CONSTRUCTIONS OF AUSLANDER-GORENSTEIN LOCAL RINGS

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ABSTRACT. Generalizing the notion of crossed product, we provide systematic constructions of Auslander-Gorenstein local rings starting from an arbitrary Auslander-Gorenstein local ring.

1. INTRODUCTION

Auslander-Gorenstein rings (see Definition 2) appear in various areas of current research. For instance, regular 3-dimensional algebras of type A in the sense of Artin and Schelter, Weyl algebras over fields of characteristic zero, enveloping algebras of finite dimensional Lie algebras and Sklyanin algebras are Auslander-Gorenstein rings (see [2], [3], [4] and [12], respectively). However, little is known about constructions of Auslander-Gorenstein rings. It was shown in [7] that a left and right noetherian ring is an Auslander-Gorenstein ring if it admits an Auslander-Gorenstein resolution over another Auslander-Gorenstein ring. In this note, generalizing the notion of crossed product (see e.g. [8], [11] and so on), we will provide systematic constructions of Auslander-Gorenstein local rings starting from an arbitrary Auslander-Gorenstein local ring.

In order to provide the construction, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [9, 10], which we modify as follows. A ring A is said to be an extension of a ring R if A contains R as a subring, and the notation A/R is used to denote that A is an extension ring of R. A ring extension A/R is said to be Frobenius if the following conditions are satisfied: (F1) $A \in \text{Mod-}R$ and $A \in \text{Mod-}R^{\text{op}}$ are finitely generated projective; and (F2) $A \cong \text{Hom}_R(A, R)$ in Mod-A and $A \cong \text{Hom}_{R^{\text{op}}}(A, R)$ in Mod-A^{op} (see [1]). If R is a noetherian ring, a Frobenius extension A/R is a typical example of a noetherian ring A admitting Auslander-Gorenstein resolution over R, so that if R is an Auslander-Gorenstein ring then so is A with inj dim $A \leq$ inj dim R, where the equality holds whenever A/R is split, i.e., the inclusion $R \to A$ is a split monomorphism of R-R-bimodules (Proposition 7).

Generalizing the notion of crossed product, we will define new multiplications on the ring of full matrices and the group ring of finite cyclic groups. Let $n \geq 2$ be an integer and set $I(n) = \{1, \dots, n\}$. We fix a cyclic permutation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & 1 & \cdots & n-1 \end{pmatrix}$$

of $I(n)$. Then the law of composition $I(n) \times I(n) \to I(n), (i, j) \mapsto \pi^{-i}(j)$ makes $I(n)$ a cyclic group. We denote by $\Omega(n)$ the set of mappings $\omega : I(n) \times I(n) \to \mathbb{Z}$ satisfying the

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following conditions: (W1) $\omega(i,i) = 0$ for all $i \in I(n)$; (W2) $\omega(i,j) + \omega(j,k) \geq \omega(i,k)$ for all $i,j,k \in I(n)$; (W3) $\omega(i,j) + \omega(j,i) \geq 1$ unless $i = j$; and (W4) $\omega(i,j) + \omega(j,\pi(i)) = \omega(i,\pi(i))$ for all $i,j \in I(n)$. We fix $\omega \in \Omega(n)$ and a ring R together with a pair (σ, c) of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ such that $\sigma(c) = c$ and $xc = c\sigma(x)$ for all $x \in R$. For instance, for any ring R and any $\sigma \in \mathrm{Aut}(R)$, a skew power series ring $R[[t;\sigma]]$ has such a pair (σ, t) (Example 8).

Denote by $\Omega_+(n)$ the subset of $\Omega(n)$ consisting of $\omega \in \Omega(n)$ such that $\omega(1,i) = \omega(i,n) = 0$ for all $i \in I(n)$. Set $\chi(i) = \sum_{k=1}^{i} \omega(k,\pi(k))$ for $i \in I(n)$. Assume that $\omega \in \Omega_+(n)$ and that $\sigma^{\chi(n)} = \mathrm{id}_R$. Let A be a free right R-module with a basis $\{v_i\}_{i \in I(n)}$ and define a multiplication on A subject to the following axioms: (G1) $v_i v_j = v_{\pi^{-j}(i)} c^{\omega(\pi^{-j}(i),j)}$ for all $i,j \in I(n)$; and (G2) $x v_i = v_i \sigma^{-\chi(i)}(x)$ for all $x \in R$ and $i \in I(n)$. Then A is an associative ring with $1 = v_n$ and R is considered as a subring of A via the injective ring homomorphism $R \to A, x \mapsto v_n x$; A/R is a split Frobenius extension; A is commutative if R is commutative and $\sigma^{\chi(i)} = \mathrm{id_R}$ for all $i \in I(n)$; and A is local if R is local and $c \in \mathrm{rad}(R)$ (Theorem 16).

