CLUSTER-TILTED ALGEBRAS OF CANONICAL TYPE AND QUIVERS WITH POTENTIAL

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Abstract. Let $\operatorname{coh}\mathbb{X}$ be the category of coherent sheaves over a weighted projective line and $\mathcal{C}_{\mathbb{X}}$ the classical cluster category associated with $\operatorname{coh}\mathbb{X}$. It is known that the morphism spaces in $\mathcal{C}_{\mathbb{X}}$ carry a natural $\mathbb{Z}/2\mathbb{Z}$-grading. Also, by results of Keller and Amiot, it is known that in this setting cluster-tilted algebras are Jacobian algebras of graded quivers with potential. We show that if T and T' are two cluster-tilting objects in $\mathcal{C}_{\mathbb{X}}$ which are related by mutation, then the corresponding cluster-tilted algebras are related by mutation of graded quivers with potential, thus enhancing Hübner's description of the quiver with relations of the corresponding tilted algebras.

Key Words: Cluster-tilted algebras, canonical algebras, quivers with potential, weighted projective lines.

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1. Introduction

The category $\operatorname{coh}\mathbb{X}$ of coherent sheaves over weighted projective lines was introduced in [7] as a geometric tool to study the representation theory of the so-called canonical algebras. It turns out that the category $\operatorname{coh}\mathbb{X}$ is an abelian hereditary category, hence it has associated cluster category $\mathcal{C}_{\mathbb{X}}$ in the sense of [4]. The category $\mathcal{C}_{\mathbb{X}}$ was studied in more detail in [3], where it is shown that the category $\mathcal{C}_{\mathbb{X}}$ can be obtained from $\operatorname{coh}\mathbb{X}$ by a suitable enlargment of the morphism spaces. Moreover, both categories $\operatorname{coh}\mathbb{X}$ and $\mathcal{C}_{\mathbb{X}}$ are equivepped with a "mutation operation", which acts on the isomorphism classes of a distinguished class of objects: basic tilting sheaves in $\operatorname{coh}\mathbb{X}$ and basic cluster-tilting objects in $\mathcal{C}_{\mathbb{X}}$. The aim of this notes is to describe the effect of this mutation operation on the endomorphism algebras of these objects, *c.f.* Theorem 4. We do so by incorporating the machinery of graded quivers with potential and their mutations introduced in [2] following the ungraded version of [5]. We note that a description of the Gabriel quiver of these endomorphism algebras was done in [8, Kor. 4.16] at the level of $\operatorname{coh}\mathbb{X}$. Thus, Theorem 4 although a minor enhancement of *loc. cit.*, provides a very convenient way to keep track of the changes on the relations both at the level of $\mathcal{C}_{\mathbb{X}}$ and $\operatorname{coh}\mathbb{X}$. This is illustrated by an example at the end of this notes.

In Section 2 we give a brief description of the category $\operatorname{coh}\mathbb{X}$, followed by a crash-course on the theory of graded quivers with potential and their mutations. At the end of the section we explain the connection between the topics discussed beforehand. In Section 3 we state the main theorem of this notes and give an example to illustrate the phenomenon described.

An expanded version of this paper will be submitted for publication elsewhere.

2. Preliminaries

In this section we collect the concepts and results that we need throughout this notes.

2.1. **Coherent sheaves over weighted projective lines.** Let k be an algebraically closed field and choose a tuple $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_t)$ of pairwise disctinct points of $\mathbb{P}^1_k$. Also choose a *parameter sequence* $\mathbf{p} = (p_1, \ldots, p_t)$ of positive integers with $p_i \geqslant 2$ for each $i \in \{1, \ldots, t\}$. We call the triple $\mathbb{X} = (\mathbb{P}^1_k, \boldsymbol{\lambda}, \mathbf{p})$ a *weighted projective line.* The category $\operatorname{coh}\mathbb{X}$ of *coherent sheaves* over $\mathbb{X}$ is defined as follows: consider the rank 1 abelian gruop with presentation

