QUOTIENTS OF EXACT CATEGORIES BY CLUSTER TILTING SUBCATEGORIES AS MODULE CATEGORIES

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Abstract. We prove that some subquotient categories of exact categories are abelian. This generalizes a result by Koenig-Zhu in the case of (algebraic) triangulated categories. As a particular case, if an exact category $\mathcal{B}$ with enough projectives and injectives has a cluster tilting subcategory $\mathcal{M}$, then $\mathcal{B}/[\mathcal{M}]$ is abelian. More precisely, it is equivalent to the category of finitely presented modules over $\underline{\mathcal{M}}$.

1. Introduction

Recently, cluster tilting theory (see for example [1, 3, 6]) permitted to construct abelian categories from some triangulated categories. In this survey we sketch out the method we introduced in [2] to generalize this observation to exact categories.

Recall that an exact category is Frobenius if it has enough projectives and injectives and they coincide. From Happel [4, Theorem 2.6], the stable category of a Frobenius category has a structure of a triangulated category. On the other hand, by Keller-Reiten [7, Proposition 2.1], in the 2-Calabi-Yau case and then Koenig-Zhu [8, Theorem 3.3] in the general case, one can pass from triangulated categories to abelian categories by factoring out any cluster tilting subcategory. Combining these two results, we deduce that the quotient of a Frobenius category by a cluster tilting subcategory is abelian. Thus, this observation gives rise to a natural question: is the quotient of an exact category by a cluster tilting subcategory abelian? As we will see, it turns out to be true.

This new result seems *a priori* less surprising than the one in triangulated categories because these ones are intuitively further to abelian categories. Nevertheless, most triangulated categories appearing in representation theory turn out to be in fact *algebraic* (*i.e.* stable categories of Frobenius categories). In this respect, the case of exact categories can be seen as a generalization of the result concerning triangulated categories, as well as a more natural version.

2. Notations

Let $\mathcal{B}$ be a Krull-Schmidt exact category with enough projectives and injectives and $\mathcal{M}$ be a full *rigid* subcategory of $\mathcal{B}$ (*i.e.* $\operatorname{Ext}^1_{\mathcal{B}}(X,X) = 0$ for any $X \in \mathcal{M}$).

Denote by $\mathcal{P}$ (resp. $\mathcal{I}$) the subcategory of projective (resp. injective) objects in $\mathcal{B}$. For any object $X, Y \in \mathcal{B}$ and a full subcategory $\mathcal{C}$ of $\mathcal{B}$, denote by $[\mathcal{C}](X,Y)$ the set of morphisms in $\operatorname{Hom}_{\mathcal{B}}(X,Y)$ which factor through objects of $\mathcal{C}$. If $\mathcal{P} \subseteq \mathcal{C}$ (resp. $\mathcal{I} \subseteq \mathcal{C}$),

The detailed version [2] of this paper has been submitted for publication.

the *(co-)stable category* $\underline{\mathcal{C}}$ (resp. $\overline{\mathcal{C}}$) of $\mathcal{C}$ is the quotient category $\mathcal{C}/[\mathcal{P}]$ (resp. $\mathcal{C}/[\mathcal{I}]$), *i.e.* the category which has the same objects than $\mathcal{C}$ and morphisms are defined as

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)/[\mathcal{P}](X,Y)$$
$$(\text{resp. } \overline{\mathrm{Hom}}_{\mathcal{C}}(X,Y) := \mathrm{Hom}_{\mathcal{C}}(X,Y)/[\mathcal{I}](X,Y)).$$

Denote by Mod$\mathcal{C}$ the category of contravariant additive functors from $\mathcal{C}$ to mod$k$ for any category $\mathcal{C}$ where k is a field. Let mod$\mathcal{C}$ be the full subcategory of Mod$\mathcal{C}$ consisting of objects A admitting an exact sequence:

$$\mathrm{Hom}_{\mathcal{C}}(-,C_1) \xrightarrow{\beta} \mathrm{Hom}_{\mathcal{C}}(-,C_0) \xrightarrow{\alpha} A \to 0$$

where $C_0, C_1 \in \mathcal{C}$.

Denote by $\overline{\Omega}\mathcal{M}$ the class of objects $X \in \mathcal{B}$ such that there exists a short exact sequence

$$0 \to M \to I \to X \to 0$$

where $M \in \mathcal{M}$, and I is injective.