2. Preliminaries

For a ring R we denote by $\mathrm{rad}(R)$ the Jacobson radical of R, by $R^\times$ the set of units in R, by $\mathrm{Z}(R)$ the center of R, by $\mathrm{Aut}(R)$ the group of ring automorphisms of R, for $\sigma \in \mathrm{Aut}(R)$ by R^σ the subring of R consisting of all $x \in R$ with $\sigma(x) = x$, and for $n \geq 2$ by $\mathrm{M}_n(R)$ the ring of $n \times n$ full matrices over R. We denote by Mod-R the category of right R-modules. Left R-modules are considered as right R^{op}-modules, where R^{op} denotes the opposite ring of R. In particular, we denote by inj dim R (resp., inj dim R^{op}) the injective dimension of R as a right (resp., left) R-module and by $\mathrm{Hom}_R(-,-)$ (resp., $\mathrm{Hom}_{R^{\mathrm{op}}}(-,-)$) the set of homomorphisms in Mod-R (resp., Mod-R^{op}).

We start by recalling the notion of Auslander-Gorenstein rings.

Proposition 1 (Auslander). *Let R be a left and right noetherian ring. Then for any $n \geq 0$ the following are equivalent.*

1. *In a minimal injective resolution $I^\bullet$ of R in* Mod-R, flat dim $I^i \leq i$ *for all* $0 \leq i \leq n$.
2. *In a minimal injective resolution $J^\bullet$ of R in* Mod-R^{op}, flat dim $J^i \leq i$ *for all* $0 \leq i \leq n$.
3. *For any $1 \leq i \leq n+1$, any $M \in$* mod-R *and any submodule X of* $\mathrm{Ext}^i_R(M,R) \in$ mod-R^{op} *we have* $\mathrm{Ext}^j_{R^{\mathrm{op}}}(X,R) = 0$ *for all* $0 \leq j < i$.
4. *For any $1 \leq i \leq n+1$, any $X \in$* mod-R^{op} *and any submodule M of* $\mathrm{Ext}^i_{R^{\mathrm{op}}}(X,R) \in$ mod-R *we have* $\mathrm{Ext}^j_R(M,R) = 0$ *for all* $0 \leq j < i$.

Definition 2 ([4]). For a left and right noetherian ring R we say that R satisfies the Auslander condition if it satisfies the equivalent conditions in Proposition 1 for all $n \geq 0$, and that R is an Auslander-Gorenstein ring if inj dim R = inj dim $R^{\mathrm{op}} < \infty$ and if it satisfies the Auslander condition.

Next, we recall the notion of Frobenius extensions of rings due to Nakayama and Tsuzuku [9, 10] which we modify as follows (see [1, Section 1]).

Definition 3 ([1]). A ring A is said to be an extension of a ring R if A contains a ring R as a subring, and the notation A/R is used to denote that A is an extension of a ring R. A ring extension A/R is said to be Frobenius if the following conditions are satisfied:
(F1) $A \in \text{Mod-}R$ and $A \in \text{Mod-}R^{\text{op}}$ are finitely generated projective; and
(F2) $A \cong \text{Hom}_R(A, R)$ in $\text{Mod-}A$ and $A \cong \text{Hom}_{R^{\text{op}}}(A, R)$ in $\text{Mod-}A^{\text{op}}$.

It should be noted that if A/R is a Frobenius extension then so is $A^{\text{op}}/R^{\text{op}}$. The next proposition is well-known and easily verified.

