DECOMPOSING TENSOR PRODUCTS FOR CYCLIC AND DIHEDRAL GROUPS

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ABSTRACT. We give a new formula for the decomposition of a tensor product of indecomposable modules of cyclic two-groups. This formula is also shown to describe the decomposition of tensor products of an important class of modules of dihedral two-groups.

1. Introduction

In this note, we give a new, closed formula for the decomposition of a tensor product of indecomposable modules of cyclic 2-groups, and show how this formula also describes the decomposition of tensor products of a class of D_{2^l}-modules. The problem of decomposing such a tensor product of modules of cyclic p-groups in characteristic p has been treated by several authors (e.g. [4, 6, 5, 1]). However, to date, all solutions have been recursive, and rather involved. Concentrating on the case $p = 2$ is a simplification which makes it possible to give a closed decomposition formula.

Our interest in this problem originated in the study of tensor products of modules of dihedral 2-groups. Thus, we show that the decomposition formula for modules of cyclic 2-groups also describes the decompositions of tensor products of the D_{2^l}-modules induced from the maximal cyclic subgroup.

Throughout this text, k denotes a field of characteristic 2. The dihedral group of order $2q$ is written as $D_{2q} = \langle \sigma, \tau \mid \sigma^2 = \tau^2 = (\sigma\tau)^q = 1 \rangle$. Here q will always be a 2-power, $q \geqslant 2$. The unique cyclic subgroup of index 2 in D_{2q} is $H_q = \langle \sigma\tau \rangle \lhd D_{2q}$.

The indecomposable modules of kC_q are classified by their dimensions; that is, up to isomorphism, for each $i \in \{1, \ldots, q\}$ there exists a unique indecomposable kC_q-module of dimension i. Fix a set of representatives $\{V_i\}_{i \leqslant q}$ such that $\dim V_i = i$. Every projective indecomposable module is isomorphic to V_q, and the tensor product of a projective with any other module is again projective. We recall that every non-projective kC_q-module is Ω-periodic of period at most 2. Indeed, for each $i < q$, the formula $\Omega(V_i) \simeq V_{q-i}$ holds.

There is a unique projection $C_{2q} \twoheadrightarrow C_q$. This surjection, via the usual inflation operation, induces a full embedding of module categories $\operatorname{mod} kC_q \hookrightarrow \operatorname{mod} kC_{2q}$, $V_i \mapsto V_i$, respecting the tensor product. Thus V_i is viewed as a module for all cyclic 2-groups of order greater than or equal to i.

2. Decomposition formula for tensor products of modules of cyclic 2-groups

The following result makes it possible to compute the decomposition of a tensor product of any two kC_q-modules recursively.

This paper is a summary of results that will be published elsewhere.

Proposition 1. *Let $i, j \leqslant q$. Then* $V_i \otimes V_j \simeq \Omega(V_{q-i} \otimes V_j) \oplus \max\{i+j-q, 0\} V_q$.

If $q/2 \le i < q$ then $q - i \le q/2$ hence, by applying Proposition 1, we can transfer the problem of finding the decomposition of $V_i \otimes V_j$ to the smaller module category $\operatorname{mod} kC_{q/2}$. This gives an inductive process which halts when one of the factors is projective, in which case the product can be immediately computed. Example 2 below illustrates the procedure. To avoid any ambiguity, we write Ω_q to indicate the Heller translate in $\operatorname{mod} kC_q$.

