τ-TILTING MODULES FOR SELF-INJECTIVE NAKAYAMA ALGEBRAS

TAKAHIDE ADACHI

Abstract. In this paper, we study τ-tilting modules over Nakayama algebras. First, for self-injective Nakayama algebras, we give a classification of τ-tilting modules. Secondly, for Nakayama algebras, we give a combinatorial method to provide Hasse quivers of support τ-tilting modules.

1. Introduction

In tilting theory of algebras, tilting modules are important objects. As a way to construct tilting modules, there is the notion of tilting mutations introduced by Riedtmann-Schofield [8]. Roughly speaking, tilting mutations are operations which construct new tilting modules by replacing indecomposable direct summands of given tilting modules. However, it is known that tilting mutations have the following disadvantage. Namely, any basic almost complete tilting module can be completed to a basic tilting module in at most two different ways [8, 9]. This means that tilting mutations are not always defined. To overcome the disadvantage of tilting modules, the notion of τ-tilting modules was introduced in [2]. The authors showed any basic almost support τ-tilting module can be completed to a basic support τ-tilting module in exactly two different ways. Moreover, for a given algebra Λ, it is shown that there are bijections between support τ-tilting Λ-modules, two-term silting complexes for Λ (see [1, 7]), and cluster-tilting objects in a 2-CY triangulated category $\mathcal{C}$ if Λ is an associated 2-CY tilted algebra to $\mathcal{C}$ (see [4, 6]). Thus it is important to give a classification of support τ-tilting Λ-modules.

In this paper, we study τ-tilting modules over Nakayama algebras. First, we classify τ-tilting modules over self-injective Nakayama algebras. We shall give a bijection between τ-tilting modules and proper support τ-tilting modules. In this case, proper support τ-tilting modules are reduced to tilting modules over path algebras of Dynkin quivers of type A. A classification of tilting modules of the path algebras is well-known (*e.g.* triangulations of polygons). Thus we can easily obtain proper support τ-tilting modules.

Secondly, we give a combinatorial method to provide Hasse quivers of support τ-tilting modules over Nakayama algebras. Then Rejection Lemma of Drozd-Kirichenko plays important role. The rejection lemma gives a connection of indecomposable modules between an algebra and its factor algebra by some ideal. Any Nakayama algebra is given by a sequence of Drozd-Kirichenko rejection from some semisimple algebra. We study a connection of support τ-tilting modules between two algebras of Drozd-Kirichenko rejection. Using the connection, we construct Hasse quivers of Nakayama algebras from some semisimple algebra.

The detailed version of this paper will be submitted for publication elsewhere.

Notation. Throughout this paper, K is an algebraically closed field, and Λ is a basic finite dimensional K-algebra. We denote by $\mathrm{mod}\Lambda$ the category of finitely generated right Λ-modules, and by $\mathrm{ind}\Lambda$ the set of isomorphism classes of indecomposable Λ-modules. For two sets X and Y, we denote by $X \sqcup Y$ the disjoint union of X and Y. We denote by C_n the cyclic quiver and by $\vec{A}_n$ the Dynkin quiver of type A with linear orientation:

$$\begin{tikzcd}
 & 1 \arrow[dr, "\alpha_1"] & \\
n \arrow[ur, "\alpha_n"] & & 2 \arrow[d, "\alpha_2"] \\
n-1 \arrow[u, "\alpha_{n-1}"] & & 3 \arrow[dl, "\alpha_3"] \\
 & \cdots \arrow[ul, "\alpha_{n-2}"] &
\end{tikzcd}
\qquad
n \xleftarrow{\alpha_{n-1}} n-1 \xleftarrow{\alpha_{n-2}} \cdots \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1$$

2. Preliminaries

Let Λ be a basic finite dimensional K-algebra with a complete set $\{e_1, e_2, \cdots, e_n\}$ of primitive orthogonal idempotents, and $E_\Lambda := \{\sum_{j \in J} e_j \mid \emptyset \neq J \subset \{1, 2, \cdots, n\}\}$. For a module $M \in \mathrm{mod}\Lambda$, we denote by $|M|$ the number of nonisomorphic indecomposable direct summands of M. We write by τ_Λ the Auslander-Reiten translation of Λ, and by $\langle e \rangle$ a two-sided ideal of Λ generated by $e \in \Lambda$.

In this section, we recall definitions and basic properties of τ-tilting modules.

Definition 1. Let Λ be a finite dimensional K-algebra, and $M \in \mathrm{mod}\Lambda$ a module.

(1) We call M *τ-rigid* Λ-module if $\mathrm{Hom}_\Lambda(M, \tau_\Lambda M) = 0$.
(2) We call M *τ-tilting* Λ-module if it is τ-rigid and $|M| = |\Lambda|$.
(3) We call M *support τ-tilting* Λ-module if there exists an idempotent $e \in \Lambda$ such that M is a τ-tilting $(\Lambda/\langle e \rangle)$-module. In this case, if $e \neq 0$, we call M *proper support τ-tilting* Λ-module.

