ON TORIC RINGS ARISING FROM CYCLIC POLYTOPES

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Abstract. Let d and n be positive integers with $n \geq d+1$ and $\mathcal{P} \subset \mathbb{R}^d$ an integral cyclic polytope of dimension d with n vertices. For a field K, let $K[\mathcal{P}] = K[\mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P}]$ denote its associated semigroup K-algebra, where $\mathcal{A}_\mathcal{P} = \{(1,\alpha) \in \mathbb{R}^{d+1} : \alpha \in \mathcal{P}\} \cap \mathbb{Z}^{d+1}$. In this draft, we study when $K[\mathcal{P}]$ is normal or very ample. Moreover, we also consider the problem when $K[\mathcal{P}]$ is Cohen–Macaulay by discussing Serre's condition (R_1) and we give a complete characterization when $K[\mathcal{P}]$ is Gorenstein. In addition, we investigate the normality of the other semigroup K-algebra $K[Q]$ arising from an integral cyclic polytope, where Q is a semigroup generated only with its vertices.

1. Introduction

This draft is based on a joint work with Takayuki Hibi, Lukas Katthän and Ryota Okazaki.

The cyclic polytope is one of the most distinguished polytopes and played the essential role in the classical theory of convex polytopes. Let d and n be positive integers with $n \geq d+1$ and $\tau_1, \ldots, \tau_n$ real numbers with $\tau_1 < \cdots < \tau_n$. We write $C_d(\tau_1, \ldots, \tau_n) \subset \mathbb{R}^d$ for the convex hull of $\{(\tau_i, \tau_i^2, \ldots, \tau_i^d) \in \mathbb{R}^d : i = 1, \ldots, n\}$. The convex polytope $C_d(\tau_1, \ldots, \tau_n) \subset \mathbb{R}^d$ is called a *cyclic polytope* of dimension d with n vertices. In particular, we say that it is an *integral cyclic polytope* if $\tau_1, \ldots, \tau_n$ are all integers. A cyclic polytope is a simplicial polytope and its combinatorial type is independent of a choice of $\tau_1, \ldots, \tau_n$. Moreover, it is well known that a cyclic polytope is a convex polytope which attains the upper bound in the Upper Bound Theorem.

In this draft, we focus on *integral* cyclic polytopes and discuss some properties on toric rings arising from integral cyclic polytopes.

In general, for an integral convex polytope $\mathcal{P}$, let $\mathcal{P}^* \subset \mathbb{R}^{N+1}$ be the convex hull of $\{(1,\alpha) \in \mathbb{R}^{N+1} : \alpha \in \mathcal{P}\}$ and $\mathcal{A}_\mathcal{P} = \mathcal{P}^* \cap \mathbb{Z}^{N+1}$. Then $\mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P}$ is an affine semigroup. Let K be a field. Then we set

$$K[\mathcal{P}] := K[\mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P}],$$

i.e., $K[\mathcal{P}]$ is an affine semigroup K-algebra associated with $\mathcal{P}$, and we call it a *toric ring* arising from $\mathcal{P}$.

For an integral convex polytope $\mathcal{P} \subset \mathbb{R}^N$, we say that $\mathcal{P}$ ($K[\mathcal{P}]$ or $\mathbb{R}_{\geq 0}\mathcal{A}_\mathcal{P}$) is *normal* if it satisfies

$$\mathbb{R}_{\geq 0}\mathcal{A}_\mathcal{P} \cap \mathbb{Z}\mathcal{A}_\mathcal{P} = \mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P}.$$

The detailed version of this paper will be submitted for publication elsewhere.

We say that $\mathcal{P}$ ($K[\mathcal{P}]$ or $\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}}$) is *very ample* if the set

$$(\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}\mathcal{A}_{\mathcal{P}}) \setminus \mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$$

is finite. Thus, normal integral convex polytopes are always very ample.

Let, as before, d and n be positive integers with $n \geq d+1$. Given integers $\tau_1, \ldots, \tau_n$ with $\tau_1 < \cdots < \tau_n$, one of our goals is to classify the integers $\tau_1, \ldots, \tau_n$ with $\tau_1 < \cdots < \tau_n$ for which $C_d(\tau_1, \ldots, \tau_n)$ is normal. Even though to find such a complete classification seems to be difficult, many fascinating problems arise in the natural way. Our first main result is that if $\tau_{i+1} - \tau_i \geq d^2 - 1$ for $1 \leq i < n$, then $C_d(\tau_1, \ldots, \tau_n)$ is normal. Moreover, it is also shown that if $d \geq 4$ and $\tau_3 - \tau_2 = 1$ or $\tau_{n-1} - \tau_{n-2} = 1$, then $C_d(\tau_1, \ldots, \tau_n)$ is non-very ample.

