DIMENSIONS OF TRIANGULATED CATEGORIES WITH RESPECT TO SUBCATEGORIES

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ABSTRACT. We introduce the concept of the dimension of a triangulated category with respect to a fixed full subcategory. For the bounded derived category of an abelian category, upper bounds of the dimension with respect to a contravariantly finite subcategory are given. Our methods not only recover some known results on the dimensions of derived categories in the sense of Rouquier, but also apply to various commutative and non-commutative noetherian rings.

1. INTRODUCTION

This is a joint work with T. Aihara, O. Iyama, R. Takahashi and M. Yoshiwaki [1]. The notion of the dimension of a triangulated category has been introduced by Rouquier [14] based on work of Bondal and Van den Bergh [9] on Brown representability. It measures how many extensions are needed to build the triangulated category out of a single object, up to finite direct sum, direct summand and shift. First of all, we recall its definition.

Definition 1. Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}, \mathcal{Y}$ be subcategories of $\mathcal{T}$.

(1) We denote by $\mathcal{X} * \mathcal{Y}$ the subcategory of $\mathcal{T}$ consisting of objects M that admit triangles $X \to M \to Y \to X[1]$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. Then $(\mathcal{X} * \mathcal{Y}) * \mathcal{Z} = \mathcal{X} * (\mathcal{Y} * \mathcal{Z})$ holds by octahedral axiom.

(2) Set $\langle \mathcal{X} \rangle := \operatorname{add}\{X[i] \mid X \in \mathcal{X}, i \in \mathbb{Z}\}$. For a positive integer n, let

$$\langle \mathcal{X} \rangle_n^{\mathcal{T}} = \langle \mathcal{X} \rangle_n := \operatorname{add}(\underbrace{\langle \mathcal{X} \rangle * \langle \mathcal{X} \rangle * \cdots * \langle \mathcal{X} \rangle}_{n}).$$

Clearly $\langle \mathcal{X} \rangle_n$ is closed under shifts. For an object M of $\mathcal{T}$, we set

$$\langle M \rangle_n := \langle \operatorname{add} M \rangle_n.$$

(3) The *(triangle) dimension* of $\mathcal{T}$ is defined as

$$\operatorname{tri.dim} \mathcal{T} := \inf\{n \geq 0 \mid \mathcal{T} = \langle M \rangle_{n+1},\ \exists M \in \mathcal{T}\}.$$

We give an example.

Example 2. Let R be an artinian local ring with a maximal ideal $\mathfrak{m}$ and a residue class field $k = R/\mathfrak{m}$. Since R is artin, there exists a positive integer ℓ such that $\mathfrak{m}^\ell = 0$. In this case, We have $\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) \leq \ell - 1$. Indeed, let X be a bounded complex on R. Then the short exact sequence $0 \to \mathfrak{m}^i X \to \mathfrak{m}^{i-1} X \to \mathfrak{m}^{i-1} X/\mathfrak{m}^i X \to 0$ of complexes induces the exact triangle $\mathfrak{m}^i X \to \mathfrak{m}^{i-1} X \to \mathfrak{m}^{i-1} X/\mathfrak{m}^i X \to \mathfrak{m}^i X[1]$ for each i. Since $\mathfrak{m}^{i-1}X/\mathfrak{m}^i X$ is annihilated by $\mathfrak{m}$, it is isomorphic to $\bigoplus_i k^{\oplus}[i]$, and we have $\mathfrak{m}^{i-1}X/\mathfrak{m}^i X \in \langle k \rangle_1$. On

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the other hand, we can see that $\mathfrak{m}^{\ell-i}X$ belongs to $\langle k\rangle_i$ by induction on i. Thus we get $X = \mathfrak{m}^0 X$ belongs to $\langle k\rangle_\ell$.

We give the definition of the *(triangle) dimension* of triangulated category with respect to a subcategory.

Definition 3. Let $\mathcal{T}$ be a triangulated category and $\mathcal{X}$ be a fullsubcategory of $\mathcal{T}$. Then we define

$$\mathcal{X}\text{-}\operatorname{tri.dim}\mathcal{T} := \inf\{n \geq 0 \mid \mathcal{T} = \langle\mathcal{X}\rangle_{n+1}\}.$$

2. Main results

First of this section, we give some basic definitions and preliminary results.

