CLASSIFYING SERRE SUBCATEGORIES VIA ATOM SPECTRUM

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Abstract. We introduce the atom spectrum of an abelian category as a topological space consisting of all the equivalence classes of monoform objects. In terms of the atom spectrum, we give a classification of Serre subcategories of an arbitrary noetherian abelian category.

Key Words: Serre subcategory, Atom spectrum, Monoform object.

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1. Introduction

Classification of subcategories has been studied by a number of authors, for example, [2], [3], [4], [7], and [1]. Subcategories themselves are interesting objects. Moreover we expect that the structure of subcategories reflects some important properties of the whole category.

Throughout this report, we fix an abelian category $\mathcal{A}$. First of all, we recall the definition of a Serre subcategory.

Definition 1. A full subcategory $\mathcal{X}$ of $\mathcal{A}$ is called a *Serre subcategory* if it is closed under subobjects, quotient objects, and extensions.

Remark 2. This condition is equivalent to that for any short exact sequence

$$0 \to L \to M \to N \to 0$$

in $\mathcal{A}$, M belongs to $\mathcal{X}$ if and only if L and N belong to $\mathcal{X}$.

A prototype of classifications of subcategories is the following theorem shown by Gabriel [2]. For a ring R, denote by $\operatorname{Mod} R$ the category of all the R-modules and by $\operatorname{mod} R$ the category of finitely generated R-modules. We say that a subset Φ of $\operatorname{Spec} R$ is *closed under specialization* if for any $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$, $\mathfrak{p} \subset \mathfrak{q}$ and $\mathfrak{p} \in \Phi$ imply $\mathfrak{q} \in \Phi$.

Theorem 3 (Gabriel [2])**.** *Let R be a commutative noetherian ring. Then we have the following bijection*

$$\begin{array}{ccc} \{\textit{Serre subcategories of } \operatorname{mod} R\} & \to & \{\Phi \subset \operatorname{Spec} R \mid \Phi \textit{ is closed under specialization}\} \\ \mathcal{X} & \mapsto & \displaystyle\bigcup_{M \in \mathcal{X}} \operatorname{Supp} M. \end{array}$$

In this report, we generalize this theorem to any abelian category with some noetherian property.

The detailed version of this paper has been submitted for publication elsewhere.

2. Monoform objects

The key notion of this report is that of monoform objects. We recall the definition of them.

Definition 4. A nonzero object H in $\mathcal{A}$ is called *monoform* if for any nonzero subobject L of H, there does not exist a nonzero subobject of H which is isomorphic to a subobject of H/L.

The following theorem states an important relationship between monoform objects and Serre subcategories.

Theorem 5. *Let M be an object in $\mathcal{A}$. M is monoform if and only if M does not belong to the smallest Serre subcategory containing all the objects of the form M/N where N is a nonzero subobject of M.*

Proposition 6. *Let H be a monoform object in $\mathcal{A}$. Then the following hold.*

(1) *Any nonzero subobject of H is also monoform.*
(2) *H is uniform, that is, for any nonzero subobjects L_1 and L_2 of H, $L_1 \cap L_2 \neq 0$.*

Definition 7. For monoform objects H and H' in $\mathcal{A}$, we say that H is *atom-equivalent* to H' if there exists a nonzero subobject of H which is isomorphic to a subobject of H'.

Remark 8. In fact, the relation of atom equivalence is an equivalence relation between monoform objects in $\mathcal{A}$ since any monoform object is uniform.

Now we define the notion of atoms, which was originally introduced by Storrer [6] in the case of module categories.

Definition 9. Denote by $\operatorname{ASpec} \mathcal{A}$ the quotient set (or quotient class) of the set of monoform objects in $\mathcal{A}$ by atom equivalence. We call it the *atom spectrum* of $\mathcal{A}$. Elements of $\operatorname{ASpec} \mathcal{A}$ are called *atoms* in $\mathcal{A}$. The equivalence class of a monoform object H in $\mathcal{A}$ is denoted by $\overline{H}$.