In the rest of the paper, we denote by $\mathrm{tilt}\Lambda$ (respectively, τ-$\mathrm{tilt}\Lambda$, $\mathrm{s}\tau$-$\mathrm{tilt}\Lambda$, $\mathrm{ps}\tau$-$\mathrm{tilt}\Lambda$) the set of isomorphism classes of basic tilting (respectively, τ-tilting, support τ-tilting, proper support τ-tilting) Λ-modules.

Lemma 2. [2, Proposition 2.3] *For any proper support τ-tilting Λ-module M, there uniquely exists an idempotent $e \in E_\Lambda$ such that M is a τ-tilting $(\Lambda/\langle e \rangle)$-module. We write by e_M the above idempotent e.*

The following is straightforward.

Proposition 3. *The following hold.*

(1) τ-$\mathrm{tilt}\Lambda = \mathrm{tilt}\Lambda$ *if Λ is a hereditary algebra.*
(2) $\mathrm{s}\tau$-$\mathrm{tilt}\Lambda = \tau$-$\mathrm{tilt}\Lambda \sqcup \mathrm{ps}\tau$-$\mathrm{tilt}\Lambda$.
(3) $\mathrm{ps}\tau$-$\mathrm{tilt}\Lambda = \displaystyle\bigsqcup_{e \in E_\Lambda} \tau$-$\mathrm{tilt}(\Lambda/\langle e \rangle)$.

By the proposition above, we have important observations.

Remark 4. We can decompose sτ-tiltΛ as the disjoint union of τ-tiltΛ and psτ-tiltΛ. Moreover, proper support τ-tilting Λ-modules are reduced to τ-tilting modules over smaller algebras. To determine sτ-tiltΛ, it is thus important to construct τ-tilting Λ-modules.

The following lemma will be useful.

Lemma 5. [2, Lemma 2.1] *Let I be a two-sided ideal of Λ, and $M, N \in \mathrm{mod}(\Lambda/I)$. Then the following hold.*

(1) *If $\mathrm{Hom}_\Lambda(N, \tau_\Lambda M) = 0$, then $\mathrm{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$.*
(2) *Assume that $I = \langle e \rangle$ for an idempotent $e \in \Lambda$. Then $\mathrm{Hom}_\Lambda(N, \tau_\Lambda M) = 0$ if and only if $\mathrm{Hom}_{\Lambda/I}(N, \tau_{\Lambda/I} M) = 0$.*

We call $M \in \mathrm{mod}\Lambda$ *almost support* τ-tilting Λ-module if there exists an idempotent $e \in \Lambda$ such that M is a τ-rigid $(\Lambda/\langle e \rangle)$-module and $|M| = |\Lambda| - |e\Lambda| - 1$.

Proposition 6. [2, Theorem 2.18] *Any basic almost support τ-tilting Λ-module can be completed to a basic support τ-tilting module in exactly two different ways.*

For any $M, N \in$ sτ-tiltΛ, we write $M \geq N$ if $\mathrm{Fac}(M) \supseteq \mathrm{Fac}(N)$.

Proposition 7. [2, Theorem 2.7] *Let Λ be a finite dimensional K-algebra. Then $\geq$ gives a partial order on* sτ-tiltΛ*.*

By the proposition above, we have an associated Hasse quiver. We recall Hasse quivers.

Definition 8. We define the *Hasse quiver* of sτ-tiltΛ as follows:

- The vertices set is sτ-tiltΛ.
- We draw an arrow from M to N if $M > N$ and there exists no $L \in$ sτ-tiltΛ such that $M > L > N$.

We denote by $\Gamma($sτ-tilt$\Lambda)$ the Hasse quiver of sτ-tiltΛ.

3. Main result I

In this section, we study τ-tilting modules over self-injective Nakayama algebras. As an application of this section, we can easily obtain support τ-tilting modules over self-injective Nakayama algebras.

Throughout this section, the following notation is used. Let $\Lambda := \Lambda_n^r$ be a connected self-injective Nakayama algebra with $|\Lambda| = n$ and the Loewy length $\ell(\Lambda) = r$. Then we have $\Lambda \simeq KC_n/R^r$, where C_n is the cyclic quiver and R is the arrow ideal of KC_n (see [3, V.3.8 Proposition]).

We define an automorphism $\phi : \Lambda \to \Lambda$ by $\phi(e_i) = e_{i+1}$ and $\phi(\alpha_i) = \alpha_{i+1}$ for any $i \in \{1, 2, \cdots, n\}$. Then ϕ induces a functor as follows.

Lemma 9. *The automorphism $\phi : \Lambda \to \Lambda$ induces an equivalence of categories $\Phi : \mathrm{mod}\Lambda \to \mathrm{mod}\Lambda$ such that $\Phi(e_i\Lambda) \simeq e_{i+1}\Lambda$ for any $i \in \{1, 2, \cdots, n\}$. Moreover, for any nonprojective module $M \in \mathrm{mod}\Lambda$, we have $\Phi(M) \simeq \tau M$.*

Let Ψ be a quasi-inverse of Φ. Then we have $\Psi(e_i\Lambda) \simeq e_{i-1}\Lambda$ and $\Psi(M) \simeq \tau^- M$ for any $i \in \{1, 2, \cdots, n\}$ and nonprojective module $M \in \mathrm{mod}\Lambda$. By Remark 4, it is important to construct τ-tilting modules for given an algebra. Our main result of this section is to construct τ-tilting Λ-modules from proper support τ-tilting Λ-module. Proper support τ-tilting Λ-modules are reduced to tilting modules over path algebras of Dynkin quivers of type A with linear orientation. A classification of tilting $(K\vec{A}_l)$-modules is already well-known for any integer $l > 0$. Indeed, there is a bijection

$$\mathrm{tilt}(K\vec{A}_l) \longleftrightarrow \{\text{ triangulations of } (l+2)\text{-gon }\}.$$

Thus we can easily obtain proper support τ-tilting modules over a self-injective Nakayama algebra.

