QUIVER VARIETIES AND QUANTUM CLUSTER ALGEBRAS

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Abstract. Inspired by a previous work [Nak11] of Nakajima, we consider a class of (equivariant) perverse sheaves on acyclic graded quiver varieties and study the Fourier-Sato-Deligne transform from representation theoretical point of view. In particular, we get a monoidal categorification of quantum cluster algebra with specific coefficient. As a corollary, the strong positivity conjecture is verified. This is based on a talk in the 45th Symposium on Ring Theory and Representation Theory in Shinshu University and a preprint [KQ12].

1. Introduction

Cluster algebras were invented by Fomin and Zelevinsky in [FZ02] with an aim to provide concrete and combinatorial formalism for the study of Lusztig's dual canonical basis and total positivity. They are commutative algebras generated by certain combinatorially defined generators (*the cluster variables*). The quantum deformations were defined in[BZ05]. Fomin and Zelevinsky stated their original motivation as follows:

> "We conjecture that the above examples can be extensively generalized: for any simply-connected connected semisimple group G, the coordinate rings $\mathbb{C}[G]$ and $\mathbb{C}[G/N]$, as well as coordinate rings of many other interesting varieties related to G, have a natural structure of a cluster algebra. This structure should serve as an algebraic framework for the study of dual canonical bases in these coordinate rings and their q-deformations. In particular, we conjecture that all monomials in the variables of any given cluster (the *cluster monomials*) belong to this dual canonical basis."

However, despite the many successful applications of (quantum) cluster algebras to other areas (cf. the introductory survey by Keller [Kel12] and Geiss, Leclerc and Schröer [GLS12]), the link between (quantum) cluster monomials and the dual canonical basis of quantum groups remains largely elusive.

Also, the following positivity conjecture has attracted a lot of interest since the invention of cluster algebras.

Conjecture 1 (Laurent positivity conjecture)**.** *With respect to any given seed, each cluster variable expands into a Laurent polynomial with non-negative integer coefficients.*

This conjecture has been proved for cluster algebras arising from surfaces by Musiker, Schiffler, and Williams [MSW11], for cluster algebras containing a bipartite seed by Nakajima [Nak11], and the quantized version for quantum cluster algebras with respect to an acyclic initial seed by [Qin12a]. Recently, Efimov [Efi11]obtained further partial results

The detailed version of this paper [KQ12] has been submitted for publication elsewhere.

on this conjecture for quantum cluster algebras containing an acyclic seed using mixed Hodge modules.

In [HL10], Hernandez and Leclerc proposed monoidal categorification of cluster algebra.

Definition 2 (monoidal categorification)**.** Let $\mathcal{A}$ be a cluster algebra (of geometric type). Let $\mathcal{C}$ be a monoidal abelian category. We say that $\mathcal{C}$ is a monoidal categorification of $\mathcal{A}$ if the Grothendieck ring $K_0(\mathcal{C})$ of $\mathcal{C}$ is isomorphic to $\mathcal{A}$ as ring and the basis of $K_0(\mathcal{C})$ which consists of simple objects of $\mathcal{C}$ includes the set of cluster monomials[1].

The existence of monoidal categorification of cluster algebra yields the following consequence on cluster algebras (with geometric coefficients).

Conjecture 3 (strong positivity conjecture)**.** *Let $\mathcal{A}$ be a cluster algebra (with geometric coefficients $\mathbb{ZP}$). Then there exists a $\mathbb{Z}$-basis $\mathcal{B}$ of $\mathcal{A}$ which contains the set of cluster monomials and has non-negative structure constants.*

In [HL10] they gave a conjecture on the monoidal categorification of T-system cluster algebra (with level ℓ) using the tensor subcategory $\mathcal{C}_\ell$ of finite dimensional representations of (untwisted) quantum affine algebras and proved $\ell = 1$ case for A_n and D_4.In [Nak04, Nak11], Nakajima studied finite dimensional representation of quantum affine algebra via perverse sheaves on graded quiver varieties and gave a proof of $\ell = 1$ case for bipartite quiver.

In [KQ12, Qin12b], we studied graded quiver varieties which are associated with acyclic quiver and generalized the Nakajima's proof for $\ell = 1$ cases using the Nakayama functor on quiver representations.