$$\mathbb{L} = \mathbb{L}(\mathbf{p}) = \langle \vec{x}_1, \ldots, \vec{x}_t \mid p_1\vec{x}_1 = \cdots = p_t\vec{x}_t =: \vec{c}\rangle$$

and the $\mathbb{L}$-graded algebra

$$S = S(\mathbf{p}, \boldsymbol{\lambda}) = k[x_1, \ldots, x_t]/\langle x_i^{p_i} - \lambda_i' x_2^{p_2} - \lambda_i'' x_1^{p_1} \mid i \in \{3, \ldots, t\}\rangle$$

where $\deg x_i = \vec{x}_i$ and $\lambda_i = [\lambda_i' : \lambda_i''] \in \mathbb{P}^1_k$ for each $i \in \{1, \ldots, t\}$. Note that the ideal which defines S is generated by homogeneous polynomials of degree $\vec{c}$. Then $\operatorname{coh}\mathbb{X}$ is the quotient of the category $\operatorname{mod}^{\mathbb{L}} S$ of finitely generated $\mathbb{L}$-graded S-modules by it's Serre subcategoy $\operatorname{mod}_0^{\mathbb{L}} S$ of finite lenght $\mathbb{L}$-graded S-modules. We refer the reader to [7] and [11] for basic results and properties of the category $\operatorname{coh}\mathbb{X}$.

The category $\operatorname{coh}\mathbb{X}$ enjoys several nice properties; it is an abelian, hereditary k-linear category with finite dimensional Hom and Ext spaces. Given a sheaf E, shifting the grading induces *twisted sheaves* $E(\vec{x})$ for each $\vec{x} \in \mathbb{L}$. In particular, twisting the grading by the dualizing element $\vec{\omega} := \sum_{i=1}^t (\vec{c} - \vec{x}_i) - 2\vec{c}$ gives the following version of *Serre's duality*:

$$\operatorname{Ext}^1_{\mathbb{X}}(E, F) \cong D\operatorname{Hom}_{\mathbb{X}}(F, E(\vec{\omega}))$$

for any E and F in $\operatorname{coh}\mathbb{X}$. This implies that $\operatorname{coh}\mathbb{X}$ has almost-split sequences and that the Auslander-Reiten translation is given by the auto-equivalence $\tau E = E(\vec{\omega})$. The free module S induces a structue sheaf in $\operatorname{coh}\mathbb{X}$ which we denote by $\mathcal{O}$. We recall that there are two group homomorphisms

$$\deg, \operatorname{rk} : K_0(\mathbb{X}) \to \mathbb{Z}$$

which, together with the function

$$\operatorname{slope} = \frac{\deg}{\operatorname{rk}} : K_0(\mathbb{X}) \to \mathbb{Q} \cup \{\infty\},$$

play an important role in the theory We refer the reader to [7] for precise definitions.

Definition 1. A sheaf T is called a *tilting sheaf* if $\operatorname{Ext}^1_{\mathbb{X}}(T, T) = 0$ and it is maximal with this property or, equivalently, the number of pairwise non-isomorphic indecomposable direct summands of T equals $2 + \sum_{i=1}^t (p_i - 1)$, the rank of the Grothendieck group of $\operatorname{coh}\mathbb{X}$.

The connection between the category $\operatorname{coh}\mathbb{X}$ and canonical algebras is explained by the following proposition:

Proposition 2. [7, Prop. 4.1] *Let T be the following vector bundle:*

$$
\begin{tikzcd}
 & \mathcal{O}(\vec{x}_1) \arrow[r] & \cdots \arrow[r] & \mathcal{O}((p_1-1)\vec{x}_1) \arrow[ddr] & \\
 & \mathcal{O}(\vec{x}_2) \arrow[r] & \cdots \arrow[r] & \mathcal{O}((p_2-1)\vec{x}_2) \arrow[dr] & \\
\mathcal{O} \arrow[uur] \arrow[ur] \arrow[dr] & & \vdots & & \mathcal{O}(\vec{c}) \\
 & \mathcal{O}(\vec{x}_t) \arrow[r] & \cdots \arrow[r] & \mathcal{O}((p_t-1)\vec{x}_t) \arrow[ur] &
\end{tikzcd}
$$

Then T is a titling bundle *and* $\mathrm{End}_{\mathbb{X}}(T)$ *is the canonical algebra of parameter sequence* $\boldsymbol{\lambda}$ *and weight sequence* $\mathbf{p}$. *The tilting bundle T is called the* canonical configuration *in* $\operatorname{coh}\mathbb{X}$.

