DERIVED EQUIVALENCE CLASSIFICATION OF GENERALIZED MULTIFOLD EXTENSIONS OF PIECEWISE HEREDITARY ALGEBRAS OF TREE TYPE

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Abstract. We give a derived equivalence classification of algebras of the form $\hat{A}/\langle\phi\rangle$ for some piecewise hereditary algebra A of tree type and some automorphism ϕ of $\hat{A}$ such that $\phi(A^{[0]}) = A^{[n]}$ for some positive integer n.

Introduction

Throughout this note we fix an algebraically closed field $\Bbbk$, and assume that all algebras are basic and finite-dimensional $\Bbbk$-algebras and that all categories are $\Bbbk$-categories.

Let A be an algebra and n a positive integer. Then an algebra of the form $T^n_\psi(A) := \hat{A}/\langle\hat{\psi}\nu_A^n\rangle$ for some automorphism ψ of A is called a *twisted n-fold extension* of A. Further an algebra of the form $\hat{A}/\langle\phi\rangle$ for some automorphism ϕ of $\hat{A}$ with jump n is called a *generalized n-fold extension* of A, where ϕ is called an automorphism with *jump n* if $\phi(A^{[0]}) = A^{[n]}$. Since obviously $\hat{\psi}\nu_A^n$ is an automorphism with jump n, we see that twisted n-fold extensions are generalized n-fold extensions. An algebra is called a *generalized* (*resp. twisted*) *multifold extension* if it is a generalized (resp. twisted) n-fold extension for some positive integer n. In [3], we gave the derived equivalence classification of twisted multifold extensions of piecewise hereditary algebras of tree type by giving a complete invariant. In this note we extend this result to generalized multifold extensions of piecewise hereditary algebras of tree type.

1. Preliminaries

For a category R we denote by R_0 and R_1 the class of objects and morphisms of R, respectively. A category R is said to be *locally bounded* if it satisfies the following:

- Distinct objects of R are not isomorphic;
- $R(x, x)$ is a local algebra for all $x \in R_0$;
- $R(x, y)$ is finite-dimensional for all $x, y \in R_0$; and
- The set $\{y \in R_0 \mid R(x, y) \neq 0 \text{ or } R(y, x) \neq 0\}$ is finite for all $x \in R_0$.

A category is called *finite* if it has only a finite number of objects.

A pair (A, E) of an algebra A and a complete set $E := \{e_1, \dots, e_n\}$ of orthogonal primitive idempotents of A can be identified with a locally bounded and finite category R by the following correspondences. Such a pair (A, E) defines a category $R_{(A,E)} := R$ as follows: $R_0 := E$, $R(x, y) := yAx$ for all $x, y \in E$, and the composition of R is

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defined by the multiplication of A. Then the category R is locally bounded and finite. Conversely, a locally bounded and finite category R defines such a pair (A_R, E_R) as follows: $A_R := \bigoplus_{x,y\in R_0} R(x,y)$ with the usual matrix multiplication (regard each element of A as a matrix indexed by R_0), and $E_R := \{(\mathbb{1}_x \delta_{(i,j),(x,x)})_{i,j\in R_0} \mid x \in R_0\}$. We always regard an algebra A as a locally bounded and finite category by fixing a complete set A_0 of orthogonal primitive idempotents of A.

Definition 1.1. Let A be an algebra.

(1) The *repetition* $\hat{A}$ of A is a $\Bbbk$-category defined as follows ($\hat{A}$ turns out to be locally bounded):

- $\hat{A}_0 := A_0 \times \mathbb{Z} = \{x^{[i]} := (x,i) \mid x \in A_0, i \in \mathbb{Z}\}$.
- $\hat{A}(x^{[i]}, y^{[j]}) := \begin{cases} \{f^{[i]} \mid f \in A(x,y)\} & \text{if } j = i, \\ \{\phi^{[i]} \mid \phi \in DA(y,x)\} & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases}$ for all $x^{[i]}, y^{[j]} \in \hat{A}_0$.
- For each $x^{[i]}, y^{[j]}, z^{[k]} \in \hat{A}_0$ the composition $\hat{A}(y^{[j]}, z^{[k]}) \times \hat{A}(x^{[i]}, y^{[j]}) \to \hat{A}(x^{[i]}, z^{[k]})$ is given as follows.
  - (i) If $i = j, j = k$, then this is the composition of A $A(y,z) \times A(x,y) \to A(x,z)$.
  - (ii) If $i = j, j+1 = k$, then this is given by the right A-module structure of DA: $DA(z,y) \times A(x,y) \to DA(z,x)$.
  - (iii) If $i+1 = j, j = k$, then this is given by the left A-module structure of DA: $A(y,z) \times DA(y,x) \to DA(z,x)$.
  - (iv) Otherwise, the composition is zero .