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2. Quantum cluster algebras

2.1. **Quantum cluster algebras.** We briefly recall the definition of (quantum) cluster algebras. For more details, see [KQ12]. A quiver $Q = (Q_0, Q_1)$ is an oriented graph where Q_0 is a set of vertices and Q_1 a set of arrows. For each arrow α, we denote its outgoing vertex by $\mathrm{out}(h)$ and its incoming vertex by $\mathrm{in}(h)$. For a quiver Q, we associate doubled quiver H by adding opposite arrows $\overline{Q_1} := \{\overline{\alpha} \colon \mathrm{in}(\alpha) \to \mathrm{out}(\alpha) \mid \alpha \in Q_1\}$. We say $(Q_0, \overline{Q_1})$ as opposite quiver. Sometimes we also denote Q_0 and Q_1 by I and Ω respectively. We say that Q is p-acyclic if Q does not contain oriented cycles whose length are less than p and is acyclic if Q does not contain any oriented cycles. For 2-acyclic quiver Q (with frozen vertices), we can define cluster algebra $\mathcal{A}(Q)$ (with geometric coefficients).

[1]We remark that the correspondence between the set of isomorphism class of prime real simple objects in $\mathcal{C}$ and the set of cluster variables is required in [HL10]. Under the monoidal categorification, the correspondence can be shown in [GLS11c, Corollary 8.6]

Berenstein and Zelevinsky[BZ05] have introduced a quantum analogue of cluster algebra with geometric coefficients using quantum torus.

Let v be a formal parameter and we consider a ring $\mathbb{Z}[v^{\pm 1}]$ or $\mathbb{Q}[v^{\pm 1}]$. Let $m \geq n$ be be two positive integers. Let Λ be an $m \times m$ skew-symmetric integer matrix and $\tilde{B}$ an $m \times n$ integer matrix. The upper $n \times n$ submatrix of $\tilde{B}$, denoted by B, is called principal part of $\tilde{B}$.

Definition 4. The pair $(\Lambda, \widetilde{B})$ is called compatible if we have $\Lambda(-\widetilde{B}) = \begin{bmatrix} D \\ 0 \end{bmatrix}$ for some $n \times n$ diagonal matrix D whose diagonal entries are strictly positive integers. It is called a unitary compatible pair if moreover D is the identity matrix 1_n. The matrix Λ is called the Λ-matrix of $(\Lambda, \widetilde{B})$ and the matrix $\widetilde{B}$ is called the B-matrix of $(\Lambda, \widetilde{B})$.

We write $\Lambda(g, h)$ for $g^t \Lambda h, g, h \in \mathbb{Z}^m$, where g^t is the transpose of $g \in \mathbb{Z}^m$ as matrix.

Definition 5. The quantum torus $\mathcal{T} = \mathcal{T}(\Lambda)$ over $\mathbb{Z}[v^{\pm}]$ is the Laurent polynomial ring $\mathbb{Z}[v^{\pm 1}][x_1^{\pm 1}, ..., x_m^{\pm 1}]$, endowed with the following twisted product $*$ such that we have

$$x^g x^h = v^{\Lambda(g,h)} x^{g+h}$$

for any $g, h \in \mathbb{Z}^m$. Here for any $g = (g_i) \in \mathbb{Z}^m$, x^g denote the monomial $\prod_{1 \leq i \leq m} x_i^{g_i}$.

For $\epsilon \in \{\pm 1\}$, we define $m \times m$-matrix $E_\epsilon = (e_{ij})$ and $n \times n$-matrix $F_\epsilon = (f_{ij})$ as follows.

$$e_{ij} = \begin{cases} \delta_{ij} & \text{if } j \neq k \\ -1 & \text{if } i = j = k \\ \max(0, -\epsilon b_{ik}) & \text{if } i \neq k, j = k, \end{cases}$$

$$f_{ij} = \begin{cases} \delta_{ij} & \text{if } i \neq k \\ -1 & \text{if } i = j = k \\ \max(0, \epsilon b_{kj}) & \text{if } i = k, j \neq k. \end{cases}$$

Fix a compatible pair $(\Lambda, \widetilde{B})$ and the quantum torus $\mathcal{T} = \mathcal{T}(\Lambda)$. Since quantum torus is a Ore domain, we can consider its fraction skew-field $\mathcal{F}$ and $\mathcal{T}$ can be considered as a subalgebra of $\mathcal{F}$.

Definition 6. (1)A quantum seed is a tuple $(\Lambda, \widetilde{B}, (x_i)_{1 \leq i \leq m})$ where $\{x_i\}_{1 \leq i \leq m} \subset \mathcal{F}$ and $(\Lambda, \widetilde{B})$ a compatible pair.