There is an involutive operation on the set of isomorphism classes of basic tilting sheaves called *mutation*, *c.f.* [8, Def. 2.9]. Let $T = T_1 \oplus \cdots \oplus T_n$ be a basic tilting sheaf and $k \in \{1, \ldots, n\}$. The *mutation at k* of T is the basic tilting sheaf $\mu_k(T) = T_k' \oplus \bigoplus_{i \neq k} T_i$ where $T_k' = \ker \alpha \oplus \operatorname{coker} \alpha^*$ and

$$\alpha : \bigoplus_{i \to k} T_i \to T_k \quad \text{and} \quad \alpha^* : T_k \to \bigoplus_{k \to j} Tj.$$

Note that α is a monomorphism (resp. epimorphism) if and only if α^* is a monomorphism (resp. epimorphism), *c.f.* [8, Prop. 2.6, Prop. 2.8].

2.2. **Graded quivers with potential and their mutations.** Quivers with potentials and their Jacobian algebras where introduced in [5] as a tool to prove several of the conjectures of [6] about cluster algebras in a rather general setting. Their graded counter part, which is the one we are concerned with, was introduced in [2].

Let $Q = (Q_0, Q_1)$ be a finite quiver without loops or two cycles and $d : Q_1 \to \mathbb{Z}/2\mathbb{Z}$ a *degree function* on the set of arrows of Q. Thus, the complete path algebra $\widehat{kQ}$ has a natural $\mathbb{Z}/2\mathbb{Z}$-grading. A *potential* is a (possibly infinite) linear combination of cyclic paths in Q; we are only interested in potentials which are homogeneous as elements of $\widehat{kQ}$. For a cyclic path $a_1 \cdots a_d$ in Q and $a \in Q_1$, let

$$\partial_a(a_1 \cdots a_d) = \sum_{a_i = a} a_{i+1} \cdots a_d a_1 \cdots a_{i-i}$$

and extend it by lineary to an arbitrary potential. The maps ∂_a are called *cyclic derivatives.*

Definition 3. A graded quiver with potential (graded QP for short) is a quadruple (Q, W, d) where (Q, d) is a $\mathbb{Z}/2\mathbb{Z}$-graded finite quiver without loops and two cycles and W is a homogeneous potential for Q. The *graded Jacobian algebra* of (Q, W, d) is the graded algebra

$$\mathrm{Jac}(Q, W, d) \cong \frac{\widehat{kQ}}{\partial(W)}$$

where $\partial(W)$ is the closure in $\widehat{kQ}$ of the ideal generated by the set $\{\partial_a(W) \mid a \in \tilde{Q}_1\}$.

For each vertex of Q there is a pair of well defined involutive operations on the right equivalence-classes of graded QPs, [5, Def. 4.2], called *left and right mutations*. They differ of each other at the level of the grading only, and as their non-graded versions they consist of a mutation step and a reduction step.