Proposition 4. *Let A/R be a ring extension with $\phi : A \xrightarrow{\sim} \text{Hom}_R(A, R)$ in $\text{Mod-}A$. Then the following hold.*

(1) *There exists a ring homomorphism $\theta : R \to A$ such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. In particular, ϕ is an isomorphism of R-A-bimodules if and only if $\theta(x) = x$ for all $x \in R$.*
(2) *If $A \in \text{Mod-}R$ is finitely generated projective then so is $\text{Hom}_R(A, R) \in \text{Mod-}R^{\text{op}}$ and $A \xrightarrow{\sim} \text{Hom}_{R^{\text{op}}}(\text{Hom}_R(A, R), R), a \mapsto (h \mapsto h(a))$, which is an isomorphism of A-R-bimodules.*
(3) *If $A \in \text{Mod-}R$ is finitely generated projective, and if ϕ is an isomorphism of R-A-bimodules, then $A \in \text{Mod-}R^{\text{op}}$ is finitely generated projective and we have an isomorphism of A-R-bimodules $\psi : A \xrightarrow{\sim} \text{Hom}_{R^{\text{op}}}(A, R)$ with $\psi(a)(b) = \phi(b)(a)$ for all $a, b \in A$, so that A/R is a Frobenius extension.*

Definition 5. Let A/R be a Frobenius extension with $\phi : A \xrightarrow{\sim} \text{Hom}_R(A, R)$ in $\text{Mod-}A$ and $\theta : R \to A$ a ring homomorphism such that $x\phi(1) = \phi(1)\theta(x)$ for all $x \in R$. In general, $\theta(R) \neq R$ ([1]). Following [9, 10], we say that A/R is a Frobenius extension of second kind if θ induces a ring automorphism of R and that A/R is a Frobenius extension of first kind if $\theta(x) = x$ for all $x \in R$, i.e., ϕ is an isomorphism of R-A-bimodules.

Definition 6 ([1]). A ring extension A/R is said to be split if the inclusion $R \to A$ is a split monomorphism of R-R-bimodules.

Proposition 7 ([1]). *For any Frobenius extension A/R the following hold.*

(1) *If R is an Auslander-Gorenstein ring then so is A with* $\text{inj dim } A \leq \text{inj dim } R$.
(2) *Assume that A/R is split. If A is an Auslander-Gorenstein ring then so is R with* $\text{inj dim } R = \text{inj dim } A$.

We end this section with recalling the notion of skew power series rings.

Example 8. Let R be a ring and $\sigma \in \text{Aut}(R)$. Let $R[t; \sigma]$ be a free right R-module with a basis $\{t^p\}_{p \geq 0}$ and define a multiplication on $R[t; \sigma]$ subject to the following axioms: (P1) $t^p t^q = t^{p+q}$ for all $p, q \geq 0$; and (P2) $xt^p = t^p \sigma^p(x)$ for all $x \in R$ and $p \geq 0$. Then $R[t; \sigma]$ is an associative ring with $1 = t^0$ and t^p is the pth power of $t = t^1$ for all $p \geq 2$. We consider R as a subring of $R[t; \sigma]$ via the injective ring homomorphism $R \to R[t; \sigma], x \mapsto t^0 x$.

Next, setting $(t^p) = \sum_{q \geq p} t^q R$ for $p \geq 1$, we have a descending chain of two-sided ideals $(t) \supset (t^2) \supset \cdots$ in $R[t; \sigma]$ and set $R[[t; \sigma]] = \varprojlim R[t; \sigma]/(t^p)$. Namely, $R[[t; \sigma]]$ is the ring of formal power series and contains $R[t; \sigma]$ as a subring. Also, every $\tau \in \text{Aut}(R)$ with $\tau\sigma = \sigma\tau$ is extended to a ring automorphism of $R[[t; \sigma]]$ such that $\sum_{p \geq 0} t^p x_p \mapsto$

$\sum_{p\geq 0} t^p\tau(x_p)$ which we denote again by τ. In particular, $\sigma \in \mathrm{Aut}(R[[t;\sigma]])$ with $\sigma(t) = t$ and $at = t\sigma(a)$ for all $a \in R[[t;\sigma]]$.

3. Structure system

Throughout the rest of this note, we set $I(n) = \{1, \dots, n\}$ with $n \geq 2$ and fix a cyclic permutation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ n & 1 & \cdots & n-1 \end{pmatrix}$$

of $I(n)$. Then $\pi^{-i}(j) = \pi^{-j}(i)$ for all $i, j \in I(n)$ and the law of composition

$$I(n) \times I(n) \to I(n), (i,j) \mapsto \pi^{-i}(j)$$

makes $I(n)$ a cyclic group.