(2) For a quantum seed and $1 \leq k \leq n$, we define quantum seed mutation $\mu_k(\Lambda, \widetilde{B}, (x_i)_{1 \leq i \leq m}) = (\Lambda', \widetilde{B'}, (x'_i)_{1 \leq i \leq m})$ as follows.

$$\Lambda' = E_\epsilon(B)^t \Lambda E_\epsilon(B),$$
$$\widetilde{B'} = E_\epsilon(B)^t \widetilde{B} F_\epsilon(B),$$
$$x'_i = \begin{cases} x_i & \text{if } i \neq k \\ x'_k & \text{if } i = k, \end{cases}$$

where x'_k is defined in the following equation.

$$x_k x'_k = v^{\Lambda(e_k, \sum_{1\le i\le m}[b_{ik}]_+ e_i)} \prod_{1\le i\le m} x_i^{[b_{ik}]_+} + v^{\Lambda(e_k, \sum_{1\le i\le m}[-b_{ik}]_+ e_i)} \prod_{1\le i\le m} x_i^{[-b_{ik}]_+}.$$

(3) Let $\mathbb{T}_n$ be the regular n-tree with distinct colors $\{1, \cdots, n\}$ at each vertices. Quantum cluster pattern is an assigment of quantum seed from $\mathbb{T}_n$ such that we have

$$(\Lambda(t'), \widetilde{B}(t'), (x_i(t'))_{1\le i\le m}) = \mu_k(\Lambda(t), \widetilde{B}(t), (x_i(t))_{1\le i\le m})$$

for each edge $t - t'$ which is colored by k.

(4) For a quantum cluster pattern, we set $\mathcal{X}_q = \bigcup_t \{x_i(t)\}_{1\le i\le m}$ and call by the set of quantum cluster variables. The quantum cluster algebra $\mathcal{A}_q$ is the $\mathbb{Z}[v^{\pm 1}]$-subalgebra which is generated by $\mathcal{X}_q$.

Quantum Laurent phenomena say that $\mathcal{A}_q$ is a subalgebra of the quantum torus $\mathcal{T}$ (cf. [BZ05]). For a quiver whose principal part is acyclic, it is known that quantum Laurent expansion at initial seed can be written as a generating function the Serre polynomial of the quiver Grassmannian associated with the correponding cluster tilting object. For more details, see [Qin12a]. For a Hodge-theoretic interpretation of quantum Laurent phoenomena, see [Efi11].

3. Quiver varieties

3.1. **Definition.** For a quiver Q, we consider a repetition quiver $\widehat{Q}$ as follows:

$\widehat{Q}_0 = Q_0 \times (1 + 2\mathbb{Z})$

$\widehat{Q}_1 = \{(\alpha, n)\colon (\mathrm{out}(\alpha), n) \to ()\}_{\alpha\in Q_1, n\in\mathbb{Z}} \cup \{\sigma(\alpha, n)\colon (\mathrm{in}(\alpha), n) \to (\mathrm{out}(\alpha), n-2)\}$

For an acyclic quiver Q, we consider a repetition quiver $\widehat{\Gamma} = (\widehat{\Gamma}_0, \widehat{\Gamma_1})$ with $\widehat{\Gamma}_0 = Q_0 \times \mathbb{Z}$ which contains $\widehat{Q}$ as a fullsubquiver on $Q_0 \times (1 + 2\mathbb{Z})$. We also add new arrows $\{\sigma(\alpha, n)\colon (\mathrm{in}(\alpha), 2n+1) \to (\mathrm{out}(\alpha), 2n-1)\}_{\alpha\in Q_1, n\in\mathbb{Z}}$ and $\{a(i, n)\colon (i, 2n+1) \to (i, 2n)\}_{i\in Q_0, n\in\mathbb{Z}} \cup \{b(i, n)\colon (i, 2n) \to (i, 2n-1)\}_{i\in Q_0, n\in\mathbb{Z}}$. Let $\mathcal{R}$ be a mesh category supported only on $Q_0 \times (1 + 2\mathbb{Z})$ and $\mathcal{S}$ be a fullsubcategory of $\mathcal{R}$ supported on $Q_0 \times 2\mathbb{Z}$.

We consider a finite dimensional $\widehat{\Gamma_0}$-graded vector space $V \oplus W$ where V is the $\widehat{Q_0}$-component and W is the $Q_0 \times 2\mathbb{Z}$-component.

Let $\mathrm{Rep}_{V\oplus W}(\mathcal{R})$ be a variety of representations of $\mathcal{R}$-module whose dimension vector is $V \oplus W$. A point $(B, \alpha, \beta) \in \mathrm{Rep}_{V\oplus W}(\mathcal{R})$ is said to be stable (resp. costable) if the following condition holds:

If a $\widehat{Q_0}$-graded subspace V' of V is B-invariant and contained in $\mathrm{Ker}(\beta)$ (resp. contains $\mathrm{Im}(\alpha)$), then $V' = 0$ (resp. $V' = V$).

We denote by $\mathrm{Rep}_{V\oplus W}(\mathcal{R})^{st}$ the (possibly empty) set of stable points.