In the rest of the paper, we denote by $\mathrm{mod}_{\mathrm{np}}\Lambda$ the full subcategory of $\mathrm{mod}\Lambda$ consisting Λ-modules which does not have nonzero projective direct summands. we let $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda := \mathrm{ps}\tau\text{-tilt}\Lambda \cap \mathrm{mod}_{\mathrm{np}}\Lambda$, and $\tau\text{-tilt}_{\mathrm{np}}\Lambda := \tau\text{-tilt}\Lambda \cap \mathrm{mod}_{\mathrm{np}}\Lambda$. We decompose $M \in \mathrm{mod}\Lambda$ as $M = M_{\mathrm{np}} \oplus M_{\mathrm{pr}}$, where $M_{\mathrm{np}} \in \mathrm{mod}_{\mathrm{np}}\Lambda$ and M_{pr} is a maximal projective direct summand of M.

we state our main theorem of this section.

Theorem 10. *Let* $\Lambda := \Lambda_n^r$.

(1) *There is a bijection*

$$\tau\text{-tilt}\Lambda \longleftrightarrow \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda$$

given by $\tau\text{-tilt}\Lambda \ni M \mapsto M_{\mathrm{np}} \in \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda$ *and* $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda \ni M \mapsto M \oplus \Phi(e_M\Lambda) \in \tau\text{-tilt}\Lambda$.

(2) *Moreover, if* $r \geq n$, *we have* $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda = \mathrm{ps}\tau\text{-tilt}\Lambda$. *Namely, (1) gives a bijection*

$$\tau\text{-tilt}\Lambda \longleftrightarrow \mathrm{ps}\tau\text{-tilt}\Lambda.$$

As an immediate consequence of Theorem 10, we have the following corollary.

Corollary 11. *The following hold.*

(1) *If* $r \geq n$, *we have*

$$\begin{aligned} \mathrm{s}\tau\text{-tilt}\Lambda &= \{M,\ M \oplus \Phi(e_M\Lambda) \mid M \in \mathrm{ps}\tau\text{-tilt}\Lambda\} \\ &= \bigsqcup_{e \in E_\Lambda} \{M,\ M \oplus \Phi(e\Lambda) \mid M \in \mathrm{tilt}(\Lambda/\langle e\rangle)\}. \end{aligned}$$

(2) *If* $r < n$, *we have*

$$\mathrm{s}\tau\text{-tilt}\Lambda = (\mathrm{ps}\tau\text{-tilt}\Lambda \setminus \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda) \sqcup \{M,\ M \oplus \Phi(e_M\Lambda) \mid M \in \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda\}.$$

In the rest of this section, we give the proof of Theorem 10.

Proposition 12. *If* M *is in* $\mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda$, *then* $M \oplus \Phi(e_M\Lambda)$ *is a* τ*-tilting* Λ*-module.*

Proof. Let $M \in \mathrm{mod}_{\mathrm{np}}\Lambda$ be a τ-tilting $(\Lambda/\langle e\rangle)$-module, where $e := e_M \in E_\Lambda$. Thus M is a τ-rigid Λ-module by Lemma 5. Moreover we have

$$\mathrm{Hom}_\Lambda(\Phi(e\Lambda), \tau_\Lambda M) \simeq \mathrm{Hom}_\Lambda(\Psi\Phi(e\Lambda), \Psi(\tau_\Lambda M)) \simeq \mathrm{Hom}_\Lambda(e\Lambda, M) = 0$$

and

$$|M \oplus \Phi(e\Lambda)| = |M| + |\Phi(e\Lambda)| = |M| + |e\Lambda| = |\Lambda|$$

by Lemma 9 and $M \in \mathrm{mod}_{\mathrm{np}}\Lambda$. Thus $M \oplus \Phi(e_M\Lambda)$ is a τ-tilting Λ-module. □

Conversely, we shall construct a proper support τ-tilting Λ-module for a given τ-tilting Λ-module.