(2) We define an automorphism ν_A of $\hat{A}$, called the *Nakayama automorphism* of $\hat{A}$, by $\nu_A(x^{[i]}) := x^{[i+1]}$, $\nu_A(f^{[i]}) := f^{[i+1]}$, $\nu_A(\phi^{[i]}) := \phi^{[i+1]}$ for all $i \in \mathbb{Z}, x \in A_0, f \in A_1, \phi \in \bigcup_{x,y\in A_0} DA(y,x)$.

(3) For each $n \in \mathbb{Z}$, we denote by $A^{[n]}$ the full subcategory of $\hat{A}$ formed by $x^{[n]}$ with $x \in A$, and by $\mathbb{1}^{[n]} : A \xrightarrow{\sim} A^{[n]} \hookrightarrow \hat{A}, x \mapsto x^{[n]}$, the embedding functor.

We cite the following from [3, Lemma 2.3].

Lemma 1.2. *Let $\psi\colon A \to B$ be an isomorphism of algebras. Denote by $\psi_x^y\colon A(y,x) \to B(\psi y, \psi x)$ the isomorphism defined by ψ for all $x, y \in A$. Define $\hat{\psi}\colon \hat{A} \to \hat{B}$ as follows.*

- *For each $x^{[i]} \in \hat{A}$, $\hat{\psi}(x^{[i]}) := (\psi x)^{[i]}$;*
- *For each $f^{[i]} \in \hat{A}(x^{[i]}, y^{[i]})$, $\hat{\psi}(f^{[i]}) := (\psi f)^{[i]}$; and*
- *For each $\phi^{[i]} \in \hat{A}(x^{[i]}, y^{[i+1]})$, $\hat{\psi}(\phi^{[i]}) := (D((\psi_x^y)^{-1})(\phi))^{[i]} = (\phi \circ (\psi_x^y)^{-1})^{[i]}$.*

Then

(1) *$\hat{\psi}$ is an isomorphism.*

(2) *Given an isomorphism $\rho\colon \hat{A} \to \hat{B}$, the following are equivalent.*
  - (a) *$\rho = \hat{\psi}$;*
  - (b) *ρ satisfies the following.*
    - (i) *$\rho\nu_A = \nu_B\rho$;*
    - (ii) *$\rho(A^{[0]}) = A^{[0]}$;*

(iii) *The diagram*

$$\begin{array}{ccc} A & \xrightarrow{\psi} & B \\ {\scriptstyle \mathbf{1}^{[0]}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{1}^{[0]}} \\ A^{[0]} & \xrightarrow[\rho]{} & B^{[0]} \end{array}$$

is commutative; and

(iv) $\rho(\phi^{[0]}) = (\phi \circ (\psi_x^y)^{-1})^{[0]}$ *for all* $x, y \in A$ *and all* $\phi \in DA(y, x)$.

An algebra is called a *tree algebra* if its ordinary quiver is an oriented tree. Let R be a locally bounded category with the Jacobson radical J and with the ordinary quiver Q. Then by definition of Q there is a bijection $f\colon Q_0 \to R_0, x \mapsto f_x$ and injections $\bar{a}_{y,x}\colon Q_1(x, y) \to J(f_x, f_y)/J^2(f_x, f_y)$ such that $\bar{a}_{y,x}(Q_1(x, y))$ forms a basis of $J(f_x, f_y)/J^2(f_x, f_y)$, where $Q_1(x, y)$ is the set of arrows from x to y in Q for all $x, y \in Q_0$. For each $\alpha \in Q_1(x, y)$ choose $a_{y,x}(\alpha) \in J(f_x, f_y)$ such that $a(\alpha) + J^2(f_x, f_y) = \bar{a}_{y,x}(\alpha)$. Then the pair (f, a) of the bijection f and the family a of injections $a_{y,x}\colon Q_1(x, y) \to J(f_x, f_y)$ $(x, y \in Q_0)$ uniquely extends to a full functor $\Phi\colon \Bbbk Q \to R$, which is called a *display functor* for R.

A path μ from y to x in a quiver with relations (Q, I) is called *maximal* if $\mu \notin I$ but $\alpha\mu, \mu\beta \in I$ for all arrows $\alpha, \beta \in Q_1$. For a k-vector space V with a basis $\{v_1, \dots, v_n\}$ we denote by $\{v_1^*, \dots, v_n^*\}$ the basis of DV dual to the basis $\{v_1, \dots, v_n\}$. In particular if $\dim_k V = 1$, $v^* \in DV$ is defined for all $v \in V \backslash \{0\}$.