Example 2. Consider the module $V_{18} \otimes V_6$, a tensor product of indecomposable modules of kC_{32}. Applying Proposition 1, we see that

$$V_{18} \otimes V_6 \simeq \Omega_{32}(V_{14} \otimes V_6). \tag{2.1}$$

Viewing $V_{14} \otimes V_6$ as a module for C_{16} and again applying Proposition 1, we obtain

$$V_{14} \otimes V_6 \simeq \Omega_{16}(V_2 \otimes V_6) \oplus 4V_{16}. \tag{2.2}$$

Now $V_2 \otimes V_6 \in \operatorname{mod} kC_8$, and

$$V_2 \otimes V_6 \simeq \Omega_8(V_2 \otimes V_2). \tag{2.3}$$

In $\operatorname{mod} kC_2$, V_2 is projective, so $V_2 \otimes V_2 \simeq 2V_2$. Applying in turn Equations (2.3), (2.2) and (2.1), we obtain the decomposition

$$\begin{aligned} V_{18} \otimes V_6 &\simeq \Omega_{32}(\Omega_{16}(\Omega_8(2V_2)) \oplus 4V_{16}) \\ &\simeq 2V_{22} \oplus 4V_{16}. \end{aligned}$$

The idea behind our decomposition formula is to record the successive applications of Proposition 1 in numerical sequences, which are then used to compute the indecomposable summands of the tensor product. Let x be any positive integer. Set $v(x) = \min\{y \in \mathbb{N} \mid 2^y \geqslant x\}$ and $x' = 2^{v(x)} - x$. A sequence $(x_n)_{n \geqslant 0}$ is defined recursively by $x_0 = x$ and $x_{n+1} = x'_n$. Let $r \in \mathbb{N}$ be the first number such that x_r is a 2-power. Then $(x_n)_{n=0}^r$ is strictly decreasing, whereas $x_n = 0$ for all $n > r$.

Now, given $i, j \in \mathbb{N}$, set $[i,j]_0 = (i_0, j_0) = (i, j)$ and, if $[i,j]_n = (i_a, j_b)$,

$$[i,j]_{n+1} = \begin{cases} (i_{a+1}, j_b) & \text{if } i_a \geqslant j_b, \\ (i_a, j_{b+1}) & \text{if } i_a < j_b. \end{cases} \tag{2.4}$$

This defines a sequence $([i,j]_n)_{n=0}^w = \left(([i,j]_n^{(1)}, [i,j]_n^{(2)})\right)_{n=0}^w$, where w is the smallest number such that $\max\left\{[i,j]_w^{(1)}, [i,j]_w^{(2)}\right\}$ is a 2-power. Now, set $m_n = 2^{v(x_n)}$, for $x_n = \max\left\{[i,j]_n^{(1)}, [i,j]_n^{(2)}\right\}$, $n \in \{0, \ldots, w\}$. Finally, for all $n \leqslant w$, let

$$\alpha_n = \max\left\{0\,,\ [i,j]_n^{(1)} + [i,j]_n^{(2)} - m_n\right\} \quad \text{and} \tag{2.5}$$

$$\beta_n = \sum_{u=0}^{n} (-1)^u m_u\,. \tag{2.6}$$

Theorem 3. *For all $i, j \in \mathbb{N}$,*

$$V_i \otimes V_j \simeq \bigoplus_{n=0}^{w} \alpha_n V_{\beta_n} \,.$$

It may be noted that while the numbers i_n are, for simplicity of presentation, recursively defined, they may all be read off from the binary expansion of the number i in a non-recursive manner.

Example 4. Consider the case $i = 20$ and $j = 51$. We have

$$i_0 = 20, \qquad i_1 = 32 - i_0 = 12, \qquad i_2 = 16 - i_1 = 4,$$

and

$$j_0 = 51, \qquad j_1 = 64 - j_0 = 13, \qquad j_2 = 16 - j_1 = 3, \qquad j_3 = 4 - j_2 = 1.$$

