SKEW REES RINGS WHICH ARE MAXIMAL ORDERS

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ABSTRACT. Let R be a Noetherian prime Goldie ring, σ be an automorphism of R and X be an invertible ideal of R. In this paper, we define the $(\sigma;X)-$maximal order and show that a skew Rees ring $R[Xt;\sigma]$ is a maximal order if and only if R is a $(\sigma;X)-$maximal order, which is proved by using the complete description of $v-$ideals of $R[Xt;\sigma]$. We give some examples of $(\sigma;X)-$maximal orders which are not maximal orders (event not $\sigma-$maximal orders) and also of $\sigma-$maximal orders but not $(\sigma;X)-$maximal orders.

1. INTRODUCTION

Throughout this paper, R is a Noetherian prime ring with quotient ring Q (in another word, R is a Noetherian order in a simple Artinian ring Q), σ is an automorphism of R and X is an invertible ideal of R.

Put

$$S = R[Xt,\sigma] = R \oplus Xt \oplus X^2t^2 \oplus .... \oplus X^nt^n \oplus ...$$

which is a subset of the skew polynomial ring $R[t,\sigma]$ in an indeterminate t. If S is a ring, then it is called a *skew Rees ring* associated to X. In this case, S and $R[t;\sigma]$ have the same quotient ring $Q(S) = Q(R[t;\sigma])$ which is a simple Artinian ring.

The aim of this paper is to obtain a necessary and sufficient conditions for S to be a maximal order and to describe the structure of $v-$ideals of S (Theorem 9 and Proposition 11). As applications, we give a necessary and sufficient conditions for S to be a generalized Asano ring and a unique factorization ring in the sense of [1], respectively (Corollary 12). These are done by using a complete description of $v-$ideals in $Q(S)$.

Furthermore we give some examples of rings which are $(\sigma;X)-$maximal orders but not maximal orders (even not $\sigma-$maximal orders). This means S is a maximal order but $R[t;\sigma]$ is not a maximal order. We also give examples of rings which are $\sigma-$maximal orders but not $(\sigma;X)-$maximal orders.

Generalized Rees rings were studied in [8] and [15] under *PI* conditions and in the book [16], they summarized them from torsion theoretical view points under *PI* conditions. Recently Akalan proved in [2] that if R is generalized Asano ring with *PI* conditions, then so is S, which motivates us to study skew Rees rings. Note we do not assume in this paper that R satisfies *PI* conditions.

In [2] Akalan defined generalized Dedekind prime ring R. It turns out that R is a generalized Dedekind ring if and only if it is a maximal order and any $v-$ideal is invertible. In this paper, we say that R is a *generalized Asano ring* if it is a generalized Dedekind ring in the sense of [2], because one-sided $v-$ideals are not necessarily projective.

We refer the readers to the books [12] or [13] for order theory.

The detailed version of this paper will be submitted for publication elsewhere.

2. $(\sigma;X)-$Maximal Orders

First we introduce some notation. For any (fractional) right $R-$ideal I and left $R-$ideal J, let

$$(R:I)_l = \{q \in Q \mid qI \subseteq R\} \text{ and } (R:J)_r = \{q \in Q \mid Jq \subseteq R\}$$

which is a left (right) $R-$ideal, respectively and

$$I_v = (R:(R:I)_l)_r \quad \text{and} \quad {}_vJ = (R:(R:J)_r)_l,$$

which is a right (left) $R-$ideal containing $I(J)$. $I(J)$ is called a *right (left) $v-$ideal* if $I_v = I$ (${}_vJ = J$). In case I is a two-sided $R-$ideal, it is said to be a *$v-$ideal* if $I_v = I = {}_vI$, and if $I \subseteq R$, we just say I is a *$v-$ideal* of R. An $R-$ideal A is said to be *$v-$invertible* if ${}_v((R:A)_lA) = R = (A(R:A)_r)_v$. We start with the following elementary lemma, which is frequently used in the paper.

Lemma 1. *Let A be an $R-$ideal and I be a right $R-$ideal.*

(1) *If A is $v-$invertible, then $O_r(A) = R = O_l(A)$ and $(R:A)_l = A^{-1} = (R:A)_r$, where $A^{-1} = \{q \in Q \mid AqA \subseteq A\}$.*
(2) *$(IA_v)_v = (IA)_v$. If A is $v-$invertible, then $(I_vA_v)_v = (IA)_v$.*

The following proposition is one of the crucial properties which shows a relation between ideals of R and of S.