Lemma 1.3. *Let A be a tree algebra and $\Phi : \Bbbk Q \to A$ a display functor with $I := \operatorname{Ker}\Phi$. Then*

(1) Φ *uniquely induces the display functor* $\hat{\Phi} : \Bbbk\hat{Q} \to \hat{A}$ *for* $\hat{A}$, *where*

(i) $\hat{Q} = (\hat{Q}_0, \hat{Q}_1, \hat{s}, \hat{t})$ *is defined as follows:*

- $\hat{Q}_0 := Q_0 \times \mathbb{Z} = \{x^{[i]} := (x, i) \mid x \in Q_0, i \in \mathbb{Z}\}$,
- $Q_1 \times \mathbb{Z} := \{\alpha^{[i]} := (\alpha, i) \mid \alpha \in Q_1, i \in \mathbb{Z}\}$,
  $\hat{Q}_1 := (Q_1 \times \mathbb{Z}) \sqcup \{\mu^{*[i]} \mid \mu$ *is a maximal path in* $(Q, I), i \in \mathbb{Z}\}$,
- $\hat{s}(\alpha^{[i]}) := s(\alpha)^{[i]}$, $\hat{t}(\alpha^{[i]}) := t(\alpha)^{[i]}$ *for all* $\alpha^{[i]} \in Q_1 \times \mathbb{Z}$, *and if μ is a maximal path from y to x in (Q, I) then,* $\hat{s}(\mu^{*[i]}) := x^{[i]}$, $\hat{t}(\mu^{*[i]}) := y^{[i+1]}$.

(ii) $\hat{\Phi}$ *is defined by* $\hat{\Phi}(x^{[i]}) := (\Phi x)^{[i]}$, $\hat{\Phi}(\alpha^{[i]}) := (\Phi\alpha)^{[i]}$, *and* $\hat{\Phi}(\mu^{*[i]}) := (\Phi(\mu)^*)^{[i]}$ *for all* $i \in \mathbb{Z}$, $x \in Q_0$, $\alpha \in Q_1$ *and maximal paths* μ *in* (Q, I).

(2) *We define an automorphism* ν_Q *of* $\hat{Q}$ *by* $\nu_Q(x^{[i]}) := x^{[i+1]}$, $\nu_Q(\alpha^{[i]}) := \alpha^{[i+1]}$, $\nu_Q(\mu^{*[i]}) := \mu^{*[i+1]}$ *for all* $i \in \mathbb{Z}, x \in Q_0, \alpha \in Q_1$, *and maximal paths* μ *in* (Q, I).

(3) $\operatorname{Ker}\hat{\Phi}$ *is equal to the ideal $\hat{I}$ defined by the full commutativity relations on $\hat{Q}$ and the zero relations $\mu = 0$ for those paths μ of $\hat{Q}$ for which there is no path* $\hat{t}(\mu) \rightsquigarrow \nu_Q(\hat{s}(\mu))$. *(Therefore note that if a path $\alpha_n \cdots \alpha_1$ is in I, then $\alpha_n^{[i]} \cdots \alpha_1^{[i]}$ is in $\hat{I}$ for all $i \in \mathbb{Z}$.)*

Let R be a locally bounded category. A morphism $f\colon x \to y$ in R_1 is called a *maximal nonzero morphism* if $f \neq 0$ and $fg = 0, hf = 0$ for all $g \in \operatorname{rad} R(z, x), h \in \operatorname{rad} R(y, z), z \in R_0$.

Lemma 1.4. *Let A be an algebra and $x^{[i]}, y^{[j]} \in \hat{A}_0$. Then there exists a maximal nonzero morphism in $\hat{A}(x^{[i]}, y^{[j]})$ if and only if $y^{[j]} = \nu_A(x^{[i]})$.*

Proof. This follows from the fact that $\hat{A}(\text{-}, x^{[i+1]}) \cong D\hat{A}(x^{[i]}, \text{-})$ for all $i \in \mathbb{Z}, x \in A_0$. $\square$

Lemma 1.5. *Let A be an algebra. Then the actions of $\phi\nu_A$ and $\nu_A\phi$ coincide on the objects of $\hat{A}$ for all $\phi \in \operatorname{Aut}(\hat{A})$.*

Proof. Let $x^{[i]} \in \hat{A}_0$. Then there is a maximal nonzero morphism in $\hat{A}(x^{[i]}, \nu_A(x^{[i]}))$ by Lemma 1.4. Since ϕ is an automorphism of $\hat{A}$, there is a maximal nonzero morphism in $\hat{A}(\phi(x^{[i]}), \phi(\nu_A(x^{[i]})))$. Hence $\phi(\nu_A(x^{[i]})) = \nu_A(\phi(x^{[i]}))$ by the same lemma. $\square$

The following is immediate by the lemma above.