Proposition 13. *Assume that $M \in \mathrm{mod}\Lambda$ is not in $\mathrm{mod}_{\mathrm{np}}\Lambda$. If M is a τ-tilting Λ-module, then M_{np} is a proper support τ-tilting Λ-module.*

Proof. Let M is a τ-tilting Λ-module and not in $\mathrm{mod}_{\mathrm{np}}\Lambda$. We decompose M as $M = M_{\mathrm{np}} \oplus M_{\mathrm{pr}}$ and assume $M_{\mathrm{pr}} = e\Lambda$, where $e \in E_\Lambda$ is an idempotent. Then M_{np} is trivially a τ-rigid Λ-module. Since M is a τ-tilting Λ-module, we have

$$\mathrm{Hom}_\Lambda(\phi^{-1}(e)\Lambda, M_{\mathrm{np}}) \simeq \mathrm{Hom}_\Lambda(\Phi(\phi^{-1}(e)\Lambda), \Phi(M_{\mathrm{np}})) \simeq \mathrm{Hom}_\Lambda(e\Lambda, \tau_\Lambda M) = 0$$

by Lemma 9, and

$$|M_{\mathrm{np}}| = |M| - |e\Lambda| = |\Lambda| - |\phi^{-1}(e)\Lambda|.$$

Thus M_{np} is a τ-tilting $(\Lambda/\langle\phi^{-1}(e)\rangle)$-module or proper support τ-tilting Λ-module by Lemma 5. $\square$

By Proposition 12 and 13, there is a bijection

$$\tau\text{-tilt}\Lambda \setminus \tau\text{-tilt}_{\mathrm{np}}\Lambda \longleftrightarrow \mathrm{ps}\tau\text{-tilt}_{\mathrm{np}}\Lambda.$$

To complete the proof of Theorem 10, we have only to show that any τ-tilting Λ-module always has a nonzero projective Λ-module as a direct summand.

We need the following lemma.

Lemma 14. *Let $X, Y \in \mathrm{mod}\Lambda$ be indecomposable with the Loewy length $\ell(X) \geq \ell(Y)$, and P_X a projective cover of X. Then $\mathrm{Hom}_\Lambda(X, Y) = 0$ if and only if $\mathrm{Hom}_\Lambda(P_X, Y) = 0$.*

Proposition 15. *Each τ-tilting Λ-module has a nonzero projective Λ-module as a direct summand.*

Proof. Let $M = X \oplus N$ be a τ-tilting Λ-module such that X is indecomposable and the Loewy length $\ell(X) \geq \ell(N)$. In particular, we have $\ell(N) = \ell(\tau N)$ because Λ is Nakayama. By the definition, N is an almost support τ-tilting Λ-module. Assume that M has no projective Λ-module as a direct summand. Since M is τ-rigid, $\mathrm{Hom}_\Lambda(X, \tau N)$ vanishes. By Lemma 14, we have $\mathrm{Hom}_\Lambda(P_X, \tau N) = 0$, where P_X is a projective cover of X. Since Λ is a Nakayama algebra, P_X is indecomposable. Therefore we have

$$|P_X \oplus N| = |P_X| + |N| = |P_X| + |M| - |X| = |M| = |\Lambda|.$$

Namely, $P_X \oplus N$ is a τ-tilting Λ-module. Moreover, N is a support τ-tilting Λ-module by Proposition 13. This means that almost support τ-tilting Λ-module N has pairwise nonisomorphic 3 support τ-tilting Λ-modules $N, X \oplus N$ and $P_X \oplus N$. By Proposition 6, this is contradiction. $\square$

Now we are ready to prove Theorem 10.

Proof of Theorem 10. (1) It follows from Proposition 12, 13 and 15.

(2) One can show that any proper support τ-tilting Λ-module has no projective Λ-module as a direct summand. $\square$

As an application of Theorem 10, we can easily calculate τ-tilting modules over self-injective Nakayama algebras.

Finally, we give a example.

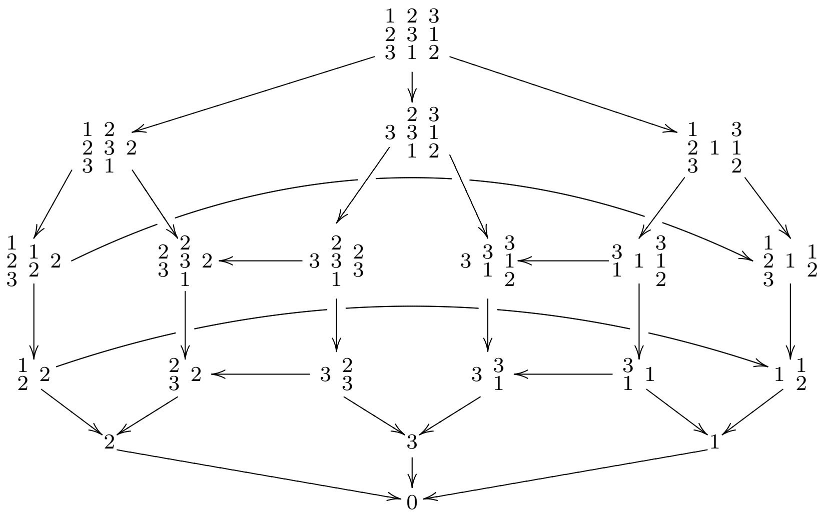
Example 16. Let $\Lambda := \Lambda_3^3$. To obtain τ-tilting Λ-modules, we need to know factor algebras $\Lambda/\langle e\rangle$ for any idempotent $e \in E_\Lambda$. Indeed, we have $\Lambda/\langle e_i\rangle \simeq K\vec{A}_2$, $\Lambda/\langle e_i + e_j\rangle \simeq K\vec{A}_1$, and $\Lambda/\langle e_1+e_2+e_3\rangle = \{0\}$ for $i, j \in \{1, 2, 3\}$. Thus proper support τ-tilting modules are given as follows:

$$\begin{aligned}
\tau\text{-tilt}(\Lambda/\langle e_3\rangle) &= \operatorname{tilt}(K\vec{A}_2) = \{\begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2, \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1\}\\
\tau\text{-tilt}(\Lambda/\langle e_2+e_3\rangle) &= \operatorname{tilt}(K\vec{A}_1) = \{1\}\\
\tau\text{-tilt}(\Lambda/\langle e_1+e_2+e_3\rangle) &= \{0\}
\end{aligned}$$

and cyclic permutation. By Theorem 10, we have

$$\begin{aligned}
\mathrm{s}\tau\text{-tilt}\Lambda = \{ & \{0\}, 1, 2, 3, \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2, \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1, \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3, \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2, \begin{smallmatrix}3\\1\end{smallmatrix} \oplus 1, \begin{smallmatrix}3\\1\end{smallmatrix} \oplus 3\}\\
\sqcup \{ & \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix}, 1 \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix} \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix}, 2 \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix} \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}, 3 \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix} \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix},\\
& \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}1\\2\end{smallmatrix} \oplus 1 \oplus \begin{smallmatrix}1\\2\\3\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix}, \begin{smallmatrix}2\\3\end{smallmatrix} \oplus 2 \oplus \begin{smallmatrix}2\\3\\1\end{smallmatrix}, \begin{smallmatrix}3\\1\end{smallmatrix} \oplus 1 \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix}, \begin{smallmatrix}3\\1\end{smallmatrix} \oplus 3 \oplus \begin{smallmatrix}3\\1\\2\end{smallmatrix}\}
\end{aligned}$$

and the Hasse quiver $\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda)$ as follows:



4. Main result II

In this section, we give a combinatorial method to provide Hasse quivers of support τ-tilting modules over Nakayama algebras. Then Rejection Lemma of Drozd-Kirichenko plays important role.

Let Λ be a finite dimensional K-algebra (not necessarily Nakayama). The following lemma is called Rejection Lemma of Drozd-Kirichenko[5].

Lemma 17 (Rejection Lemma of Drozd-Kirichenko)**.** *Let Λ be a finite dimensional K-algebra, and Q a projective-injective indecomposable summand of Λ. Then the following hold.*

(1) *$I := \operatorname{soc}(Q)$ is a two-sided ideal of Λ.*

(2) *There exists a one-to-one correspondence between* $\mathrm{ind}(\Lambda/I)$ *and* $\mathrm{ind}(\Lambda) \setminus \{Q\}$.

From now, we always assume that Q is a projective-injective indecomposable summand of Λ, and $I := \mathrm{soc}(Q)$. In the rest of the paper, we denote by $\mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)$ (respectively, $\mathrm{s}\tau\text{-tilt}^{I}(\Lambda/I)$) the subset of $\mathrm{s}\tau\text{-tilt}(\Lambda/I)$ consisting Λ-modules which have Q/I as a direct summand (respectively, does not have I as a composition factor). We let $\mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I) := \mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I) \cap \mathrm{s}\tau\text{-tilt}^{I}(\Lambda/I)$ and $\mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda := \{M \in \mathrm{s}\tau\text{-tilt}\Lambda \mid \mathrm{bas}(M \otimes_{\Lambda} \Lambda/I) \in \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)\}$, where $\mathrm{bas}(X)$ means a basic part of $X \in \mathrm{mod}\Lambda$.

The following theorem is very important.

Theorem 18. *Let Λ be a finite dimensional K-algebra, Q be a projective-injective indecomposable summand of Λ, and $I := \mathrm{soc}(Q)$.*

(1) *The map $M \mapsto \mathrm{bas}(M \otimes_{\Lambda} \Lambda/I)$ gives a surjection*

$$\mathrm{s}\tau\text{-tilt}\Lambda \twoheadrightarrow \mathrm{s}\tau\text{-tilt}(\Lambda/I)$$

which preserves the partial orders. Moreover, the restriction gives a bijection

$$\mathrm{s}\tau\text{-tilt}\Lambda \setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda \longleftrightarrow \mathrm{s}\tau\text{-tilt}(\Lambda/I) \setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)$$

where the inverse is given by

$$\begin{gathered}\mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I) \setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I) \ni Q/I \oplus U \mapsto Q \oplus U \in \mathrm{s}\tau\text{-tilt}\Lambda \setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda \\ \mathrm{s}\tau\text{-tilt}(\Lambda/I) \setminus \mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I) \ni N \mapsto N \in \mathrm{s}\tau\text{-tilt}\Lambda \setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda.\end{gathered}$$

(2) *We have*

$$\mathrm{s}\tau\text{-tilt}\Lambda = (\mathrm{s}\tau\text{-tilt}\Lambda \setminus \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda) \sqcup \{N, Q \oplus N \mid N \in \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)\}.$$

By Theorem 18, we can recover $\mathrm{s}\tau\text{-tilt}\Lambda$ from $\mathrm{s}\tau\text{-tilt}(\Lambda/I)$. Moreover, since the map preserves the partial orders, Hasse quivers of $\mathrm{s}\tau\text{-tilt}\Lambda$ and $\mathrm{s}\tau\text{-tilt}(\Lambda/I)$ are almost same. Thus, as a result of Theorem 18, we have two corollaries for a construction of the Hasse quiver $\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda)$.