We denote by $\Omega(n)$ the set of mappings $\omega : I(n) \times I(n) \to \mathbb{Z}$ satisfying the following conditions:

(W1) $\omega(i,i) = 0$ for all $i \in I(n)$;
(W2) $\omega(i,j) + \omega(j,k) \geq \omega(i,k)$ for all $i,j,k \in I(n)$;
(W3) $\omega(i,j) + \omega(j,i) \geq 1$ unless $i = j$; and
(W4) $\omega(i,j) + \omega(j,\pi(i)) = \omega(i,\pi(i))$ for all $i,j \in I(n)$.

Example 9. Let $n = 4$. Then, setting

$$(\omega(i,j))_{1\leq i,j\leq 4} = \begin{pmatrix} 0 & 4 & 4 & 3 \\ 1 & 0 & 2 & -1 \\ -1 & 3 & 0 & -1 \\ 2 & 4 & 6 & 0 \end{pmatrix},$$

we have $\omega \in \Omega(4)$.

Lemma 10. *For any $\omega \in \Omega(n)$ the following hold.*

(1) $\omega(\pi(i),\pi(j)) = \omega(i,j) - \omega(i,\pi(i)) + \omega(j,\pi(j))$ *for all* $i,j \in I(n)$.
(2) $\omega(1,i) = 0$ *for all* $i \in I(n)$ *if and only if* $\omega(i,n) = 0$ *for all* $i \in I(n)$.

We denote by $\Omega_+(n)$ the subset of $\Omega(n)$ consisting of $\omega \in \Omega(n)$ such that $\omega(1,i) = \omega(i,n) = 0$ for all $i \in I(n)$ (cf. Lemma 10(2)).

Example 11. Let $n = 4$. Then, setting

$$(\omega(i,j))_{1\leq i,j\leq 4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 2 & 3 & 0 \end{pmatrix},$$

we have $\omega \in \Omega_+(4)$.

Lemma 12. *For any $\omega \in \Omega_+(n)$ the following hold.*

(1) $\omega(i,\pi(i)) = \omega(i,1) = \omega(n,\pi(i)) \geq 1$ *unless* $i = 1$.
(2) $\omega(i,j) + \omega(\pi^j(i),k) = \omega(i,\pi^{-j}(k)) + \omega(\pi^{-j}(k),j)$ *for all* $i,j,k \in I(n)$.

We denote by $X_+(n)$ the set of mappings $\chi : I(n) \to \mathbb{Z}$ satisfying the following conditions:

(X1) $\chi(1) < \chi(2) < \cdots < \chi(n)$;

(X2) $\chi(i) + \chi(n-i+1) = \chi(n)$ for all $i \in I(n)$; and

(X3) $\chi(j-i) \le \chi(j) - \chi(i) \le \chi(j-i+1)$ for all $i, j \in I(n)$ with $i < j$.

Remark 13. For any $\chi : I(n) \to \mathbb{Z}$ satisfying the condition (X2) we have $\chi(1)+\chi(n) = \chi(n)$ and hence $\chi(1) = 0$.

Example 14. Let $n = 4$. Then, setting $\chi(1) = 0,\ \chi(2) = 3,\ \chi(3) = 5$ and $\chi(4) = 8$, we have $\chi \in X(4)$.

Proposition 15. *For any $\omega \in \Omega_+(n)$, setting $\chi(i) = \sum_{k=1}^{i} \omega(k, \pi(k))$ for $i \in I(n)$, we have $\chi \in X_+(n)$ and*

$$\omega(i,j) = \begin{cases} \chi(i) - \chi(j) + \chi(j-i+1) & \text{if } i \le j, \\ \chi(i) - \chi(j) - \chi(i-j) & \text{if } i > j \end{cases}$$

for all $i, j \in I(n)$, so that we have a bijection $\Omega_+(n) \xrightarrow{\sim} X_+(n), \omega \mapsto \chi$.