Let $\mathcal{P}$ be an integral cyclic polytope. We will also consider the Cohen–Macaulayness and the Gorensteinness of the toric ring $K[\mathcal{P}]$. By proving that $K[\mathcal{P}]$ always satisfies Serre's condition (R_1), it follows that $K[\mathcal{P}]$ is Cohen–Macaulay if and only if $K[\mathcal{P}]$ is normal. Thus the characterization of Cohen–Macaulayness of integral cyclic polytopes is nothing but that of normality. Moreover, it turns out that $K[\mathcal{P}]$ is Gorenstein if and only if one has $d = 2, n = 3$ and $(\tau_2 - \tau_1, \tau_3 - \tau_2) = (2, 1)$ or $(1, 2)$.

In addition, we also define another toric rings arising from integral cyclic polytopes. Let

$$Q := Q_d(\tau_1, \ldots, \tau_n) := \mathbb{Z}_{\geq 0}\{(1, \tau_i, \tau_i^2, \ldots, \tau_i^d) \in \mathbb{Z}^{d+1} : i = 1, \ldots, n\}.$$

Then we write $K[Q]$ for the toric ring associated with the configuration Q. In other words, $K[Q]$ is a toric ring arising from the matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \tau_1 & \tau_2 & \cdots & \tau_n \\ \vdots & \vdots & \vdots & \vdots \\ \tau_1^d & \tau_2^d & \cdots & \tau_n^d \end{pmatrix}, \tag{1.1}$$

which is nothing but the Vandermonde matrix. We will show that if $d \geq 2$ and $n = d+2$, then $K[Q]$ is not normal.

2. Normal cyclic polytopes and non-very ample cyclic polytopes

The first main result of this draft is the following

Theorem 1 ([5, Theorem 2.1 and Theorem 3.1]). *Let d and n be positive integers with $n \geq d+1$ and $C_d(\tau_1, \ldots, \tau_n)$ an integral cyclic polytope, where $\tau_1 < \cdots < \tau_n$.*
(a) For each $1 \leq i \leq n-1$, if

$$\tau_{i+1} - \tau_i \geq d^2 - 1,$$

then $C_d(\tau_1, \ldots, \tau_n)$ is normal.
(b) Let $d \geq 4$. If

$$\textit{either} \quad \tau_3 - \tau_2 = 1 \quad \textit{or} \quad \tau_{n-1} - \tau_{n-2} = 1$$

is satisfied, then $C_d(\tau_1, \ldots, \tau_n)$ is not very ample.

Remark 2. Since each lattice length of an edge $\mathrm{conv}(\{(\tau_i, \tau_i^2, \ldots, \tau_i^d), (\tau_j, \tau_j^2, \ldots, \tau_j^d)\})$ of $\mathcal{P}$ coincides with $\tau_j - \tau_i$, where $i < j$, it follows immediately from [4, Theorem 1.3 (b)] that $\mathcal{P}$ is normal if $\tau_{i+1} - \tau_i \geq d(d+1)$ for $1 \leq i \leq n-1$. Thus, our constraint $\tau_{i+1} - \tau_i \geq d^2 - 1$ on integral cyclic polytopes is better than a general case, but this bound is still very rough. For example, $C_3(0,1,2,3)$ is normal, while we have $\tau_2 - \tau_1 = \tau_3 - \tau_2 = \tau_4 - \tau_3 = 1 < 8$. Similarly, $C_4(0,1,3,5,6)$ is also normal, although one has $\tau_2 - \tau_1 = \tau_5 - \tau_4 = 1$ and $\tau_3 - \tau_2 = \tau_4 - \tau_3 = 2$.

On the case where $d = 2$, it is well known that there exists a unimodular triangulation for every integral convex polytope of dimension 2. Therefore, integral convex polytopes of dimension 2 are always normal.

On the case where $d = 3$ and $d = 4$, exhaustive computational experiences lead us to give the following

Conjecture 3. *(a) All cyclic polytopes of dimension 3 are normal.*
(b) A cyclic polytope of dimension 4 is normal if and only if we have

$$\tau_3 - \tau_2 \geq 2 \quad \textit{and} \quad \tau_{n-1} - \tau_{n-2} \geq 2.$$

3. Cohen–Macaulay toric rings and Gorenstein toric rings arising from cyclic polytopes

Recall that a Noetherian ring R is said to satisfy (S_n) if

$$\mathrm{depth} R_{\mathfrak{p}} \geq \min\{n, \dim R_{\mathfrak{p}}\}$$

for all $\mathfrak{p} \in \mathrm{Spec}(R)$, and satisfy (R_n) if $R_{\mathfrak{p}}$ is a regular local ring for all $\mathfrak{p} \in \mathrm{Spec}(R)$ with $\dim R_{\mathfrak{p}} \leq n$. The conditions (S_n) and (R_n) are called Serre's conditions.

The well-known criterion for normality of a Noetherian ring, Serre's Criterion (cf. [2, Theorem 2.2.22]), says that a Noetherian ring is normal if and only if it satisfies (R_1) and (S_2).