Let (Q, W, d) be graded QP with W homogeneous of degree r and $k \in Q_0$. The *non-reduced left mutation at k* of (Q, W, d) is the graded QP $\tilde{\mu}_k^L(Q, W, d) = (Q', W', d')$ defined as follows:

(1) The quivers Q and Q' have the same vertex set.
(2) All arrows of Q which are not adjacent to k are also arrows of Q' and of the same degree.
(3) Each arrow $a : i \to k$ of Q is replaced in Q' by an arrow $a^* : k \to i$ of degree $d(a) + r$.
(4) Each arrow $b : k \to j$ of Q is replaced in Q' by an arrow $b^* : j \to k$ of degree $d(b)$.
(5) Each composition $i \xrightarrow{a} k \xrightarrow{b} j$ in Q is replaced in Q' by an arrow $[ba] : i \to j$ of degree $d(a) + d(b)$.
(6) The new potential is given by

$$W' = [W] + \sum_{i \xrightarrow{a} k \xrightarrow{b} j} [ba]a^*b^*$$

where $[W]$ is the potential obtained from W by replacing each composition $i \xrightarrow{a} k \xrightarrow{b} j$ which appears in W with the corresponding arrow $[ba]$ of Q'.

By [2, Thm. 4.6], there exist a graded QP (Q'_{red}, W'_{red}, d') which is right equivalent to (Q', W', d'), *c.f.* [5, Def. 4.2], and such that Q' has neither loops or two cycles. The *left mutation at k* of (Q', W', d') is then defined as

$$\mu_k^L(Q, W, d) := (Q'_{red}, W'_{red}, d').$$

Note that right equivalent quivers with potential have the same Jacobian algebras. The *right mutation at k* $\mu_k^R(Q, W, d)$ of (Q, W, d) is defined almost identically (reduction step included), just by replacing (iii) and (iv) above by

(iii') Each arrow $a : i \to k$ of Q is replaced in Q' by an arrow $a^* : k \to i$ of degree $d(a)$.
(iv') Each arrow $b : k \to j$ of Q is replaced in Q' by an arrow $b^* : j \to k$ of degree $d(b) + r$.

2.3. **Graded QPs and cluster-tilted algebras of canonical type.** Let $\mathcal{C} = \mathcal{C}_{\mathbb{X}}$ be the cluster-category of $\mathbb{X}$, *c.f.* [4]. It follows from [3, Prop. 2.3] that $\mathcal{C}$ can be taken as the category whose objects are precisely the objects of $\operatorname{coh}\mathbb{X}$, but whose morphism spaces are given by

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) := \operatorname{Hom}_{\mathbb{X}}(X, Y) \oplus \operatorname{Ext}^1_{\mathbb{X}}(X, \tau^{-1}Y).$$

Moreover, isomorphism classes in $\operatorname{coh}\mathbb{X}$ and $\mathcal{C}_{\mathbb{X}}$, and tilting sheaves in $\operatorname{coh}\mathbb{X}$ are precisely the so-called *cluster-tilting objects* in $\mathcal{C}$, *i.e.* objects $T \in \mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(T, T[1]) = 0$ and such that if $X \in \mathcal{C}$ is such that $\operatorname{Hom}_{\mathcal{C}}(T \oplus X, (T \oplus X)[1]) = 0$ then $X \in \operatorname{add} T$. For a detailed study of the combinatorics of cluster-tilting objects we refer the reader to [4] and [10] for their higher counterparts.

We recall from [9] that if Λ is a finite dimensional algebra of finite global dimension n, the $n+1$-preprojective algebra of Λ is the graded algebra

$$\Pi_{n+1}(\Lambda) := \bigoplus_{i=0}^{\infty} \mathrm{Ext}^n_\Lambda(D\Lambda, \Lambda).$$

Let $T \in \mathrm{coh}\,\mathbb{X}$ be a basic tilting sheaf. The endomorphism algebra $\mathrm{End}_\mathbb{X}(T)$ has global dimension less or equal than 2, thus, by [9, Thm. 6.11(a)], the 3-preprojective algebra of Λ can be realized as a graded Jacobian algebra using the following simple construction: Let Q be the Gabriel quiver of the basic algebra $\mathrm{End}_\mathbb{X}(T)$ so that

$$\mathrm{End}_\mathbb{X}(T) \cong \frac{kQ}{\langle r_1, \dots r_s\rangle}$$

where $\{r_1, \dots, r_s\}$ is a set of minimal relations. Consider the quiver

$$\tilde{Q} = Q \amalg \{r_i^* : t(r_i) \to s(r_i) \mid r_i : s(r_i) \dashrightarrow t(r_i)\},$$