Proposition 2. (1) *$S = R[Xt;\sigma]$ is a ring if and only if $\sigma(X) = X$. In this case, S is also Noetherian.*
(2) *Suppose $\sigma(X) = X$.*
  (i) *Let $\mathfrak{a}$ be an deal of R. Then*

$$\mathfrak{a}[Xt;\sigma] = \mathfrak{a} \oplus \mathfrak{a}Xt \oplus \mathfrak{a}X^2t^2 \oplus \ldots \oplus \mathfrak{a}X^nt^n \oplus \ldots$$

  is an ideal of S if and only if $X\sigma(\mathfrak{a}) = \mathfrak{a}X$.
  (ii) *Let $\mathfrak{a}$ be an $R-$ideal in Q with $X\sigma(\mathfrak{a}) = \mathfrak{a}X$. Then $\mathfrak{a}[Xt;\sigma]$ is an $S-$ideal in $Q(S)$.*

In the remainder of this paper, we assume that $S = R[Xt;\sigma]$ is a ring and put $T = Q[t;\sigma]$, the skew polynomial ring over Q. Note that T is a principal ideal ring ([3, Corollary 6.2.2] or [12, Corollary 2.3.7]) and we use this property to study $S-$ideal.

Lemma 3. *Let I be a right $S-$ideal and J be a left $S-$ideal. Then*

(1) *$(T:IT)_l = T(S:I)_l$ and $(T:TJ)_r = (S:J)_rT$.*
(2) *$(IT)_v = I_vT$ and ${}_v(TJ) = T_vJ$.*
(3) *If I' is a right ideal of T, then $I' = (I' \cap S)T$. If I' is an essential right ideal, then $(I' \cap S)_v = I' \cap S$.*

It is very important to investigate prime v-ideals P of S and there are two case whether $P \cap R$ is (0) or not. In case $P \cap R = (0)$, we have the following by using Lemma 3.

Lemma 4. *Let $T = Q[t;\sigma]$. There is a $(1-1)-$correspondence between*

$$\mathrm{Spec}_0(S) = \{P: \textit{ prime ideal of } S \mid P \cap R = (0)\} \quad \textit{and} \quad \mathrm{Spec}(T)$$

via $P \longmapsto PT$, $P' \longmapsto P' \cap S$. In particular, P is a $v-$ideal.

To express the case $P \cap R \neq (0)$, we need some preliminaries. Let $\mathfrak{a}$ be a right $R-$ideal. Then $\mathfrak{a}[Xt;\sigma] = \mathfrak{a} \oplus \mathfrak{a}Xt \oplus ... \oplus \mathfrak{a}X^n t^n \oplus ...$ is a right $S-$ideal. Similarly for any left $R-$ideal $\mathfrak{b}$, $S\mathfrak{b} = \mathfrak{b} \oplus tX\mathfrak{b} \oplus ... \oplus t^n X^n \mathfrak{b} \oplus ...$ is a left $S-$ideal.

Lemma 5. *Let $\mathfrak{a}$ be a right $R-$ideal and $\mathfrak{b}$ be a left $R-$ideal. Then*

$$(S : \mathfrak{a}[Xt;\sigma])_l = S(R : \mathfrak{a})_l \ \ \textit{and} \ \ (S : S\mathfrak{b})_r = (R : \mathfrak{b})_r S$$

In particular, $(\mathfrak{a}[Xt;\sigma])_v = \mathfrak{a}_v[Xt;\sigma]$ and $_v(S\mathfrak{b}) = S_v\mathfrak{b}$.

It is well known that σ is naturally extended to an automorphism of $Q(R[t;\sigma])$ by $\sigma(f(t)) = tf(t)t^{-1}$ for any $f(t) \in R[t;\sigma]$. Note that σ induces an automorphism of S. Let $\mathfrak{a}$ be an ideal of R. We showed in Proposition 2 that $\mathfrak{a}[Xt;\sigma]$ is an ideal of S if and only if $X\sigma(\mathfrak{a}) = \mathfrak{a}X$ which is crucial property for S to be a maximal order. In general, a subset I of $Q(S)$ is said to be $(\sigma;X)-$*invariant* if $X\sigma(I) = IX$.

R is said to be a $(\sigma;X)-$*maximal order* if $O_l(\mathfrak{a}) = R = O_r(\mathfrak{a})$ for any $(\sigma;X)-$invariant ideal of R. If R is a $(\sigma;X)-$maximal order, then it is proved that $O_l(\mathfrak{a}) = R = O_r(\mathfrak{a})$ for any $(\sigma;X)-$invariant $R-$ideal $\mathfrak{a}$. Hence $(R:\mathfrak{a})_l = \mathfrak{a}^{-1} = (R:\mathfrak{a})_r$ where $\mathfrak{a}^{-1} = \{q \in Q \mid \mathfrak{a}q\mathfrak{a} \subseteq \mathfrak{a}\}$ and $\mathfrak{a}_v = \mathfrak{a}^{-1-1} = {}_v\mathfrak{a}$ follows.

