyclic paths small cycles. A subset $C \subset Q_1$ is called a cut if it contains precisely one arrow from every small cycle. The symmetry group of C is defined as

$$S_C := \{g \in G \mid gC = C\} \leq G.$$

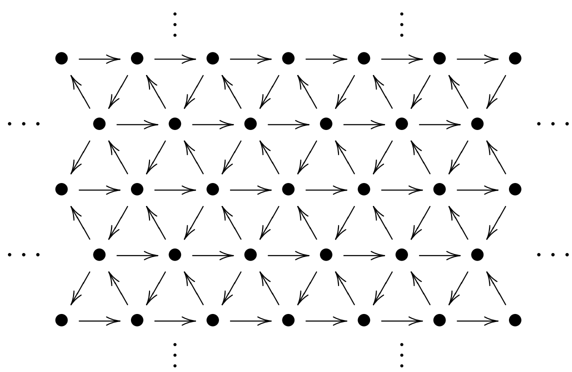
We say that a cut C is acyclic if all paths in $Q_C := (Q_0, Q_1 \setminus C)$ have length bounded by some $N \geq 0$, and periodic if S_C is cofinite in G. If both these conditions are satisfied and $B \leq S_C$ is cofinite we say that

$$\Gamma(B)_C := \Gamma(B)/\langle \overline{a} \mid \overline{a} \in C/B \rangle$$

is n-representation finite of type $\widetilde{A}$. The name is justified by the following Theorem.

Theorem 5. *If C is an acyclic periodic cut and $B \leq S_C$ is cofinite, then $\Gamma(B)_C$ is n-representation finite.*

We remark that if $n = 1$, then $\Gamma(B)_C$ is a path algebra of an acyclic quiver of type $\widetilde{A}$ constructed exactly as explained above. For $n = 2$, Q_0 is a triangular lattice in the plane and Q is



where the small cycles are formed by the small triangles.

Finally we shall generalize the alternating orientation of A_∞^∞. To do this define $\omega : G \to \mathbb{Z}/(n+1)\mathbb{Z}$ by $\omega(f_i) = 1$ and set

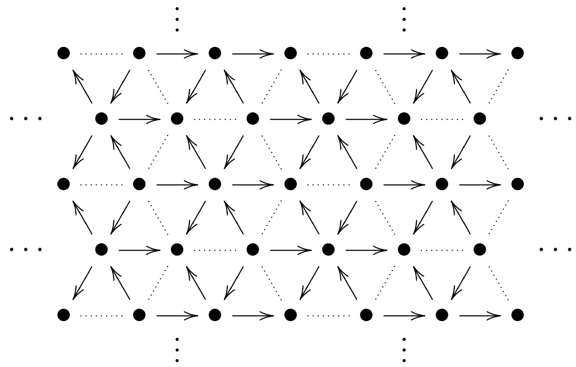
$$C := \{a_i : v \to v + f_i \mid \omega(v) = 0,\ 1 \leq i \leq n+1\}.$$

Then every path in Q of length $n+1$ intersects C and so C is acyclic. Moreover, $S_C = \ker \omega$ and so C is periodic.

For $n = 1$, Q_C is

$$\cdots\ -2 \leftarrow -1 \rightarrow 0 \leftarrow 1 \rightarrow 2\ \cdots$$

For $n = 2$, Q_C is



where the dotted lines indicate commutativity relations in $\Gamma/\langle C \rangle$.

Now let's consider $\Gamma(B)_C$ for $B = S_C$. Then we can identify Q_0/B with $\mathbb{Z}/(n+1)\mathbb{Z}$ via ω and C/B consists of all arrows from $n+1$ to 1. Hence $\Gamma(B)_C$ is the Beilinson algebra:

$$1 \;\begin{matrix}\xrightarrow{a_1}\\ \vdots \\ \xrightarrow[a_{n+1}]{}\end{matrix}\; 2 \;\begin{matrix}\xrightarrow{a_1}\\ \vdots \\ \xrightarrow[a_{n+1}]{}\end{matrix}\; 3 \cdots n \;\begin{matrix}\xrightarrow{a_1}\\ \vdots \\ \xrightarrow[a_{n+1}]{}\end{matrix}\; n+1\,, \qquad a_i a_j = a_j a_i.$$

and for $n = 1$ we obtain the Kronecker algebra:

$$1 \rightrightarrows 2.$$

References

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[3] O. Iyama, S. Oppermann, *Stable categories of higher preprojective algebras*, arXiv:0912.3412.

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