If any $M \in \mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)$ has I as a composition factor, we have $\mathrm{s}\tau\text{-tilt}^{I}_{Q/I}\Lambda = \emptyset$. Thus we have a bijection between $\mathrm{s}\tau\text{-tilt}\Lambda$ and $\mathrm{s}\tau\text{-tilt}(\Lambda/I)$.

Corollary 19. *If Q/I has I as a composition factor, then the map of Theorem 18 is a bijection. In particular, there exists a quiver isomorphism*

$$\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda) \longrightarrow \Gamma(\mathrm{s}\tau\text{-tilt}(\Lambda/I)).$$

Assume that $X \geq N$ in $\mathrm{s}\tau\text{-tilt}(\Lambda/I)$ and $N \in \mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)$. Then we remark that X is also in $\mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)$.

Corollary 20. $\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda)$ *is obtained from* $\Gamma(\mathrm{s}\tau\text{-tilt}(\Lambda/I))$ *by the following two steps: First we replace any arrow $X \to N$ in $\Gamma(\mathrm{s}\tau\text{-tilt}(\Lambda/I))$ satisfying $N \in \mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)$ as follows:*

- *If X is in* $\mathrm{s}\tau\text{-tilt}_{Q/I}(\Lambda/I)$ *but not in* $\mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)$,

$$\textit{replacing}\quad \begin{tikzcd} X \arrow[d] \\ N \end{tikzcd} \quad \textit{by a subquiver} \quad \begin{tikzcd} Q\oplus (X/(Q/I)) \arrow[d] \\ Q\oplus N \arrow[d] \\ N \end{tikzcd}\ .$$

- *If X is in* $\mathrm{s}\tau\text{-tilt}^{I}_{Q/I}(\Lambda/I)$,

$$\textit{replacing}\quad \begin{tikzcd} X \arrow[d] \\ N \end{tikzcd} \quad \textit{by a subquiver} \quad \begin{tikzcd} & Q\oplus X \arrow[dl] \arrow[dr] & \\ X \arrow[dr] & & Q\oplus N. \arrow[dl] \\ & N & \end{tikzcd}$$

Finally we replace other vetices by the bijection of Theorem 18(1).

From now, we assume that Λ is Nakayama with $n = |\Lambda|$. Let Q be a projective-injective indecomposable summand of Λ, and $I := \mathrm{soc}(Q)$. If the Loewy length of Q is bigger than n or $\ell(Q/I) \geq n$, then Q/I is sincere. Namely, Q/I has I as a composition factor. Then we have a quiver isomorphism $\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda) \to \Gamma(\mathrm{s}\tau\text{-tilt}(\Lambda/I))$ by Corollary 19.

On the other hand, if the Loewy length of Q is not bigger than n or $\ell(Q/I) < n$, then Q/I does not have I as a composition factor. In this case, by Corollary 20, we can construct the Hasse quiver of Λ from Λ/I.

Since Nakayama algebras have a projective-injective indecomposable module and its factor algebras is also Nakayama (see [3, V.3.3 Lemma and V.3.4 Lemma]), we can iteratively apply the rejection lemma to Nakayama algebras.

Let $\Lambda_0 := \Lambda$ be a Nakayama algebra with $n = |\Lambda|$. By iteratively applying the rejection lemma, we have a sequence of Nakayama algebras

$$\cdots \twoheadrightarrow \Lambda_{-2} \twoheadrightarrow \Lambda_{-1} \twoheadrightarrow \Lambda_0 \twoheadrightarrow \Lambda_1 \twoheadrightarrow \cdots \twoheadrightarrow \Lambda_m = K^n$$

such that $\Lambda_i := \Lambda_{i-1}/I_{i-1}$ and Λ_m is a semisimple algebra K^n, where Q_i is a projective-injective indecomposable Λ_i-module, $I_i := \mathrm{soc}(Q_i)$ and $m > 0$ is an integer. Thus we always can construct the Hasse quiver of any Nakayama algebra from some semisimple algebra by the observation above.

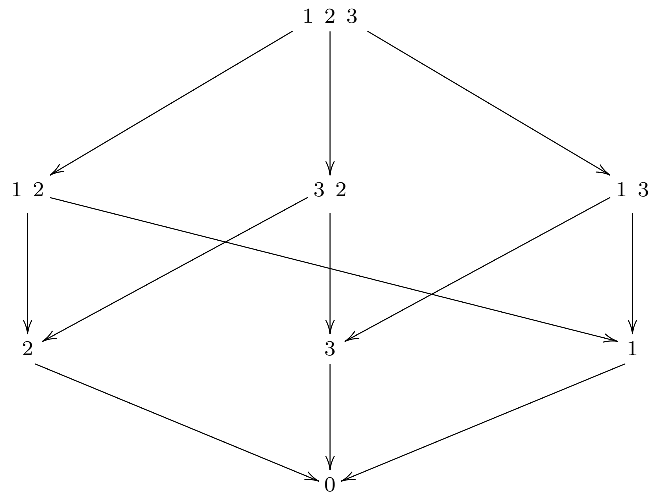
Theorem 21. *Let Λ be a Nakayama algebra with $n = |\Lambda|$. Then $\Gamma(\mathrm{s}\tau\text{-tilt}\Lambda)$ is obtained from $\Gamma(\mathrm{s}\tau\text{-tilt}(K^n))$.*