Proposition 1.6. *Let A be an algebra, n an integer, and ϕ an automorphism of $\hat{A}$. Then the following are equivalent:*

(1) *ϕ is an automorphism with jump n;*
(2) *$\phi(A^i) = A^{[i+n]}$ for some integer i; and*
(3) *$\phi(A^j) = A^{[j+n]}$ for all integers j.*

In the sequel, we always assume that n is a positive integer when we consider a morphism with jump n. Let Q be a quiver. We denote by $\bar{Q}$ the underlying graph of Q, and call Q *finite* if both Q_0 and Q_1 are finite sets. Each automorphism of Q is regarded as an automorphism of $\bar{Q}$ preserving the orientation of Q, thus $\operatorname{Aut}(Q)$ can be regarded as a subgroup of $\operatorname{Aut}(\bar{Q})$. Suppose now that Q is a finite oriented tree. Then it is also known that $\operatorname{Aut}(Q) \leq \operatorname{Aut}_0(\bar{Q}) := \{f \in \operatorname{Aut}(\bar{Q}) \mid \exists x \in Q_0, f(x) = x\}$. We say that Q is an *admissibly oriented* tree if $\operatorname{Aut}(Q) = \operatorname{Aut}_0(\bar{Q})$. We quote the following from [3, Lemma 4.1]:

Lemma 1.7. *For any finite tree T there exists an admissibly oriented tree Q with a unique source such that $\bar{Q} = T$.*

We recall the following (cf. [3, Section 4.1]):

Definition 1.8. Let R be a locally bounded category. The *formal additive hull* $\operatorname{add} R$ of R is a category defined as follows.

- $(\operatorname{add} R)_0 := \{\bigoplus_{i=1}^n x_i := (x_1, \dots, x_n) \mid n \in \mathbb{N},\ x_1, \dots, x_n \in R_0\}$;
- For each $x = \bigoplus_{i=1}^m x_i, y = \bigoplus_{j=1}^m y_i \in (\operatorname{add} R)_0$,

$$(\operatorname{add} R)(x, y) := \{(\mu_{j,i})_{j,i} \mid \mu_{j,i} \in R(x_i, y_j) \text{ for all } i = 1, \dots, m, j = 1, \dots, n\}; \text{and}$$

- The composition is given by the matrix multiplication.

It is well known that the *Yoneda functor* $Y_R\colon \operatorname{add} R \to \operatorname{prj} R$, $\bigoplus_{i=1}^n x_i \mapsto \bigoplus_{i=1}^n R(\text{-}, x_i)$ is an equivalence. Let $F\colon R \to S$ be a functor of locally bounded categories. Then F naturally induces functors $\operatorname{add} F\colon \operatorname{add} R \to \operatorname{add} S$ and $\tilde{F} := \mathcal{K}^{\mathrm{b}}(\operatorname{add} F)\colon \mathcal{K}^{\mathrm{b}}(\operatorname{add} R) \to \mathcal{K}^{\mathrm{b}}(\operatorname{add} S)$, which are isomorphisms if F is an isomorphism.

2. Reduction to hereditary tree algebras

Proposition 2.1. *Let A be a piecewise hereditary algebra of tree type $\bar{Q}$ for an admissibly oriented tree Q, and n a positive integer. Then we have the following:*

(1) *For any $\phi \in \mathrm{Aut}(\hat{A})$ with jump n, there exists some $\psi \in \mathrm{Aut}(\widehat{\Bbbk Q})$ with jump n such that $\hat{A}/\langle\phi\rangle$ is derived equivalent to $\widehat{\Bbbk Q}/\langle\psi\rangle$; and*

(2) *If we set $\phi' := \nu_{\hat{A}}^n \hat{\phi}_0 \in \mathrm{Aut}(\hat{A})$, where $\phi_0 := (\mathbb{1}^{[0]})^{-1}\nu^{-n}\phi|_{A^{[0]}}\mathbb{1}^{[0]}$, then there exists some $\psi' \in \mathrm{Aut}(\widehat{\Bbbk Q})$ with jump n such that $\hat{A}/\langle\phi'\rangle$ is derived equivalent to $\widehat{\Bbbk Q}/\langle\psi'\rangle$, and that the actions of ψ and ψ' coincide on the objects of $\widehat{\Bbbk Q}$.*