Let $D_{\sigma,X}(R)$ be the set of all $(\sigma;X)-$invariant $v-$ideals. For any $\mathfrak{a},\mathfrak{b} \in D_{\sigma,X}(R)$, we define $\mathfrak{a} \circ \mathfrak{b} = (\mathfrak{a}\mathfrak{b})_v$. Then we have the following whose proof is similar to one in the maximal orders ([12, (2.1.2)]).

Proposition 6. *Let R be a $(\sigma;X)-$maximal order in Q. Then $D_{\sigma,X}(R)$ is an Abelian group generated by maximal $(\sigma;X)-$invariant $v-$ideals of R.*

The following lemmas show how to obtain prime ideals of S from ideals of R and how to connect ideals of S with ideals of R.

Lemma 7. *Suppose R is a $(\sigma;X)-$maximal order in Q. Let $\mathfrak{p}$ be a maximal $(\sigma;X)-$invariant $v-$ideal of R. Then $P = \mathfrak{p}[Xt;\sigma]$ is a prime ideal and it is a $v-$ideal.*

Lemma 8. *Suppose R is a $(\sigma;X)-$maximal order in Q. Let A be an ideal of S with $A = A_v$ and $\mathfrak{a} = A \cap R \neq (0)$. Then*

(1) *A and $\mathfrak{a}$ are $(\sigma;X)-$invariant.*
(2) *$A = \mathfrak{a}[Xt;\sigma]$ and is $v-$invertible.*

Theorem is proved by mainly using Lemmas 3 and 8.

Theorem 9. *Let R be a Noetherian prime ring with its quotient ring Q, σ be an automorphism of R and $S = R[Xt;\sigma]$ be a skew Rees ring associated to X, where X is an invertible ideal with $\sigma(X) = X$. Then R is a $(\sigma;X)-$maximal order if and only if $S = R[Xt;\sigma]$ is a maximal order in $Q(S)$.*

3. Applications, Examples and Conjectures

As applications of Theorem 9, we give a necessary and sufficient conditions for S to be a generalized Asano ring and a unique factorization ring (a UFR). Furthermore we give Noetherian prime rings which are $(\sigma;X)-$maximal orders (but not maximal orders) and $(\sigma;X)-$maximal orders (but not $\sigma-$maximal orders) where an order R is called a *$\sigma-$maximal order* if for any ideal $\mathfrak{a}$ with $\sigma(\mathfrak{a}) = \mathfrak{a}$, $O_l(\mathfrak{a}) = R = O_r(\mathfrak{a})$.

If R is a $(\sigma;X)-$maximal order, then S is a maximal order and so $D(S)$, the set of all $v-$ideals in $Q(S)$, is an Abelian group generated by prime $v-$ideals of S (see [12, Theorem 2.1.2]). Note that any maximal $v-$ideal of S is a prime $v-$ideal and the converse is also true. The set of principal $S-$ideals in $Q(S)$ is a subgroup $P(S)$ of $D(S)$. The factor group $D(S)/P(S)$ is called the *class group* of S and denoted by $C(S)$. Similarly $P_{\sigma,X}(R)$, the set of $(\sigma;X)-$invariant principal $R-$ideals in Q is a subgroup of $D_{\sigma,X}(R)$ and $C_{\sigma,X}(R) = D_{\sigma,X}(R)/P_{\sigma,X}(R)$ is called the *$(\sigma;X)-$class group* of R.

First we describe the structure of $v-$ideals in $Q(S)$ as follows (this is proved by using Lemma 8 and [12, (2.3.11)]):

Proposition 10. *Suppose R is a $(\sigma;X)-$maximal order and let A be a $v-$ideal in $Q(S)$. Then $A = t^n w\mathfrak{a}[Xt;\sigma]$ for some $\mathfrak{a} \in D_{\sigma,X}(R)$, $w \in Z(Q(T))$ the center of $Q(T)$ and n is an integer.*

The statement (1) of Proposition 11 follows from Lemmas 3 and 8. To prove the second statement, consider the mapping $\varphi : D_{\sigma,X}(R) \to D(S)$ given by $\varphi(\mathfrak{a}) = \mathfrak{a}[Xt;\sigma]$ for any $\mathfrak{a} \in D_{\sigma,X}(R)$.