In section 4, we see that there exists a bijection between $\operatorname{ASpec}(\operatorname{Mod} R)$ and $\operatorname{Spec} R$. Hence the atom spectrum is a generalization of the prime spectrum in the commutative ring theory.

Definition 10. Let M be an object in $\mathcal{A}$.

(1) Define the *atom support* of M by
$$\operatorname{ASupp} M = \{\overline{H} \in \operatorname{ASpec} \mathcal{A} \mid H \text{ is a subquotient of } M\}.$$
(2) Define the set of *associated atoms* of M by
$$\operatorname{AAss} M = \{\overline{H} \in \operatorname{ASpec} \mathcal{A} \mid H \text{ is a subobject of } M\}.$$

The following proposition is a generalization of a proposition which is well-known in the commutative ring theory.

Proposition 11. *Let $0 \to L \to M \to N \to 0$ be a short exact sequence in $\mathcal{A}$. Then the following hold.*

(1) $\operatorname{ASupp} M = \operatorname{ASupp} L \cup \operatorname{ASupp} N$.
(2) $\operatorname{AAss} L \subset \operatorname{AAss} M \subset \operatorname{AAss} L \cup \operatorname{AAss} N$.

3. Main theorem

In order to generalize Gabriel's theorem (Theorem 3), we need to consider a generalized condition of "closed under specialization". This condition is given by the following topology.

Definition 12. Define a topology on $\operatorname{ASpec}\mathcal{A}$ as follows: we say that a subset (or subclass) Φ of $\operatorname{ASpec}\mathcal{A}$ is open if for any α, there exists $H \in \alpha$ such that $\operatorname{ASupp} H \subset \Phi$.

Proposition 13. *Open subsets of* $\operatorname{ASpec}\mathcal{A}$ *define a topology on* $\operatorname{ASpec}\mathcal{A}$ *which has an open basis* $\{\operatorname{ASupp} M \mid M \in \mathcal{A}\}$.

We recall the definition of noetherian abelian categories.

Definition 14.
(1) An object M in $\mathcal{A}$ is called *noetherian* if for any ascending chain $L_0 \subset L_1 \subset \cdots$ of subobjects of M, there exists $n \geq 0$ such that $L_n = L_{n+1} = \cdots$.
(2) An abelian category $\mathcal{A}$ is called *noetherian* if it is skeletally small (that is, the class of isomorphism classes forms a set), and any object in $\mathcal{A}$ is noetherian.

Remark 15. The skeletally smallness is just a set-theoretical assumption. It $\mathcal{A}$ ensures that $\operatorname{ASpec}\mathcal{A}$ is a set. We do not need to assume it if we allow $\operatorname{ASpec}\mathcal{A}$ to be a proper class.

Theorem 16 ([5])**.** *Let $\mathcal{A}$ be a noetherian abelian category. Then there exists a bijection*

$$\begin{array}{rcl} \{\textit{Serre subcategories of } \mathcal{A}\} & \to & \{\textit{open subsets of } \operatorname{ASpec}\mathcal{A}\} \\ \mathcal{X} & \mapsto & \displaystyle\bigcup_{M \in \mathcal{X}} \operatorname{ASupp}\mathcal{X}. \end{array}$$

The inverse map is given by $\Phi \mapsto \{M \in \mathcal{A} \mid \operatorname{ASupp} M \subset \Phi\}$.

4. In the case of module categories

In the case of module categories, the atom spectrum is described in terms of one-sided ideals.

Proposition 17. *Let R be a ring. Then any atom in* $\operatorname{Mod} R$ *is represented by a monoform object of the form $R/\mathfrak{p}$, where $\mathfrak{p}$ is a right ideal of R. Moreover if R is right noetherian, then* $\operatorname{ASpec}(\operatorname{mod} R)$ *is homeomorphic to* $\operatorname{ASpec}(\operatorname{Mod} R)$.