Let $\mathcal{X}$ be an additive category. An $\mathcal{X}$-module is an additive contravariant functor from $\mathcal{X}$ to the category of abelian groups. A morphism between $\mathcal{X}$-modules is a natural transformation. For any object $X \in \mathcal{X}$, the functor $\operatorname{Hom}_{\mathcal{X}}(-,X)$ is an $\mathcal{X}$-module. We say that an $\mathcal{X}$-module F is *finitely presented* if there is an exact sequence $\operatorname{Hom}_{\mathcal{X}}(-,X_1) \to \operatorname{Hom}_{\mathcal{X}}(-,X_0) \to F \to 0$ with $X_0, X_1 \in \mathcal{X}$ [3, 16]. The category of finitely presented $\mathcal{X}$-modules is denoted by $\operatorname{mod}\mathcal{X}$. The assignment $X \mapsto \operatorname{Hom}_{\mathcal{X}}(-,X)$ makes a fully faithful functor $\mathcal{X} \to \operatorname{mod}\mathcal{X}$, which is called the *Yoneda embedding* of $\mathcal{X}$.

We recall here a well-known criterion for $\operatorname{mod}\mathcal{X}$ to be abelian. Let $\mathcal{X}$ be an additive category and $f : X \to Y$ be a morphism in $\mathcal{X}$. A morphism $g : Z \to X$ in $\mathcal{X}$ is called a *pseudo-kernel* if $\operatorname{Hom}_{\mathcal{X}}(-,Z) \to \operatorname{Hom}_{\mathcal{X}}(-,X) \to \operatorname{Hom}_{\mathcal{X}}(-,Y)$ is exact on $\mathcal{X}$. We say that $\mathcal{X}$ *has pseudo-kernels* if all morphisms in $\mathcal{X}$ have pseudo-kernels.

Proposition 4. [4] *Let $\mathcal{X}$ be an additive category. Then* $\operatorname{mod}\mathcal{X}$ *is an abelian category if and only if $\mathcal{X}$ has pseudo-kernels.*

We give a class of additive categories having pseudo-kernels. We say that a subcategory $\mathcal{X}$ of an additive category $\mathcal{A}$ is *contravariantly finite* if for any object $M \in \mathcal{A}$ there exist $X \in \mathcal{X}$ and a morphism $f : X \to M$ such that $\operatorname{Hom}_{\mathcal{A}}(X', f)$ is surjective for all $X' \in \mathcal{X}$ [8].

Example 5. Let $\mathcal{A}$ be an additive category and $\mathcal{X}$ be a contravariantly finite subcategory of $\mathcal{A}$. If $\mathcal{A}$ has pseudo-kernels, then $\mathcal{X}$ also has pseudo-kernels. Hence if $\mathcal{A}$ is an abelian category, then so is $\operatorname{mod}\mathcal{X}$.

Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ be a subcategory of $\mathcal{A}$. We say that $\mathcal{X}$ *generates* $\mathcal{A}$ if for any object M of $\mathcal{A}$ there is an epimorphism $X \to M$ with $X \in \mathcal{X}$.

Now we can state the main result.

Theorem 6. *Let $\mathcal{A}$ be an abelian category and $\mathcal{X}$ a contravariantly finite subcategory which generates $\mathcal{A}$. Then there is an inequality* $\mathcal{X}\text{-}\operatorname{tri.dim}\mathsf{D}^{\mathrm{b}}(\mathcal{A}) \leq \operatorname{gl.dim}(\operatorname{mod}\mathcal{X})$.

In representation theory, the notion of tilting modules/complexes plays an important role to control derived categories [13]. Its dual notion of cotilting modules was studied by Auslander and Reiten as a non-commutative generalization of canonical modules over commutative rings [5, 6, 7]. Now, we apply the results above to rings admitting cotilting modules. Let us begin with recalling the definition of a cotilting module.

Definition 7. Let A be a noetherian ring and T be a finitely generated A-module. Denote by $\mathcal{X}_T$ the subcategory of $\operatorname{mod} A$ consisting of modules X with $\operatorname{Ext}^i_A(X, T) = 0$ for all $i > 0$. We call T *cotilting* if it satisfies the following three conditions.

(1) The injective dimension of the A-module T is finite.

(2) $\operatorname{Ext}^i_A(T, T) = 0$ for all $i > 0$ (i.e., $T \in \mathcal{X}_T$).

(3) For any $X \in \mathcal{X}_T$, there exists an exact sequence $0 \to X \to T' \to X' \to 0$ in $\operatorname{mod} A$ with $T' \in \operatorname{add} T$ and $X' \in \mathcal{X}_T$.