Proposition 11. *Suppose R is a $(\sigma;X)-$maximal order. Then*
(1) $D(S) \cong D_{\sigma,X}(R) \oplus D(T)$.
(2) $C(S) \cong C_{\sigma,X}(R)$.

An order R is called a *generalized Asano ring* (a *G-Asano ring*) if it is a maximal order and every $v-$ ideal of R is invertible. Similarly R is called a *generalized $(\sigma;X)-$Asano ring (a $G-(\sigma;X)-$Asano ring)* if it is a $(\sigma;X)-$maximal order and every $(\sigma;X)-$invariant $v-$ideals of R is invertible. If R is a $G-(\sigma;X)-$Asano ring, then S is a $G-$Asano ring by Proposition 10. The converse is also true which is proved by using Lemma 5.

In [1], they defined a non-commutative unique factorization ring (a UFR). It turns out that an order is a UFR if and only if it is a maximal order and every $v-$ideal is principal. We can define, in an obvious way, the concept of a $(\sigma;X)-$UFR and it follows from Proposition 11 that R is a $(\sigma;X)-$UFR if and only if $C_{\sigma,X}(R) = (0)$. Hence we have

Corollary 12. (1) *R is a $G-$ $(\sigma;X)-$Asano ring if and only if $S = R[Xt;\sigma]$ is a $G-$Asano ring.*
(2) *R is a $(\sigma;X)-$UFR if and only if S is a UFR.*

Now we give some examples of $(\sigma;X)-$maximal orders but not maximal orders (even not $\sigma-$maximal orders). We also give examples of $\sigma-$maximal orders but not $(\sigma;X)-$maximal orders. The first example is a trivial case.

Example 1. Any Noetherian maximal order R is a $(\sigma;X)-$maximal order and a $\sigma-$maximal order. Hence S and $R[t;\sigma]$ are maximal orders (Theorem 9 and [12, Theorem 2.3.19]).

Let R be an HNP ring satisfying the following conditions :
(a) There is a cycle $\mathfrak{m}_1, \mathfrak{m}_2, ..., \mathfrak{m}_n$ $(n \geq 2)$ such that $\mathfrak{p} = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap ... \cap \mathfrak{m}_n$ is principal, say $\mathfrak{p} = aR = Ra$ for some $a \in \mathfrak{p}$.
(b) Any maximal ideal different from $\mathfrak{m}_i (1 \leq i \leq n)$ is invertible.

See [1] for examples of HNP rings satisfying conditions (a) and (b). Define an automorphism σ of R by $\sigma(r) = ara^{-1}$ for $r \in R$. Then it follows from [1] that

(1) $\sigma(\mathfrak{m}_1) = \mathfrak{m}_2, ..., \sigma(\mathfrak{m}_n) = \mathfrak{m}_1$ and
(2) $\sigma(\mathfrak{n}) = \mathfrak{n}$ for all maximal ideals $\mathfrak{n}$ with $\mathfrak{n} \neq \mathfrak{m}_i$ $(1 \leq i \leq n)$.

Example 2. Suppose R is an HNP ring with the conditions (a) and (b).

(1) Put $X = \mathfrak{n}_1^{e_1}...\mathfrak{n}_k^{e_k}$, where $\mathfrak{n}_j$ are maximal ideals different from $\mathfrak{m}_i$ $(1 \leq i \leq n)$. Then R is a $(\sigma;X)-$maximal order which is not a maximal order (in fact, it is a $G-(\sigma;X)-$Asano ring as well as a $\sigma-G-$Asano ring), but it is a $\sigma-G-$Asano ring. Hence S and $R[t;\sigma]$ are $G-$Asano rings.
(2) Put $X = \mathfrak{p}$. Then
   (i) If $n = 2$, then R is not a $(\sigma;X)-$maximal order and so S is not a maximal order.
   (ii) If $n \geq 3$, then R is a $(\sigma;X)-$maximal order and so S is a maximal order (in fact, it is a $G-$Asano ring).

As in Example 2, put $X = \mathfrak{p}$. Then since $\sigma(\mathfrak{m}_i) = X\mathfrak{m}_iX^{-1}$, we have $X\sigma^{-1}(\mathfrak{m}_i) = \mathfrak{m}_iX$ and so R is not a $(\sigma^{-1};X)-$ maximal order. Hence we have

Remark 1 Under the same notation and assumptions as in Example 2(2), $S_1 = R[Xt;\sigma^{-1}]$ is not a maximal order and $R[t;\sigma^{-1}]$ is a maximal order.

Next we give examples of rings which are $(\sigma;X)-$maximal orders but not $\sigma-$maximal orders.