Proposition 18. *Let R be a commutative ring. Then the following hold.*

(1) *For any ideal $\mathfrak{a}$ of R, $R/\mathfrak{a}$ is monoform in* $\operatorname{Mod} R$ *if and only if $\mathfrak{a}$ is a prime ideal of R.*
(2) *For any prime ideals $\mathfrak{p}$ and $\mathfrak{q}$ of R, $R/\mathfrak{p}$ is atom-equivalent to $R/\mathfrak{q}$ if and only if $\mathfrak{p} = \mathfrak{q}$. Therefore the correspondence $\mathfrak{p} \mapsto \overline{R/\mathfrak{p}}$ gives a bijection* $\operatorname{Spec} R \to \operatorname{ASpec}(\operatorname{Mod} R)$.
(3) *For any R-module M,* $\operatorname{ASupp} M = \operatorname{Supp} M$*, and* $\operatorname{AAss} M = \operatorname{Ass} M$.
(4) *For any subset Φ of* $\operatorname{Spec} R$*, Φ is open in the sense of* $\operatorname{ASpec}(\operatorname{Mod} R)$ *if and only if Φ is closed under specialization.*

Remark 19. In the case where R is noetherian, we can formulate these claims by using $\operatorname{ASpec}(\operatorname{mod} R)$ instead of $\operatorname{ASpec}(\operatorname{Mod} R)$. Then new claims which we obtain also hold.

In the case of artinian rings, monoformness is stated in terms of composition factors.

Proposition 20. *Let R be a right artinian ring. Then a finitely generated R-module M is monoform if and only if it has simple socle S such that there exists no other composition factor of M which is isomorphic to S.*

Proposition 21. *Let R be a right artinian ring and $\{S_1, \ldots, S_n\}$ be a maximal set of pairwise nonisomorphic simple modules. Then* $\operatorname{ASpec} R = \{\overline{S_1}, \ldots, \overline{S_n}\}$ *with the discrete topology.*

Example 22. Let R be the ring of lower triangular matrices over a field K, that is,

$$R = \begin{bmatrix} K & 0 \\ K & K \end{bmatrix}.$$

Then all the right ideals of R are

$$0, \begin{bmatrix} 0 & 0 \\ K & 0 \end{bmatrix}, \mathfrak{p}_a = K \begin{bmatrix} 1 & 0 \\ a & 0 \end{bmatrix} (a \in K), \mathfrak{m}_1 = \begin{bmatrix} 0 & 0 \\ K & K \end{bmatrix}, \mathfrak{m}_2 = \begin{bmatrix} K & 0 \\ K & 0 \end{bmatrix}, R.$$

All the comonoform right ideals of R are

$$\mathfrak{p}_a (a \in K), \mathfrak{m}_1, \mathfrak{m}_2.$$

Since

$$\frac{R}{\mathfrak{p}_a} \cong \begin{bmatrix} K & K \end{bmatrix}, \frac{R}{\mathfrak{m}_1} \cong \begin{bmatrix} K & 0 \end{bmatrix}, \frac{R}{\mathfrak{m}_2} \cong \frac{\begin{bmatrix} K & K \end{bmatrix}}{\begin{bmatrix} K & 0 \end{bmatrix}},$$

we have $\widetilde{\mathfrak{p}_a} = \widetilde{\mathfrak{m}_1} \neq \widetilde{\mathfrak{m}_2}$. Therefore all the Serre subcategories of $\operatorname{mod} R$ are $\{$zero objects$\}$, $\langle R/\mathfrak{m}_1 \rangle_{\mathrm{Serre}}$, $\langle R/\mathfrak{m}_2 \rangle_{\mathrm{Serre}}$, and $\operatorname{mod} R$, where $\langle R/\mathfrak{m}_i \rangle_{\mathrm{Serre}}$ is the smallest Serre subcategory containing $R/\mathfrak{m}_i$.

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