Proof. (1) We set $\phi_i := (\mathbb{1}^{[i]})^{-1}\nu^{-n}\phi|_{A^{[i]}}\mathbb{1}^{[i]} \in \mathrm{Aut}(A)$ for all $i \in \mathbb{Z}$. By [3, Lemma 5.4], there exists a tilting triple $(A, E, \Bbbk Q)$ with an isomorphism $\zeta\colon E \to \Bbbk Q$ such that E is $\langle\tilde{\eta}\rangle$-stable up to isomorphisms for all $\eta \in \mathrm{Aut}(A)$. In particular, E is $\langle\tilde{\phi}_i\rangle$-stable up to isomorphisms for all $i \in \mathbb{Z}$. Then $(\hat{A}, \hat{E}, \widehat{\Bbbk Q})$ is a tilting triple with an isomorphism $\hat{\zeta}$ by [1, Theorem 1.5] and the following holds.

Claim 1. $\hat{E}$ is $\langle\tilde{\phi}\rangle$-stable up to isomorphisms.

Indeed for each $T \in E_0$ and $i \in \mathbb{Z}$, we have

$$\begin{aligned} \tilde{\phi}\tilde{\mathbb{1}}^{[i]}(T) &= \tilde{\nu}^n\tilde{\nu}^{-n}\tilde{\phi}\tilde{\mathbb{1}}^{[i]}(T) \\ &= \tilde{\nu}^n\tilde{\mathbb{1}}^{[i]}\tilde{\phi}_i(T) \\ &= \tilde{\mathbb{1}}^{[i+n]}\tilde{\phi}_i(T). \end{aligned} \tag{2.1}$$

Since E is $\langle\tilde{\phi}_i\rangle$-stable up to isomorphisms, there is some $T' \in E$ such that $T' \cong \tilde{\phi}_i(T)$, and hence $\tilde{\mathbb{1}}^{[i+n]}\tilde{\phi}_i(T) \cong \tilde{\mathbb{1}}^{[i+n]}(T') \in \hat{E}$, as desired.

By [3, Remark 3.5], we have a $\langle\tilde{\phi}\rangle$-stable tilting subcategory $\hat{E}'$ and an isomorphism $\theta\colon \hat{E}' \xrightarrow{\sim} \hat{E}$. Therefore by [2, Proposition 5.4] $\hat{A}/\langle\phi\rangle$ and $\hat{E}'/\langle\tilde{\phi}\rangle$ are derived equivalent. If we set $\psi := (\hat{\zeta}\theta)\tilde{\phi}(\hat{\zeta}\theta)^{-1}$, then (2.1) shows that ψ is an automorphism with jump n, and that $\hat{E}'/\langle\tilde{\phi}\rangle \cong \widehat{\Bbbk Q}/\langle\psi\rangle$. Hence $\hat{A}/\langle\phi\rangle$ and $\widehat{\Bbbk Q}/\langle\psi\rangle$ are derived equivalent.

(2) Note that ϕ' is also an automorphism with jump n. By the same argument we see that $\hat{E}$ is also $\langle\tilde{\phi}'\rangle$-stable up to isomorphisms; there exists a $\langle\tilde{\phi}'\rangle$-stable tilting subcategory $\hat{E}''$ and an isomorphism $\theta'\colon \hat{E}'' \xrightarrow{\sim} \hat{E}$; and $\hat{A}/\langle\phi'\rangle$ and $\hat{E}''/\langle\tilde{\phi}'\rangle$ are derived equivalent. Set $\psi' := (\hat{\zeta}\theta')\tilde{\phi}'(\hat{\zeta}\theta')^{-1}$, then ψ' is an automorphism with jump n, $\hat{E}''/\langle\tilde{\phi}'\rangle \cong \widehat{\Bbbk Q}/\langle\psi'\rangle$, and $\hat{A}/\langle\phi'\rangle$ and $\widehat{\Bbbk Q}/\langle\psi'\rangle$ are derived equivalent. Now for $i = 0$ (2.1) shows that $\tilde{\phi}\tilde{\mathbb{1}}^{[0]}(T) = \tilde{\mathbb{1}}^{[n]}\tilde{\phi}_0(T)$ for all $T \in E_0$. Since $\phi'_0 = \phi_0$, the same calculation shows that $\tilde{\phi}'\tilde{\mathbb{1}}^{[0]}(T) = \tilde{\mathbb{1}}^{[n]}\tilde{\phi}_0(T)$ for all $T \in E_0$. Thus the actions of $\tilde{\phi}$ and $\tilde{\phi}'$ coincide on the objects of $E^{[0]}$, which shows that the actions of ψ and ψ' coincide on the objects of $\Bbbk Q^{[0]}$. Hence by Lemma 1.5 their actions coincide on the objects of $\widehat{\Bbbk Q}$. □

3. Hereditary tree algebras

Remark 3.1. Let Q be an oriented tree.