By using the combinatorial criterion of (R_1), which can be found in [1, Exercises 4.15 and 4.16], we can show

Proposition 4. *Let $\mathcal{P}$ be an integral cyclic polytope. Then $K[\mathcal{P}]$ always satisfies the condition (R_1).*

As a consequence of this proposition, we obtain

Theorem 5. *Let $\mathcal{P}$ be an integral cyclic polytope and $K[\mathcal{P}]$ its toric ring. Then the following conditions are equivalent:*

(1) *$K[\mathcal{P}]$ is normal;*
(2) *$K[\mathcal{P}]$ is Cohen–Macaulay;*
(3) *$K[\mathcal{P}]$ satisfies the condition (S_2).*

Remark 6. One can also prove that an integral cyclic polytope is normal if and only if it is seminormal. See [1, p. 66] for the definition and basic properties of seminormality. We use the notation from that book. Now, assume that $\mathcal{P}$ is not normal. Then there exists a point m in $\mathbb{R}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathbb{Z}\mathcal{A}_{\mathcal{P}}$ which is not contained in $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. This point m lies in the interior of a unique face $\mathcal{F}$ of $\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}}$. But using the same construction as above, we can show that $\mathbb{Z}(\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathcal{F}) = \mathbb{Z}^{d+1} \cap \mathcal{H}$, where $\mathcal{H}$ is the linear subspace spanned by $\mathcal{F}$. Thus $m \in \mathbb{Z}(\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathcal{F})$ is an exceptional point, and therefore $(\mathbb{Z}_{\geq 0}\mathcal{A}_{\mathcal{P}} \cap \mathcal{F})_*$ is not normal. Hence, $\mathcal{P}$ is not seminormal.

Moreover, we also obtain a complete characterization when $K[\mathcal{P}]$ is Gorenstein as follows.

Theorem 7. *Let $\mathcal{P}$ be an integral cyclic polytope and $K[\mathcal{P}]$ its toric ring. Then $K[\mathcal{P}]$ is Gorenstein if and only if*

$$d = 2,\ n = 3,\ (\tau_2 - \tau_1, \tau_3 - \tau_2) = (1, 2) \text{ or } (2, 1)$$

is satisfied.

4. The semigroup ring associated only with vertices of a cyclic polytope

Let Q denote the afiine semigroup $Q_d(\tau_1, \ldots, \tau_n)$ arising from the matrix (1.1). Let $S = K[x_1, \ldots, x_n]$ be the polynomial ring over a field K and $K[Q]$ an affine semigroup K-algebra generated by the monomials $\{t_0 t_1^{\tau_i} \cdots t_d^{\tau_i^d} : i = 1, \ldots, n\}$, which is a subring of the Laurent polynomial ring $K[t_0, t_1^{\pm}, \ldots, t_d^{\pm}]$. Let I_Q be the kernel of the surjective ring homomorphism $S \to K[Q]$ which sends each x_i to $t_0 t_1^{\tau_i} \cdots t_d^{\tau_i^d}$. The ideal I_Q is just the toric ideal associated with the matrix (1.1). In particular, it is homogeneous with respect to the usual $\mathbb{Z}$-grading on S.

When $n = d + 1$, since the matrix (1.1) is nonsingular, $K[Q]$ is regular. In particular, it is normal. When $d = 1$, the matrix (1.1) can be transformed into

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & \tau_2 - \tau_1 & \cdots & \tau_n - \tau_1 \end{pmatrix}.$$

Since I_Q is preserved even if we divide a common divisor of $(\tau_2 - \tau_1), \ldots, (\tau_n - \tau_1)$ out of the second row, we may assume the greatest common divisor of $\tau_2 - \tau_1, \ldots, \tau_n - \tau_1$ is equal to 1. The ideal I_Q is a defining ideal of a projective monomial curve in $\mathbb{P}^{n-1}$, and it is well known (cf. [3]) that the corresponding curve is normal if and only if it is a rational normal curve of degree $n - 1$, that is, $\tau_i - \tau_1 = i - 1$ for all i with $2 \leq i \leq n$ (after the above transformation and re-setting each $\tau_i - \tau_1$). Consequently, in the case $d = 1$, the ring $K[Q]$ is normal if and only if $\tau_2 - \tau_1 = \tau_3 - \tau_2 = \cdots = \tau_n - \tau_{n-1}$. Hence we assume that $d \geq 2$ and $n \geq d + 1$.

Theorem 8. *Let Q be as above.*
(a) If $d \geq 2$ and $n = d + 2$, then $K[Q]$ is not normal.
(b) When $n \geq d + 3$, if $\prod_{k=1}^{d}(\tau_{d+1} - \tau_k) \nmid \prod_{k=1}^{d}(\tau_s - \tau_k)$ for some s with $d + 2 \leq s \leq n$, then $K[Q]$ is not normal.

Remark 9. When $n = d + 2$, since I_Q is principal, $K[Q]$ is Gorenstein. Hence, it is also Cohen–Macaulay.

Conjecture 10. *Let $K[Q]$ be as above. Then*
(a) $K[Q]$ is never normal if $d \geq 2$ and $d \geq n + 3$;
(b) $K[Q]$ is never Cohen–Macaulay if $d \geq 2$ and $d \geq n + 3$.

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