Example 22. Let $\Lambda_0 := \Lambda_3^3$ be a self-injecitve Nakayama algebra. Then we have a sequence of Nakayama algebras

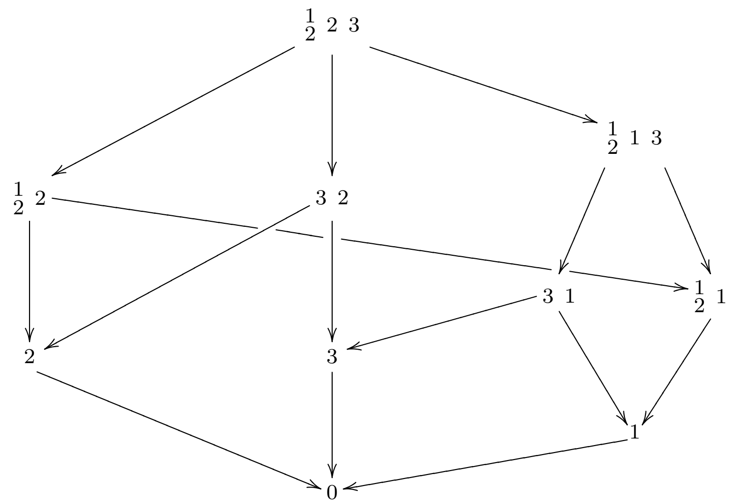
$$\Lambda_0 \twoheadrightarrow \Lambda_1 \twoheadrightarrow \Lambda_2 \twoheadrightarrow \Lambda_3 \twoheadrightarrow \Lambda_4 \twoheadrightarrow \Lambda_5 \twoheadrightarrow \Lambda_6$$

by the rejection lemma. Thus we have Hasse quivers from K^3 to Λ_3^3 by Theorem 21.

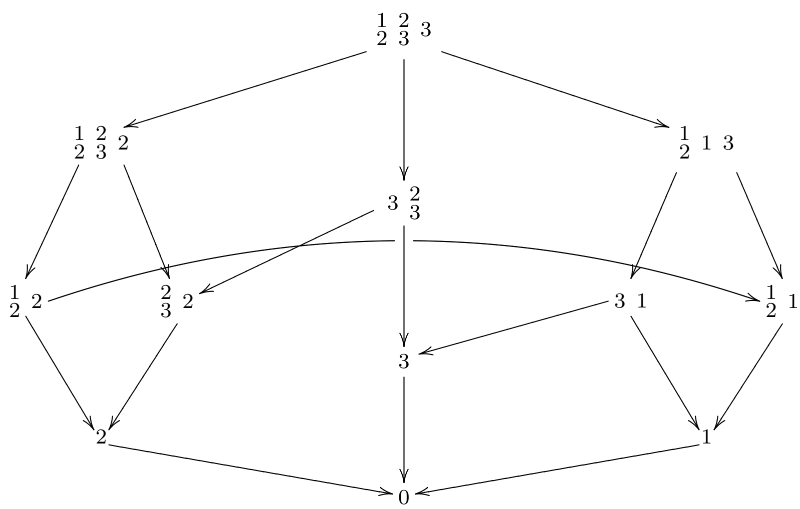
(1) $\Lambda_6 = K^3$



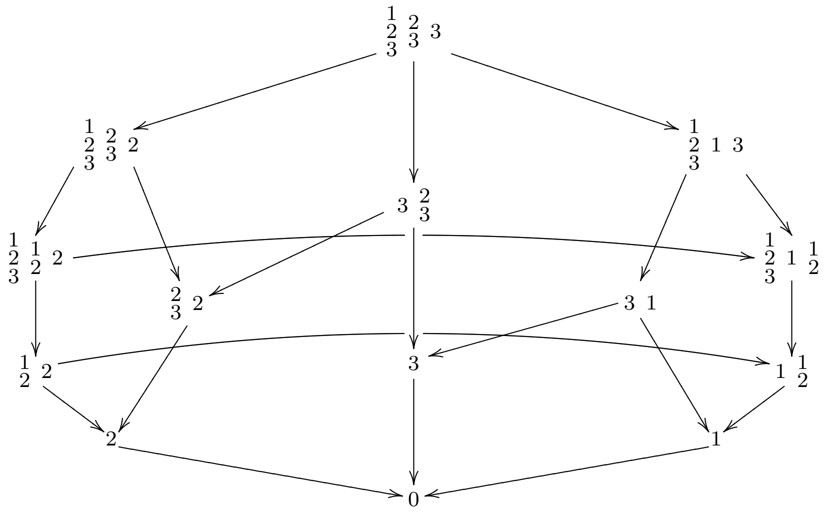
(2) $\Lambda_5 = K(\;3 \qquad 2 \longleftarrow 1\;)$