Example 8. (1) Let R be a commutative Cohen-Macaulay local ring with a canonical module ω_R. We denote by $\mathrm{CM}(R)$ the category of maximal Cohen-Macaulay R-modules. Then ω_R is a cotilting module over R and $\mathcal{X}_{\omega_R} = \mathrm{CM}(R)$ holds. Let Λ be an R-order. For any tilting Λ^{op}-module T in the sense of Miyashita [12] with $T \in \mathrm{CM}(R)$, the Λ-module $\operatorname{Hom}_R(T, \omega_R)$ is cotilting. For the cotilting Λ-module $\omega_\Lambda := \operatorname{Hom}_R(\Lambda, \omega_R)$ it holds that $\mathcal{X}_{\omega_\Lambda} = \mathrm{CM}(\Lambda)$.

(2) Let Λ be an Iwanaga-Gorenstein ring. Then Λ is a cotilting module over Λ, and hence $\mathcal{X}_\Lambda = \mathrm{CM}(\Lambda)$.

Let R and Λ be as above. We set $A := R$ or Λ. Let T be a cotilting A-module. It comes from Auslander-Buchweitz approximation theory [5], we can see that the subcategory $\mathcal{X}_T$ of $\operatorname{mod} A$ is a contravariantly finite subcategory which generates $\operatorname{mod} A$.

Immediately we have the following inequality, which is a special case of [10].

Proposition 9. *Let T be a cotilting module of A. Then one has*

$$\operatorname{gl.dim}(\operatorname{mod} \mathcal{X}_T) \leq \max\{2, \operatorname{inj.dim} T\}.$$

Let R be a commutative Cohen-Macaulay local ring with a canonical module ω_R. Since the injective dimension of ω_R is equal to the Krull dimension of R, we obtain the following corollary.

Corollary 10. *Let R be a commutative Cohen-Macaulay local ring with a canonical module. Then one has*

$$\mathrm{CM}(R)\text{-}\operatorname{tri.dim} \mathsf{D}^{\mathrm{b}}(\operatorname{mod} R) \leq \max\{1, \dim R\}.$$

References

[1] T. Aihara; T. Araya; O. Iyama; R. Takahashi; Michio Yoshiwaki, Dimensions of triangulated categories with respect to subcategories, preprint (2012), `arxiv:1204.6421`.

[2] T. Aihara; R. Takahashi, Generators and dimensions of derived categories, preprint (2011), `arxiv:1106.0205`.

[3] I. Assem; D. Simson; A. Skowroński, Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory, London Mathematical Society Student Texts 65, *Cambridge University Press, Cambridge*, 2006.

[4] M. Auslander, Coherent functors, 1966 Proc. Conf. Categorical Algebra (La Jolla, Calif., 1965) pp. 189–231 Springer, New York.

[5] M. Auslander; R.-O. Buchweitz, The homological theory of maximal Cohen-Macaulay approximations, Colloque en l'honneur de Pierre Samuel (Orsay, 1987). Mem. Soc. Math. France (N.S.) No. 38 (1989), 5–37.

[6] M. Auslander; I. Reiten, Applications of contravariantly finite subcategories, *Adv. Math.* **86** (1991), no. 1, 111–152.

[7] M. Auslander; I. Reiten, Cohen-Macaulay and Gorenstein Artin algebras, *Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991)*, 221–245, *Progr. Math.*, 95, *Birkhauser, Basel*, 1991.

[8] M. Auslander; S. O. Smalø, Almost split sequences in subcategories, *J. Algebra* **69** (1981) 426–454.

[9] A. Bondal; M. van den Bergh, Generators and representability of functors in commutative and noncommutative geometry, *Mosc. Math. J.* **3** (2003), no. 1, 1–36, 258.

[10] O. Iyama, Auslander correspondence, *Adv. Math.* **210** (2007) 51–82.

[11] H. Krause; D. Kussin, Rouquier's theorem on representation dimension, Trends in representation theory of algebras and related topics, 95–103, Contemp. Math., 406, *Amer. Math. Soc., Providence, RI*, 2006.

[12] Y. Miyashita, Tilting modules of finite projective dimension, *Math. Z.* **193** (1986), no. 1, 113–146.

[13] J. Rickard, Morita theory for derived categories, *J. London Math. Soc.* (2) **39** (1989), no. 3, 436–456.

[14] R. Rouquier, Dimensions of triangulated categories, *J. K-Theory* **1** (2008), 193–256.

[15] R. Takahashi, Classifying thick subcategories of the stable category of Cohen-Macaulay modules, *Adv. Math.* **225** (2010), no. 4, 2076–2116.

[16] Y. Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, London Mathematical Society Lecture Note Series, 146, *Cambridge University Press, Cambridge*, 1990.

[17] M. Yoshiwaki, On self-injective algebras of stable dimension zero, *Nagoya Math. J.* **203** (2011), 101–108.

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