4. Group rings

Throughout the rest of this note, we fix a ring R together with a pair (σ, c) of $\sigma \in \mathrm{Aut}(R)$ and $c \in R$ satisfying the following condition

$$(*) \quad \sigma(c) = c \quad \text{and} \quad xc = c\sigma(x) \text{ for all } x \in R.$$

Note that if $c \in R^\times$ then $\sigma(x) = c^{-1}xc$ for all $x \in R$, and that the condition (*) is satisfied if either $c = 0$ and σ is arbitrary, or $c \in \mathrm{Z}(R)$ and $\sigma = \mathrm{id}_R$. We refer to Example 8 for a non-trivial example. As usual, we require that $c^0 = 1$ even if $c = 0$. We fix $\omega \in \Omega_+(n)$ and, setting $\chi(i) = \sum_{k=1}^{i} \omega(k, \pi(k))$ for $i \in I(n)$, assume that $\sigma^{\chi(n)} = \mathrm{id}_R$.

Let A be a free right R-module with a basis $\{v_i\}_{i \in I(n)}$ and define a multiplication on A subject to the following axioms:

(G1) $v_i v_j = v_{\pi^{-j}(i)} c^{\omega(\pi^{-j}(i), j)}$ for all $i, j \in I(n)$; and

(G2) $x v_i = v_i \sigma^{-\chi(i)}(x)$ for all $x \in R$ and $i \in I(n)$.

Denoting by $\{\beta_i\}_{i \in I(n)}$ the dual basis of $\{v_i\}_{i \in I(n)}$ for the free left R-module $\mathrm{Hom}_R(A, R)$, we have $a = \sum_{i \in I(n)} v_i \beta_i(a)$ for all $a \in A$. It is not difficult to see that for any $a, b \in A$ and $i \in I(n)$ we have

$$\beta_i(ab) = \sum_{j \in I(n)} c^{\omega(i,j)} \sigma^{-\chi(j)}(\beta_{\pi^j(i)}(a)) \beta_j(b).$$

Theorem 16. *The following hold.*

1. *A is an associative ring with $1 = v_n$ and contains R as a subring via the injective ring homomorphism $R \to A, x \mapsto v_n x$.*
2. *A/R is a split Frobenius extension of first kind.*
3. *$v_i v_j = v_j v_i$ for all $i, j \in I(n)$. In particular, A is commutative if R is commutative and $\sigma^{\chi(i)} = \mathrm{id}_R$ for all $i \in I(n)$. Furthermore, for any $i \in I(n)$ with $i \ne n$ we have ${v_i}^r = c^s$ for some $2 \le r \le n$ and $s \ge 1$.*

(4) *There exists an injective ring homomorphism*

$$\rho : A \to \mathrm{M}_n(R), a \mapsto (c^{\omega(i,j)}\sigma^{-\chi(j)}(\beta_{\pi^j(i)}(a)))_{i,j\in I(n)}$$

such that $a \in A^\times$ *for all* $a \in A$ *with* $\rho(a) \in \mathrm{M}_n(R)^\times$.

(5) *If* $c \in \mathrm{rad}(R)$ *then* $\beta_n(a) \in R^\times$ *for all* $a \in A^\times$ *and* $R/\mathrm{rad}(R) \xrightarrow{\sim} A/\mathrm{rad}(A)$ *canonically, so that if* R *is local then so is* A.

Remark 17. Every $\tau \in \mathrm{Aut}(R)$ with $\tau\sigma = \sigma\tau$ and $\tau(c) = c$ is extended to a ring automorphism of A such that $\sum_{i\in I(n)} v_i x_i \mapsto \sum_{i\in I(n)} v_i \tau(x_i)$ which we denote again by τ. In particular, $\sigma \in \mathrm{Aut}(A)$ with $\sigma(c) = c$ and $ac = c\sigma(a)$ for all $a \in A$, so that for any $v \in \mathrm{Z}(A)^\sigma$ we can replace $(R; \sigma, c)$ by $(A; \sigma, vc)$ in the construction above.

In the following, we denote by $R[\omega; \sigma, c]$ the ring A constructed above.

Example 18. If $\chi(i) = (i-1)p$ with $p \geq 1$ for all $i \in I(n)$ then

$$R[t; \sigma^p]/(t^n - c^p) \xrightarrow{\sim} R[\omega; \sigma, c], t \mapsto v_{n-1},$$

where $(t^n - c^p) = (t^n - c^p)R[t; \sigma^p]$.

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