(1) We may identify $\widehat{\Bbbk Q} = \Bbbk\hat{Q}/\hat{I}$ as stated in Lemma 1.3, and we denote by $\overline{\mu}$ the morphism $\mu + \hat{I}$ in $\widehat{\Bbbk Q}$ for each morphism μ in $\Bbbk\hat{Q}$.

(2) Let $x, y \in \hat{Q}_0$. Since $\hat{I}$ contains full commutativity relations, we have $\dim_{\Bbbk} \widehat{\Bbbk Q}(x, y) \leq 1$, and in particular $\hat{Q}$ has no double arrows.

(3) Let $\alpha\colon x \to y$ be in $\hat{Q}_1$ and $\phi \in \mathrm{Aut}(\widehat{\Bbbk Q})$. Then there exists a unique arrow $\phi x \to \phi y$ in $\hat{Q}$, which we denote by $(\hat{\pi}\phi)(\alpha)$, and we have $\phi(\overline{\alpha}) = \phi_\alpha \overline{(\hat{\pi}\phi)(\alpha)} \in \widehat{\Bbbk Q}(\phi x, \phi y)$ for a unique $\phi_\alpha \in \Bbbk^\times := \Bbbk \setminus \{0\}$. This defines an automorphism $\hat{\pi}\phi$ of $\hat{Q}$, and thus a group homomorphism $\hat{\pi} : \mathrm{Aut}(\widehat{\Bbbk Q}) \to \mathrm{Aut}(\hat{Q})$.

(4) Similarly, let $\alpha\colon x \to y$ be in Q_1 and $\psi \in \mathrm{Aut}(\Bbbk Q)$. Then there exists a unique arrow $\psi x \to \psi y$ in Q, which we denote by $(\pi\psi)(\alpha)$. This defines an automorphism $\pi\psi$ of Q, and thus a group homomorphism $\pi : \mathrm{Aut}(\Bbbk Q) \to \mathrm{Aut}(Q)$.

We cite the following from [3, Proposition 7.4].

Proposition 3.2. *Let R be a locally bounded category, and g, h automorphisms of R acting freely on R. If there exists a map $\rho\colon R_0 \to \Bbbk^\times$ such that $\rho(y)g(f) = h(f)\rho(x)$ for all morphisms $f\colon x \to y$ in R, then $R/\langle g\rangle \cong R/\langle h\rangle$.* □

Definition 3.3. (1) For a quiver $Q = (Q_0, Q_1, s, t)$ we set $Q[Q_1^{-1}]$ to be the quiver

$$Q[Q_1^{-1}] := (Q_0, Q_1 \sqcup \{\alpha^{-1} \mid \alpha \in Q_1\}, s', t'),$$

where $s'|_{Q_1} := s$, $t'|_{Q_1} := t$, $s'(\alpha^{-1}) := t(\alpha)$ and $t'(\alpha^{-1}) := s(\alpha)$ for all $\alpha \in Q_1$. A *walk* in Q is a path in $Q[Q_1^{-1}]$.

(2) Suppose that Q is a finite oriented tree. Then for each $x, y \in Q_0$ there exists a unique shortest walk from x to y in Q, which we denote by $w(x, y)$. If $w(x, y) = \alpha_n^{\varepsilon_n} \cdots \alpha_1^{\varepsilon_1}$ for some $\alpha_1, \cdots, \alpha_n \in Q_1$ and $\varepsilon_1, \ldots, \varepsilon_n \in \{1, -1\}$, then we define a subquiver $W(x, y)$ of Q by $W(x, y) := (W(x, y)_0, W(x, y)_1, s', t')$, where $W(x, y)_0 := \{s(\alpha_i), t(\alpha_i) \mid i = 1, \ldots, n\}$, $W(x, y)_1 := \{\alpha_1, \ldots, \alpha_n\}$, and s', t' are restrictions of s, t to $W(x, y)_1$, respectively. Since Q is an oriented tree, $w(x, y)$ is uniquely recovered by $W(x, y)$. Therefore we can identify $w(x, y)$ with $W(x, y)$, and define a *sink* and a *source* of $w(x, y)$ as those in $W(x, y)$.