Denote by $\mathcal{M}_L$ (resp. $\mathcal{M}_R$) the subcategory of objects X which admit short exact sequences

$$0 \to X \xrightarrow{d^0} M^0 \xrightarrow{d^1} M^1 \to 0 \quad (\text{resp. } 0 \to M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} X \to 0)$$

with $M^0, M^1, M_0, M_1 \in \mathcal{M}$. In this case, d^0 (resp. d_0) is a left (resp. right) $\mathcal{M}$-approximation of X.

3. Two quotient category: $\mathcal{M}_L/[\mathcal{M}]$ and $\mathcal{M}_R/[\overline{\Omega}\mathcal{M}]$

3.1. **Quotient category of $\mathcal{M}_L/[\mathcal{M}]$ by a rigid subcategory $\mathcal{M}$.** In this subsection, we assume that $\mathcal{M}$ is a rigid subcategory of $\mathcal{B}$ which contains $\mathcal{P}$. Now we consider the functor

$$\begin{aligned} H:\ & \mathcal{M}_L \to \mathrm{Mod}\underline{\mathcal{M}} \\ & X \mapsto \mathrm{Ext}^1_{\mathcal{B}}(-,X)|_{\mathcal{M}} \end{aligned}$$

Let $\pi : \mathcal{M}_L \to \mathcal{M}_L/[\mathcal{M}]$ be the projection functor. By definition of a rigid subcategory, $HX = 0$ if $X \in \mathcal{M}$. Hence, by the universal property of π, there exists a functor $F : \mathcal{M}_L/[\mathcal{M}] \to \mathrm{Mod}\underline{\mathcal{M}}$ such that $F\pi = H$. From the following lemma we can see directly that $F(X) \in \mathrm{mod}\underline{\mathcal{M}}$:

Lemma 1. *For any short exact sequence*

$$0 \to X \xrightarrow{d^0} M^0 \xrightarrow{d^1} M^1 \to 0$$

where $M^0, M^1 \in \mathcal{M}$, there is an exact sequence in Mod$\underline{\mathcal{M}}$

$$\underline{\mathrm{Hom}}_{\mathcal{M}}(-,M^0) \to \underline{\mathrm{Hom}}_{\mathcal{M}}(-,M^1) \to FX \to 0.$$

The functor F induces the equivalence we want:

Theorem 2. *The functor $F : \mathcal{M}_L/[\mathcal{M}] \to \mathrm{mod}\underline{\mathcal{M}}$ is an equivalence of categories.*

Moreover, we have the following corollary:

Corollary 3. *If $\mathcal{M}$ is rigid and contravariantly finite, then $\mathcal{M}_L/[\mathcal{M}]$ is abelian.*

3.2. **Quotient category of $\mathcal{M}_R$ by $\overline{\Omega}\mathcal{M}$.** In this subsection we assume that $\mathcal{M}$ is a rigid subcategory of $\mathcal{B}$ which contains $\mathcal{I}$.

We denote

$$\begin{aligned} K : \mathcal{M}_R &\to \mathrm{Mod}\overline{\mathcal{M}} \\ X &\mapsto \overline{\mathrm{Hom}}_{\mathcal{B}}(-, X). \end{aligned}$$

Let $\pi' : \mathcal{M}_R \to \mathcal{M}_R/[\overline{\Omega}\mathcal{M}]$ be the projection functor. By the universal property of π', there is a functor $G : \mathcal{M}_R/[\overline{\Omega}\mathcal{M}] \to \mathrm{Mod}\overline{\mathcal{M}}$ such that $G\pi' = K$. From the lemma we can see that $GX \in \mathrm{mod}\overline{\mathcal{M}}$:

Lemma 4. *For every short exact sequence*

$$0 \to M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} X \to 0$$

where $M_1, M_0 \in \mathcal{M}$, there is an exact sequence

$$\overline{\mathrm{Hom}}_{\mathcal{M}}(-, M_1) \to \overline{\mathrm{Hom}}_{\mathcal{M}}(-, M_0) \to GX \to 0.$$

The functor G also gives an equivalence:

Theorem 5. *The functor $G : \mathcal{M}_R/[\overline{\Omega}\mathcal{M}] \to \mathrm{mod}\overline{\mathcal{M}}$ is an equivalence of categories.*

If we denote $\overline{\mathcal{M}}^{\perp} = \{X \in \mathcal{M}_R \mid \overline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{M}, X) = 0\}$, we get the following corollary:

Corollary 6. *We have $\overline{\Omega}\mathcal{M} = \overline{\mathcal{M}}^{\perp}$.*

4. Case of n-cluster tilting subcategories and AR translation

For a subcategory $\mathcal{C}$ of $\mathcal{B}$, we define

$$\begin{aligned} {}^{\perp_m}\mathcal{C} &= \{X \in \mathcal{B} \,|\, \forall i \in \{1, \ldots, m\}, \mathrm{Ext}^i_{\mathcal{B}}(X, \mathcal{C}) = 0\} \\ \text{and} \quad \mathcal{C}^{\perp_m} &= \{X \in \mathcal{B} \,|\, \forall i \in \{1, \ldots, m\}, \mathrm{Ext}^i_{\mathcal{B}}(\mathcal{C}, X) = 0\}. \end{aligned}$$

Recall that $\mathcal{M}$ is called *n-cluster tilting*, if it satisfies the following conditions:

(1) $\mathcal{M}$ is contravariantly finite and covariantly finite in $\mathcal{B}$,
(2) $\mathcal{M} = \mathcal{M}^{\perp_{n-1}}$,
(3) $\mathcal{M} = {}^{\perp_{n-1}}\mathcal{M}$.

The previous results concern categories $\mathcal{M}_L$ and $\mathcal{M}_R$ which have not good properties in general. From now on, we suppose that $\mathcal{M}$ is n-cluster tilting for some integer $n \geq 2$ (see [5, 6]). Thus, the properties of $\mathcal{M}_L$ and $\mathcal{M}_R$ becomes much clearer:

Proposition 7. *The following equalities hold:*

$${}^{\perp_{n-2}}\mathcal{M} = \mathcal{M}_L \quad \textit{and} \quad \mathcal{M}^{\perp_{n-2}} = \mathcal{M}_R.$$

By this proposition, we obtain that both $\mathcal{M}_L$ and $\mathcal{M}_R$ are exact subcategories of $\mathcal{B}$. In particular we get

Corollary 8. *If $\mathcal{M}$ is 2-cluster tilting then $\mathcal{B}/[\mathcal{M}] \simeq \mathrm{mod}\underline{\mathcal{M}}$ is abelian.*

Now, we assume that $\mathcal{B}$ has an AR translation $\tau : \underline{B} \to \overline{B}$ with reciprocal τ^-. Following [5], we define $(n-1)$-AR translations

$$\tau_{n-1} : \underline{{}^{\perp_{n-2}}\mathcal{P}} \to \overline{\mathcal{I}^{\perp_{n-2}}} \quad \text{and} \quad \tau_{n-1}^- : \overline{\mathcal{I}^{\perp_{n-2}}} \to \underline{{}^{\perp_{n-2}}\mathcal{P}}$$

by $\tau_{n-1} = \tau\Omega^{n-2}$ and $\tau_{n-1}^- = \tau\overline{\Omega}^{n-2}$ (where Ω is the syzygy functor). In fact, the only property we need for these functors is that, if $X \in {}^{\perp_{n-2}}\mathcal{P}$ and $Y \in \mathcal{I}^{\perp_{n-2}}$, the following functorial isomorphisms hold:

(1) $\mathrm{Ext}_{\mathcal{B}}^{n-1}(X, Y) \simeq \mathrm{D}\overline{\mathrm{Hom}}_{\mathcal{B}}(Y, \tau_{n-1}X) \simeq \mathrm{D}\underline{\mathrm{Hom}}_{\mathcal{B}}(\tau_{n-1}^- Y, X)$,
(2) $\forall i \in \{1, 2, ..., n-2\}$,
$\mathrm{Ext}_{\mathcal{B}}^{n-1-i}(X, Y) \simeq \mathrm{D}\mathrm{Ext}_{\mathcal{B}}^{i}(Y, \tau_{n-1}X) \simeq \mathrm{D}\mathrm{Ext}_{\mathcal{B}}^{i}(\tau_{n-1}^- Y, X)$

where $D = \mathrm{HomExt}_k(-, k)$. This is a weak version of [5, Theorem 1.5].

From this, we deduce easily that τ_{n-1} induces an equivalence from $\underline{{}^{\perp_{n-2}}\mathcal{M}}$ to $\overline{\mathcal{M}^{\perp_{n-2}}}$ the inverse of which is $\tau_{n-1}^{-1} = \tau_{n-1}^-$.