Now we can define all sequences needed for the application of Theorem 3. First, the sequence $[i, j]$ consists of pairs (i_a, j_b), formed by applying the equation (2.4) above:

$$[i,j]_0 = (20, 51), \quad [i,j]_1 = (20, 13), \quad [i,j]_2 = (12, 13), \quad [i,j]_3 = (12, 3), \quad [i,j]_4 = (4, 3);$$

m_n is the smallest 2-power greater than or equal to the two components of $[i, j]_n$:

$$m_0 = 64, \qquad m_1 = 32, \qquad m_2 = 16, \qquad m_3 = 16, \qquad m_4 = 4;$$

$\alpha_n = [i,j]_n^{(1)} + [i,j]_n^{(2)} - m_n$ if this number is positive, otherwise $\alpha_n = 0$:

$$\alpha_0 = 7, \qquad \alpha_1 = 1, \qquad \alpha_2 = 9, \qquad \alpha_3 = 0, \qquad \alpha_4 = 3;$$

β_n is the alternating sum of the numbers $m_1, \ldots, m_n$:

$$\beta_0 = 64, \qquad \beta_1 = 32, \qquad \beta_2 = 48, \qquad \beta_3 = 32, \qquad \beta_4 = 36.$$

With Theorem 3, we conclude that

$$V_{20} \otimes V_{51} \simeq 7V_{64} \oplus V_{32} \oplus 9V_{48} \oplus 3V_{36} \,.$$

3. Application: pseudoprojective modules of dihedral 2-groups

It turns out that Theorem 3 can be used to describe tensor products of a class of modules of dihedral 2-groups. These are the so-called *pseudoprojective* modules, given as

$M(A_l B_l, 1)$ for some $l \in \mathbb{N}$ (see [3] for definition of the relevant notation). The pseudoprojective modules are band modules, given by schemas in the following way:

$$\begin{tikzcd}[column sep=small]
 & k \arrow[dl, "X"'] \arrow[dr, "Y"] & \\
k \arrow[d, "Y"'] & & k \arrow[d, "X"] \\
{} \arrow[d, dashed] & & {} \arrow[d, dashed] \\
k \arrow[dr] & & k \arrow[dl] \\
 & k &
\end{tikzcd}$$

We shall use M_d to denote the pseudoprojective module of dimension d, in other words, $M(A_l B_l^{-1}, 1) \simeq M_{2l}$.

The pseudoprojective modules are precisely the kD_{2q}-modules that are induced from the maximal cyclic subgroup $H_q \lhd D_{2q}$:

Proposition 5. *For each $i \in \{1, \ldots, q\}$, the induced module $V_i {\uparrow}_{H_q}^{D_{2q}}$ is isomorphic to M_{2i}.*

Applying Mackey's tensor product theorem (see e.g. [2, Corollary 3.3.5(i)]),

$$M_{2i} \otimes M_{2j} \simeq 2\,(V_i \otimes V_j){\uparrow}^{D_{2q}} \simeq 2 \left(\bigoplus_{n=0}^{w} \alpha_n V_{\beta_n} \right) \Bigg\uparrow^{D_{2q}} \simeq \bigoplus_{n=0}^{w} 2\alpha_n M_{2\beta_n} .$$

Similarly, for $V_{2i}, V_{2j} \in \operatorname{mod} kC_{2q}$,

$$V_{2i} \otimes V_{2j} \simeq V_i {\uparrow}_{C_q}^{C_{2q}} \otimes V_j {\uparrow}_{C_q}^{C_{2q}} \simeq 2 \left(\bigoplus_{n=0}^{w} \alpha_n V_{\beta_n} \right) \Bigg\uparrow_{C_q}^{C_{2q}} \simeq \bigoplus_{n=0}^{w} 2\alpha_n V_{2\beta_n} .$$

It follows that the decompositions of tensor products $V_{2i} \otimes V_{2j}$ and $M_{2i} \otimes M_{2j}$ are governed by the same formula. This proves the following result.

Corollary 6. *For any even numbers $i, j \in \mathbb{N}$, the decomposition formula*

$$M_i \otimes M_j \simeq \bigoplus_{n=0}^{w} \alpha_n V_{\beta_n}$$

holds, with the numbers α_n and β_n defined by Equations (2.5) and (2.6) respectively.

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