Proposition 3.4. *Let Q be a finite oriented tree and ϕ, ψ automorphisms of $\widehat{\Bbbk Q}$ acting freely on $\widehat{\Bbbk Q}$. If the actions of ϕ and ψ coincide on the objects of $\widehat{\Bbbk Q}$, then there exists a map $\rho\colon (\hat{Q}_0 =)\ \widehat{\Bbbk Q}_0 \to \Bbbk^\times$ such that $\rho(y)\psi(f) = \phi(f)\rho(x)$ for all morphisms $f\colon x \to y$ in $\widehat{\Bbbk Q}$. Hence in particular, $\widehat{\Bbbk Q}/\langle\phi\rangle$ is isomorphic to $\widehat{\Bbbk Q}/\langle\psi\rangle$.*

Proof. Assume that the actions of $\phi, \psi \in \mathrm{Aut}(\widehat{\Bbbk Q})$ coincides on the objects of $\widehat{\Bbbk Q}$. Then ϕ and ψ induce the same quiver automorphism $q = \hat{\pi}\phi = \hat{\pi}\psi$ of $\hat{Q}$, and there exist $(\phi_\alpha)_{\alpha \in \hat{Q}_1}, (\psi_\alpha)_{\alpha \in \hat{Q}_1} \in (k^\times)^{\hat{Q}_1}$ such that for each $\alpha \in \hat{Q}_1$ we have

$$\phi(\overline{\alpha}) = \phi_\alpha \overline{q(\alpha)}, \quad \psi(\overline{\alpha}) = \psi_\alpha \overline{q(\alpha)}.$$

For each path $\lambda = \alpha_n \cdots \alpha_1$ in $\hat{Q}$ with $\alpha_1, \ldots, \alpha_n \in \hat{Q}_1$ we set $\phi_\lambda := \phi_{\alpha_n} \cdots \phi_{\alpha_1}$. Then we have

$$\phi(\overline{\lambda}) = \phi_\lambda \overline{q(\lambda)},$$

where $q(\lambda) := q(\alpha_n)\cdots q(\alpha_1)$ because $\phi(\overline{\alpha_n})\cdots\phi(\overline{\alpha_1}) = \phi_{\alpha_n}\cdots\phi_{\alpha_1}\overline{q(\alpha_n)\cdots q(\alpha_1)}$.

To show the statement we may assume that $\psi_\alpha = 1$ for all $\alpha \in \hat{Q}_1$. Since for each $x, y \in \hat{Q}_0$ the morphism space $\widehat{\Bbbk Q}(x,y)$ is at most 1-dimensional and has a basis of the form $\overline{\mu}$ for some path μ, it is enough to show that there exists a map $\rho : \hat{Q}_0 \to \Bbbk^\times$ satisfying the following condition:

$$\rho(v^{[j]}) = \phi_\beta \rho(u^{[i]}) \quad \text{for all } \beta : u^{[i]} \to v^{[j]} \text{ in } \hat{Q}_1. \tag{3.1}$$

We define a map ρ as follows:
Fix a maximal path $\mu\colon y \rightsquigarrow x$ in Q. Then x is a sink and y is a source in Q. We can write μ as $\mu = \alpha_l \cdots \alpha_1$ for some $\alpha_1, \dots, \alpha_l \in Q_1$. First we set $\rho(x^{[0]}) := 1$. By induction on $0 \le i \in \mathbb{Z}$ we define $\rho(x^{[i]})$ and $\rho(x^{[-i]})$ by the following formulas:

$$\rho(x^{[i+1]}) := \phi_{\mu^{[i+1]}}\phi_{\mu^{*[i]}}\rho(x^{[i]}), \tag{3.2}$$

$$\rho(x^{[i-1]}) := \phi^{-1}_{\mu^{*[i-1]}}\phi^{-1}_{\mu^{[i]}}\rho(x^{[i]}). \tag{3.3}$$

Now for each $i \in \mathbb{Z}$ and $u \in Q_0$ if $w(u,x) = \beta_m^{\varepsilon_m}\cdots\beta_1^{\varepsilon_1}$ for some $\beta_1, \dots, \beta_m \in Q_1$ and $\varepsilon_1, \dots, \varepsilon_m \in \{1, -1\}$, then we set

$$\rho(u^{[i]}) := \phi^{-\varepsilon_1}_{\beta_1^{[i]}}\cdots\phi^{-\varepsilon_m}_{\beta_m^{[i]}}\rho(x^{[i]}). \tag{3.4}$$

We have to verify the condition (3.1).

Case 1. $\beta = \alpha^{[i]}\colon u^{[i]} \to v^{[i]}$ for some $i \in \mathbb{Z}$, and $\alpha\colon u \to v$ in Q_1. Since Q is an oriented tree, we have either $w(u,x) = w(v,x)\alpha$ or $w(v,x) = w(u,x)\alpha^{-1}$. In either case we have $\rho(v^{[i]}) = \phi_{\alpha^{[i]}}\rho(u^{[i]})$ by the formula (3.4).