Remark that

$$\begin{aligned} X \in \mathcal{M} &\Leftrightarrow \mathrm{Ext}_{\mathcal{B}}^i(X, \mathcal{M}) = 0, \ \forall i \in \{1, 2, ..., n-1\} \\ &\Leftrightarrow \begin{cases} \overline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{M}, \tau_{n-1}X) = 0 \\ \mathrm{Ext}_{\mathcal{B}}^i(\mathcal{M}, \tau_{n-1}X) = 0 \quad \text{for all } i \in \{1, 2, ..., n-2\} \end{cases} \\ &\Leftrightarrow \tau_{n-1}X \in \mathcal{M}^{\perp_{n-2}} \cap \overline{\mathcal{M}}^{\perp}. \end{aligned}$$

Moreover, as $\overline{\mathcal{M}}^{\perp} = \overline{\Omega}\mathcal{M} \subseteq \mathcal{M}^{\perp_{n-2}}$, $X \in \mathcal{M} \Leftrightarrow \tau_{n-1}X \in \overline{\mathcal{M}}$.

Now $X \in \mathcal{P}$ implies that $\mathrm{Ext}_{\mathcal{B}}^{n-1}(X, \mathcal{B}) = 0$, then $\overline{\mathrm{Hom}}_{\mathcal{B}}(\mathcal{B}, \tau_{n-1}X) = 0$, which means $\tau_{n-1}X \in \mathcal{I}$. Dually $X \in \mathcal{I}$ implies that $\tau_{n-1}^{-1}X \in \mathcal{P}$. Hence $X \in \mathcal{P} \Leftrightarrow \tau_{n-1}X \in \mathcal{I}$. We get the following proposition:

Proposition 9. *The functor τ_{n-1} induces an equivalence from $\underline{\mathcal{M}}$ to $\overline{\overline{\Omega}\mathcal{M}}$ and an equivalence from ${}^{\perp_{n-2}}\mathcal{M}/[\mathcal{M}]$ to $\mathcal{M}^{\perp_{n-2}}/[\overline{\Omega}/\mathcal{M}]$.*

Denote by $\overline{\Omega}^{-1}$ the inverse of $\overline{\Omega} : \overline{\mathcal{M}} \to \overline{\overline{\Omega}\mathcal{M}}$. Then we have

Corollary 10. *The compositions $\tau_{n-1}^{-1} \circ \overline{\Omega}$ and $\overline{\Omega}^{-1} \circ \tau_{n-1}$ induce mutually inverse equivalences between $\overline{\mathcal{M}}$ and $\underline{\mathcal{M}}$.*

According to this corollary, we can define reciprocal equivalences:

(1) $\mu : \mathrm{Mod}\underline{\mathcal{M}} \to \mathrm{Mod}\overline{\mathcal{M}}$, $\mu(C) = C \circ \tau_{n-1}^{-1} \circ \overline{\Omega}$,
(2) $\mu^{-1} : \mathrm{Mod}\overline{\mathcal{M}} \to \mathrm{Mod}\underline{\mathcal{M}}$, $\mu^{-1}(C') = C' \circ \overline{\Omega}^{-1} \circ \tau_{n-1}$.

Thus we have:

Proposition 11. *The functors μ and μ^{-1} induce mutually inverse equivalences between* $\mathrm{mod}\underline{\mathcal{M}}$ *and* $\mathrm{mod}\overline{\mathcal{M}}$.

Finally we give:

Theorem 12. *If $\mathcal{B}$ has an $(n-1)$–AR translation τ_{n-1}, then we have a diagram which is commutative up to the equivalence*

$$\begin{tikzcd}
{}^{\perp_{n-2}}\mathcal{M}/[\mathcal{M}] \arrow[r, "F"] \arrow[d, "\tau_{n-1}"'] & \mathrm{mod}\underline{\mathcal{M}} \arrow[d, "\mu"] \\
\mathcal{M}^{\perp_{n-2}}/[\overline{\Omega}\mathcal{M}] \arrow[r, "G"'] & \mathrm{mod}\overline{\mathcal{M}}.
\end{tikzcd}$$