i.e. $\tilde{Q}$ is obtained from Q by adding an arrow in the opposite direction for each relation defining $\mathrm{End}_\mathbb{X}(T)$. Thus we can define a homogeneous potential W in $\tilde{Q}$ of degree 1 by

$$W := \sum_{i=1}^{s} r_i r_i^*,$$

and there is an isomorphism of graded algebras

$$\mathrm{Jac}(\tilde{Q}, W, d) \cong \Pi_3(\mathrm{End}_\mathbb{X}(T)) \cong \mathrm{End}_\mathcal{C}(T).$$

3. Mutations of cluster-tilting objects and graded QPs

In this section we describe the effect of mutation on the endomorphism algebra of a cluster-tilting object in the cluster category $\mathcal{C}_\mathbb{X}$ using the machinery of graded quivers with potential. We must mention that this was partially done by T. Hübner in [8, Kor. 4.16] who described the effect of mutation of a tilting sheaf on it's endomorphism algebra. Since both cluster categories and (graded) quivers with potential were not available at that time and although Hübner's description of the quiver was equivalent to the one that we present, describing the relations would have beed rather complicated. Thus, even if Theorem 4 is a minor refinement of *loc. cit.*, it provides a simple algorithm to compute the relations of both the endomorphism algebra of the mutated tilting sheaf and of it's associated cluster-tilted algebra.

Theorem 4. *Let $\mathbb{X}$ be an arbitrary weighted projective line and $T = \bigoplus_{i=1}^n T_i$ a basic tilting sheaf over $\mathbb{X}$ such that $\mathrm{End}_\mathcal{C}(T) \cong \mathrm{Jac}(\tilde{Q}, W, d)$,* c.f. *Section 2.3. Let $k \in \{1, \dots, n\}$ and suppose that T_k is a formal sink of T. Then there is an isomorphism of graded algebras*

$$\mathrm{End}_\mathcal{C}(\mu_k(T)) \cong \mathrm{Jac}(\mu_k^R(\tilde{Q}, W, d)).$$

Analogously, if T_k is a formal source of T, then there is an isomorphism of graded algebras

$$\mathrm{End}_\mathcal{C}(\mu_k(T)) \cong \mathrm{Jac}(\mu_k^L(\tilde{Q}, W, d)).$$

We end this notes with an example illustrating Theorem 4, *c.f.* [3, Sec. 3].

Example 5. Let $\mathbf{p} = (3, 3, 3)$ so that

$$\mathbb{L} = \langle \vec{x}, \vec{y}, \vec{z} \mid 3\vec{x} = 3\vec{y} = 3\vec{z} =: \vec{c} \rangle$$

(we do not need to worry about $\boldsymbol{\lambda}$ in this particular case). Consider the canonical configuration T of $\operatorname{coh} \mathbb{X}$, *c.f.* Proposition 2. Then $\Lambda = \operatorname{End}_{\mathbb{X}}(T)$ is given by the quiver

$$\begin{tikzcd}
 & \mathcal{O}(\vec{x}) \arrow[rr, "x_2"] & & \mathcal{O}(2\vec{x}) \arrow[ddr, "x_3"] & \\
 & \mathcal{O}(\vec{y}) \arrow[rr, "y_2"'] & & \mathcal{O}(2\vec{y}) \arrow[dr, "y_3"'] & \\
\mathcal{O} \arrow[uur, "x_1"] \arrow[ur, "y_1"'] \arrow[dr, "z_1"'] & & & & \mathcal{O}(\vec{c}) \arrow[llll, dashed, gray, bend left] \\
 & \mathcal{O}(\vec{z}) \arrow[rr, "z_2"'] & & \mathcal{O}(2\vec{z}) \arrow[ur, "z_3"'] &
\end{tikzcd}$$