Case 2. Otherwise, we have $\beta = \lambda^{*[i]} : u^{[i]} \to v^{[i+1]}$ for some maximal path $\lambda\colon v \rightsquigarrow u$ in Q and $i \in \mathbb{Z}$. In this case the condition (3.1) has the following form:

$$\rho(v^{[i+1]}) = \phi_{\lambda^{*[i]}}\rho(u^{[i]}). \tag{3.5}$$

Two paths are said to be *parallel* if they have the same source and the same target. We prepare the following for the proof.

Claim 2. If ζ and η are parallel paths in $\hat{Q}$, then we have $\phi_\zeta = \phi_\eta$.

Indeed, since $\zeta - \eta \in \hat{I}$, we have $\phi(\overline{\zeta}) = \phi(\overline{\eta})$, which shows

$$\phi_\zeta\overline{q(\zeta)} = \phi_\eta\overline{q(\eta)}.$$

Here we have $\overline{q(\zeta)} = \psi(\overline{\zeta}) = \psi(\overline{\eta}) = \overline{q(\eta)}$, and $\psi(\overline{\zeta}) \ne 0$ because $\overline{\zeta} \ne 0$. Hence $\phi_\zeta = \phi_\eta$, as required.

We now set $d(a,b)$ to be the number of sinks in $w(a,b)$ for all $a, b \in Q_0$. By induction on $d(y,v)$ we can show that the condition (3.5) holds. □

4. Main result

Theorem 4.1. *Let A be a piecewise hereditary algebra of tree type and ϕ an automorphism of $\hat{A}$ with jump n. Then $\hat{A}/\langle\phi\rangle$ and $T^n_{\phi_0}(A)$ are derived equivalent, where we set $\phi_0 := (\mathbb{1}^{[0]})^{-1}\nu^{-n}\phi|_{A^{[0]}}\mathbb{1}^{[0]}$.*

Proof. Let T be the tree type of A. Then by Lemma 1.7 there exists an admissibly oriented tree Q with $\bar{Q} = T$. We set $\phi' := \nu_A^n \hat{\phi}_0$ $(= \hat{\phi}_0 \nu_A^n)$. Then $T_{\phi_0}^n(A) = \hat{A}/\langle \phi' \rangle$. By Proposition 2.1(2) there exist some $\psi, \psi' \in \mathrm{Aut}(\widehat{\Bbbk Q})$ both with jump n such that $\hat{A}/\langle \phi \rangle$ (resp. $\hat{A}/\langle \phi' \rangle$) is derived equivalent to $\widehat{\Bbbk Q}/\langle \psi \rangle$ (resp. $\widehat{\Bbbk Q}/\langle \psi' \rangle$), and the actions of ψ and ψ' coincide on the objects of $\widehat{\Bbbk Q}$. Then by Proposition 3.4 we have $\widehat{\Bbbk Q}/\langle \psi \rangle \cong \widehat{\Bbbk Q}/\langle \psi' \rangle$. Hence $\hat{A}/\langle \phi \rangle$ and $T_{\phi_0}^n(A)$ are derived equivalent. $\square$

Definition 4.2. Let Λ be a generalized n-fold extension of a piecewise hereditary algebra A of tree type T, say $\Lambda = \hat{A}/\langle \phi \rangle$ for some $\phi \in \mathrm{Aut}(A)$ with jump n. Further let Q be an admissibly oriented tree with $\bar{Q} = T$. Then by Proposition 2.1 there exists $\psi \in \mathrm{Aut}(\widehat{\Bbbk Q})$ with jump n such that $\hat{A}/\langle \phi \rangle$ is derived equivalent to $\widehat{\Bbbk Q}/\langle \psi \rangle$. We define the (*derived equivalence*) *type* $\mathrm{type}(\Lambda)$ of Λ to be the triple $(T, n, \overline{\pi}(\psi_0))$, where $\psi_0 := (\mathbb{1}^{[0]})^{-1} \nu_{\Bbbk Q}^{-n} \psi|_{(\Bbbk Q)^{[0]}} \mathbb{1}^{[0]}$ and $\overline{\pi}(\psi_0)$ is the conjugacy class of $\pi(\psi_0)$ in $\mathrm{Aut}(T)$. $\mathrm{type}(\Lambda)$ is uniquely determined by Λ.

By Theorem 4.1, we can extend the main theorem in [3] as follows.

Theorem 4.3. *Let Λ, Λ' be generalized multifold extensions of piecewise hereditary algebras of tree type. Then the following are equivalent:*

(i) Λ *and* Λ' *are derived equivalent.*
(ii) Λ *and* Λ' *are stably equivalent.*
(iii) $\mathrm{type}(\Lambda) = \mathrm{type}(\Lambda')$.

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