By duality, if we denote by $\mathrm{mod}'\underline{\mathcal{M}}$ (resp. $\mathrm{mod}'\overline{\mathcal{M}}$) the category of finitely copresented modules over $\underline{\mathcal{M}}$ (resp. $\overline{\mathcal{M}}$), we get the following commutative diagram:

$$\begin{tikzcd}
{}^{\perp_{n-2}}\mathcal{M}/[\Omega\mathcal{M}] \arrow[r, "\sim"] \arrow[d, "\tau_{n-1}"'] & \mathrm{mod}'\underline{\mathcal{M}} \arrow[d, "\wr"] \\
\mathcal{M}^{\perp_{n-2}}/[\mathcal{M}] \arrow[r, "\sim"] & \mathrm{mod}'\overline{\mathcal{M}}
\end{tikzcd}$$

where $\Omega\mathcal{M}$ the class of objects $X \in \mathcal{B}$ such that there exists a short exact sequence

$$0 \to X \to P \to M \to 0$$

with $M \in \mathcal{M}$ and P projective.

5. Example

In this section, we explain an example coming directly from representation theory (Auslander algebras).

Let Λ be the Auslander algebra of $k\vec{A}_3$. That is kQ/R where Q is the following quiver

$$\begin{tikzcd}
& & 3 \arrow[dl] & & \\
& 5 \arrow[dl] \arrow[rr, dotted, no head] & & 2 \arrow[ul] \arrow[dl] & \\
6 \arrow[rr, dotted, no head] & & 4 \arrow[ul] \arrow[rr, dotted, no head] & & 1 \arrow[ul]
\end{tikzcd}$$

and the ideal of relations R is generated by the mesh relations symbolized by dashed lines. Then, using the method introduced in [6, §1], one can compute a cluster tilting subcategory $\mathcal{M}$ of $\mathrm{mod}\Lambda$, and the quiver of $\mathcal{M}$ is given in Figure 1.

We can also calculate $\overline{\Omega}\mathcal{M}$ easily since in this case

$$\overline{\Omega}\mathcal{M} = \overline{\mathcal{M}}^{\perp} = \{X \in \mathrm{mod}\Lambda \mid \overline{\mathrm{Hom}}_{\Lambda}(\mathcal{M}, X) = 0\}.$$

In this example, the quiver of $\mathrm{mod}\Lambda/[\mathcal{M}]$ is the following.

$$\begin{tikzcd}[column sep=small]
& \begin{smallmatrix} & 3 \\ 5 & \end{smallmatrix} \arrow[dr] & & & & \begin{smallmatrix} 2 & \\ & 3 \end{smallmatrix} \arrow[dr] & \\
5 \arrow[ur] \arrow[rr, dotted, no head] & & \begin{smallmatrix} 4 & & 3 \\ & 5 & \end{smallmatrix} \arrow[dr] \arrow[rr, dotted, no head] & & \begin{smallmatrix} & 2 & \\ 4 & & 3 \end{smallmatrix} \arrow[ur] \arrow[rr, dotted, no head] & & 2 \\
& & & 3 \arrow[ur] & & &
\end{tikzcd}$$

The quiver of $\underline{\mathcal{M}}$ is the following.

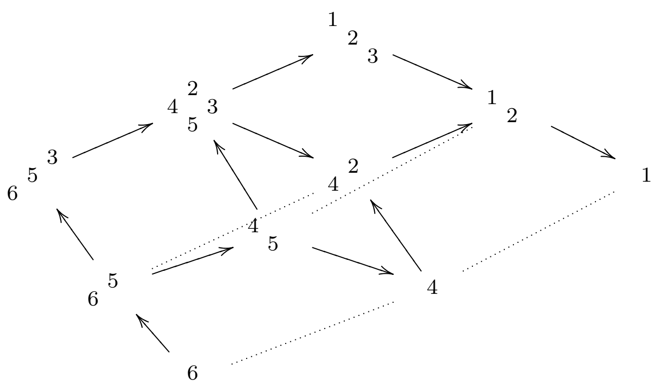
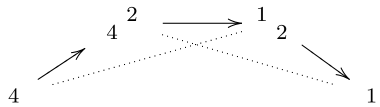


FIGURE 1. Quiver of $\mathcal{M}$



As expected, we obtain that $\mathrm{mod}\Lambda/[\mathcal{M}] \simeq \mathrm{mod}\underline{\mathcal{M}}$. One can also calculate and check the equivalence $\mathrm{mod}\Lambda/[\overline{\mathcal{M}}^{\perp}] \simeq \mathrm{mod}\overline{\mathcal{M}}$.

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