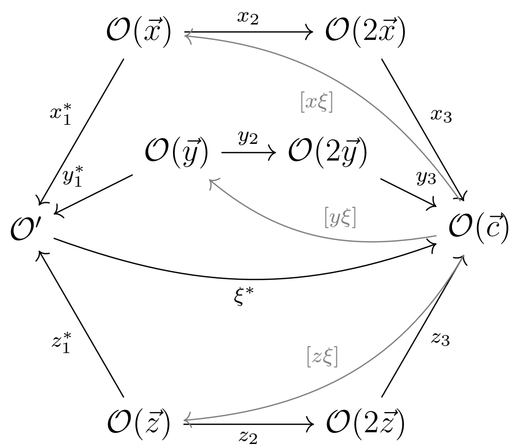
subject to the relation $x^3 + y^3 + z^3 = 0$. As explained in Section 2.3, the cluster-tilted algebra $\operatorname{End}_{\mathcal{C}}(T) \cong \Pi_3(\Lambda)$ is given by the Jacobian algebra of the graded quiver

$$\begin{tikzcd}
 & \mathcal{O}(\vec{x}) \arrow[rr, "x_2"] & & \mathcal{O}(2\vec{x}) \arrow[ddr, "x_3"] & \\
 & \mathcal{O}(\vec{y}) \arrow[rr, "y_2"'] & & \mathcal{O}(2\vec{y}) \arrow[dr, "y_3"'] & \\
\mathcal{O} \arrow[uur, "x_1"] \arrow[ur, "y_1"'] \arrow[dr, "z_1"'] & & & & \mathcal{O}(\vec{c}) \arrow[llll, gray, bend left, "\xi"] \\
 & \mathcal{O}(\vec{z}) \arrow[rr, "z_2"'] & & \mathcal{O}(2\vec{z}) \arrow[ur, "z_3"'] &
\end{tikzcd}$$

with potential

$$W = (x^3 + y^3 + z^3)\xi$$

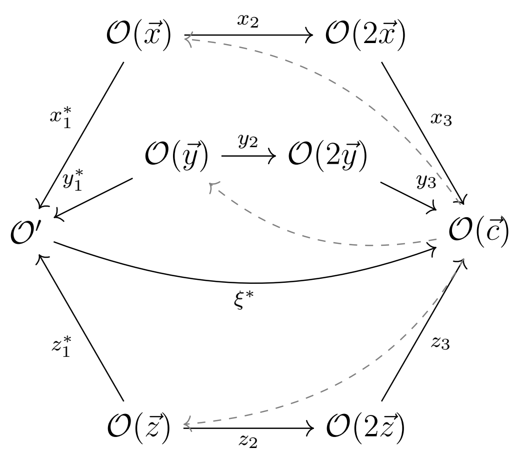
where the only arrow of degree 1 is colored gray. It is easy to see that $\mathcal{O}$ is a formal source of T since a relation in $\operatorname{End}_{\mathbb{X}}(T)$ begins at $\mathcal{O}$, *c.f.* [8, Kor. 3.5]. Then the algebra $\operatorname{End}_{\mathcal{C}}(\mu_{\mathcal{O}} T)$ is given by the Jacobian algebra of the graded quiver



with potential

$$W' = x^2[x\xi] + y^2[y\xi] + z^2[z\xi] + [x\xi]\xi^* x_1^* + [y\xi]\xi^* y_1^* + [z\xi]\xi^* z_1^*.$$

Note that we use the *left* mutation of graded quivers with potential. As explained in Section 2.3, by taking the degree zero part of $\mathrm{End}_{\mathbb{X}}(\mu_{\mathcal{O}}T)$ we obtain that $\mathrm{End}_{\mathbb{X}}(\mu_{\mathcal{O}}T)$ is isomorphic to the algebra given by the quiver



subject to the relations

$$x^2 + \xi^* x_1^* = 0$$
$$y^2 + \xi^* y_1^* = 0$$
$$z^2 + \xi^* z_1^* = 0.$$

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