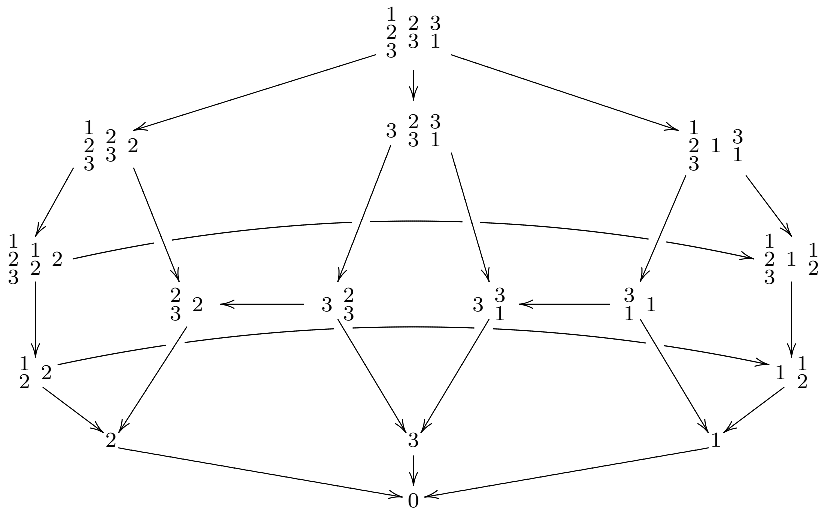
(3) $\Lambda_4 = K(\;3 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1\;)/\langle \alpha_1\alpha_2 \rangle$



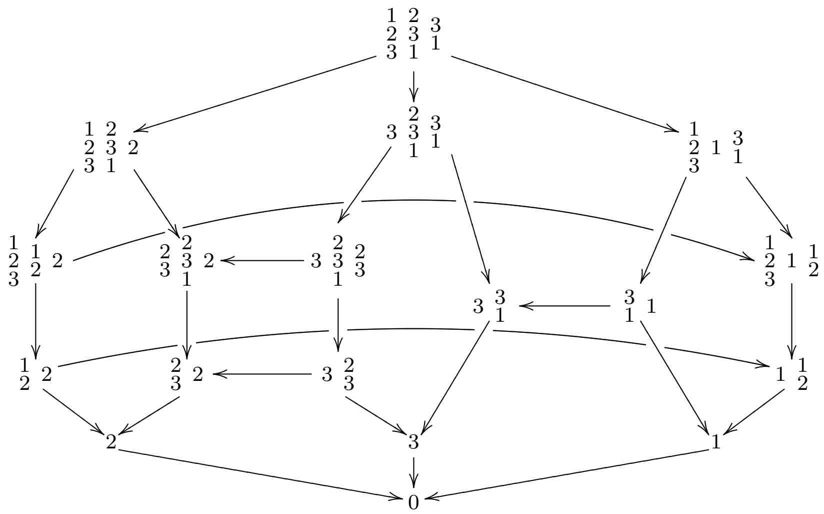
(4) $\Lambda_3 = K(\ 3 \xleftarrow{\alpha_2} 2 \xleftarrow{\alpha_1} 1\ )$



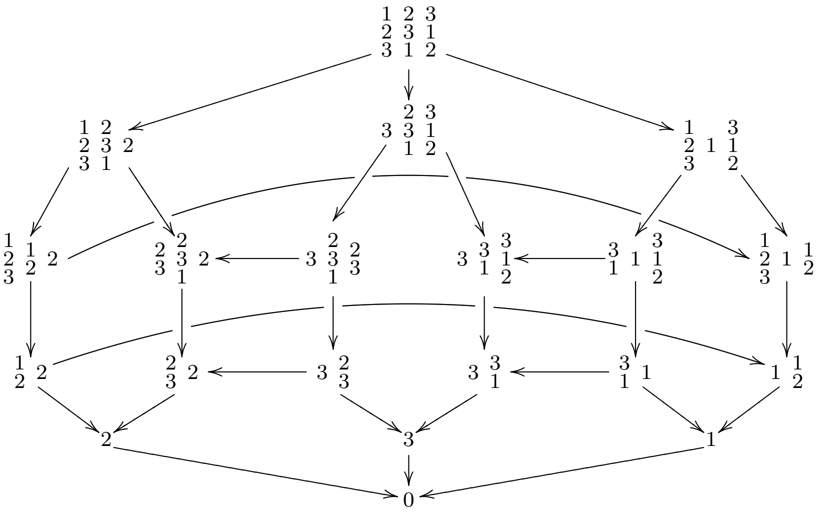
(5) $\Lambda_2 = KC_n/\langle \alpha_2\alpha_3, \alpha_3\alpha_1 \rangle$



(6) $\Lambda_1 = KC_n/\langle \alpha_1\alpha_2\alpha_3, \alpha_3\alpha_1 \rangle$



(7) $\Lambda_0 = \Lambda_3^3$



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Graduate School of Mathematics
Nagoya University
Frocho, Chikusaku, Nagoya 464-8602 Japan

E-mail address: `m09002b@math.nagoya-u.ac.jp`