Let k be a field with automorphism σ and let $K = \begin{pmatrix} k & k \\ k & k \end{pmatrix}$, the ring of 2×2 matrices over k. Then we can extend σ to an automorphism of K by $\sigma(q) = \begin{pmatrix} \sigma(a) & \sigma(b) \\ \sigma(c) & \sigma(d) \end{pmatrix}$, where $q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Let $U = K[x;\sigma]$ and $I = eK + xU$, where $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then I is a $\sigma-$invariant maximal right ideal of U with $UI = U$. We consider $R = \{u \in U \mid uI \subseteq I\}$, the idealizer of I. By [13, Theorem 5.5.10], R is an HNP ring and I is an idempotent maximal ideal of R. We note that $R = K(1-e)+eK+xU$. R has another idempotent maximal ideal $J = K(1-e)+xU$, which is a $\sigma-$invariant maximal left ideal of U with $JU = U$. Put $X = I \cap J = eK(1-e)+xU$. Since $O_r(I) = U = O_l(J)$ and $O_r(J) = x^{-1}(eK(1-e))+R = O_l(I)$, $\{I,J\}$ is a cycle and X is an invertible ideal of R by [5, Proposition 2.5].

Example 3. Under the same notation and assumptions,

(1) R is not a $\sigma-$maximal order and $R[t;\sigma]$ is not a maximal order.
(2) R is a $(\sigma;X)-$maximal order and S is a maximal order (in fact, S is a $G-$Asano ring). Furthermore
   (i) If σ is of infinite order, then XS and XtS are only prime $v-$ideals of S.
   (ii) If σ is of finite order, say n, then there are infinite number of prime $v-$ideals of S.

Remark 2 There exist some examples of maximal orders which are not $G-$Asano rings ([2, Example 3.4] and [11, Example]).

Remark 3 In Examples 2 and 3, the rings are all HNP rings. However, by using examples in [10] we can provide $(\sigma;X)-$maximal orders which are neither HNP rings nor maximal orders. We will show them in detail in the forth-coming paper.

Finally we introduce a conjecture concerning skew Rees rings.

Problem Let $S = R[Xt;\sigma,\delta]$ be a subset of an Ore extension $R[t;\sigma,\delta]$, where δ is a left σ-derivation of R. Then what is a necessary and sufficient condition for S to be a maximal order or a generalized Asano ring?

References

[1] G. Q. Abbasi, S. Kobayashi, H. Marubayashi and A. Ueda, *Noncommutative unique factorization rings*, Comm. in Algebra, **19(1)** (1991), 167–198.

[2] E. Akalan, *On generalized Dedekind prime rings*, J. of Algebra, **320** (2008), 2907–2916.

[3] G.Cauchon, *Les T-anneaux et les anneaux a identites polynomiales Noetheriens*, these, (1977).

[4] M. Chamarie, *Anneaux de Krull non commutative*, these, (1981).

[5] D. Eisenbud and J.C. Robson, *Hereditary Noetherian prime rings*, J. of Algebra, **16** (1970), 86–104.

[6] A.J. Gray, *A note on the invertible ideal theorem*, Glasgow Math. J., **25** (1984), 27–30.

[7] N. Jacobson, *Pseudo-linear transformations*, Annals of Math. **38 (2)** (1937), 484–507.

[8] L. Le Bruyn and F. Van Oystaeyen, *Generalized Rees rings and relative maximal order satisfying polynomial identities*, J. Algebra **83 (2)** (1983), 409–436.

[9] H. Marubayashi, *A Krull type generalization of HNP rings with enough invertible ideals*, Comm. in Algebra, **11(5)** (1983), 469–499.

[10] H. Marubayashi, Intan Muchtadi-Alamsyah and A. Ueda, *Skew polynomial rings which are generalized Asano prime rings*, preprint.

[11] H. Marubayashi and K. Nishida, *Overrings of Krull orders*, Osaka J. Math. **16**, (1979), 843-851.

[12] H. Marubayashi and F. Van Oystaeyen, *Prime Divisors and Noncommutative Valuation Theory*, Lecture Notes in Mathematics 2059, Springer, (2013).

[13] J.C. McConnel and J.C. Robson, *Noncommutative Noetherian Rings*, Wiley, (1987).

[14] B. Stenstrőm, *Rings of Quotients*, Springer-Verlag, (1975).

[15] F. Van Oystaeyen, *Generalized Rees rings and arithmetical graded rings*, J. Algebra **82(1)**, (1983), 185–193.

[16] F. Van Oystaeyen and A. Verschoren, *Relative Invariants of rings*, The Noncommutative Theory, Pure and Applied Mathematics, Marcel Dekker, Inc., Vol. **86** (1984).

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