Definition 7 (graded quiver varieties)**.** (1) The set-theoretical quotient $\mathcal{M}(V, W) = \mathrm{Rep}_{V\oplus W}(\mathcal{R})^{st}/G_V$ of the set of stable points with respect to the group action defined by base change of the product of general linear groups G_V is called smooth graded quiver variety.

(2) The affine algebraic-geometric quotient $\mathcal{M}_0(V, W) = \mathrm{Rep}_{V\oplus W}(\mathcal{R})//G_V$ is called affine graded quiver variety.

The smooth graded quiver variety can be defined as a homogenous spectrum of semi G_V invariants of $\mathrm{Rep}_{V\oplus W}(\mathcal{R})$ with respect to a character $\chi\colon G_V \to \mathbb{G}_m$. So there is a natural (G_W-equivariant) projective morphism $\pi\colon \mathcal{M}(V,W) \to \mathcal{M}_0(V,W)$ by general theory of geometric invariant theory. Since $\mathcal{M}_0(V,W)$ parametrizes semisimple representations, we can consider its union along all V. We denote it by $\mathcal{M}_0(W)$. The following gives a "description" of $\mathcal{M}_0(W)$ and is due to Leclerc-Plamondon [LP12] based on a result by Lusztig.

Theorem 8. *We have a natural G_W-equivariant isomorphism $\Phi_0\colon \mathcal{M}_0(W) \simeq \mathrm{Rep}_W(\mathcal{S})$.*

Let $\mathcal{P}_W$ be the set of isomorphism class of (G_W-equivariant) simple perverse sheaves on $\mathcal{M}_0(W)$ which appear in $\pi_!\mathbb{C}_{\mathcal{M}(V,W)}$ for some shifts and V and $\mathcal{Q}_W$ be the fullsubcategory of $D^b(\mathcal{M}_0(W))$ which is generated by $\mathcal{P}_W$ by shifts and direct sums. Let K_W be the quantum Grothendieck group of $\mathcal{Q}_W$ which is defined by shifts and direct sum and has a structure of $\mathbb{Z}[v^{\pm 1}]$-module.

We have a natural stratification on $\mathcal{M}_0(W)$ and the classification of $\mathcal{P}_W$ in terms of the stratification.

Let $\mathcal{M}_0^{reg}(V,W)$ be the (possible empty) open subsets of $\mathcal{M}_0(V,W)$ which consists of closed G_V-orbits whose stablizer is trivial and the dimension vector (V,W) is called dominant if $W - C_Q(z)V \geq 0$, where $C_Q(z)$ is the quantum Cartan matrix defined by

$$C_Q(z)_{ij} = \#\left\{h \in \Omega \,\middle|\, {\scriptstyle \mathrm{out}(h)=i \atop \mathrm{in}(h)=j}\right\} z - \left\{h \in \Omega \,\middle|\, {\scriptstyle \mathrm{out}(h)=j \atop \mathrm{in}(h)=i}\right\} z^{-1},$$

where $z\colon \mathbb{Z}^{\widehat{\Gamma_0}} \to \mathbb{Z}^{\widehat{\Gamma_0}}$ is the shift defined by $(zW)_i(n) = W_i(n-1)$.

Theorem 9. (1) *$\mathcal{M}_0(V,W)$ is not empty if and only if (V,W) is dominant. If (V,W) is dominant, $\mathcal{M}(V,W)$ is connected.*

(2) $\mathcal{M}_0(W) = \bigsqcup \mathcal{M}_0^{reg}(V,W)$

(3) $\mathcal{P}_W = \{\mathbf{IC}_W(V) \mid (V,W)$ *is dominant*$\}$, *where* $\mathbf{IC}_W(V) := \mathbf{IC}(\mathcal{M}_0(V,W),\mathbb{C})$ *is the intersection cohomology complex associated with the stratum* $\mathcal{M}_0^{reg}(V,W)$.

3.2. **Quantum Grothendieck ring.** Let $0 \subset W^2 \subset W$ be a $\mathcal{S}_0$-graded subspace and $W^1 = W/W^2$ and fix a splitting $W \simeq W^1 \oplus W^2$. Let $\lambda\colon \mathbb{G}_m \to G_W$ be the 1-parameter subgroup defined by $\lambda(t) = \mathrm{id}_{W^1} \oplus t\mathrm{id}_{W^2}$. Then $\mathbb{G}_m$ acts on $\mathcal{M}(V,W)$ and $\mathcal{M}_0(W)$. Let $\mathcal{T}_0(W^1,W^2)$ be the closed subvariety of $\mathcal{M}_0(W)$ which consists of points such that $\lim_{t\to 0} t\cdot[B,\alpha,\beta]$ exists. Then we have the following diagram:

$$\mathcal{M}_0(W^1)\times\mathcal{M}_0(W^2) \xleftarrow[\kappa_0]{} \mathcal{T}_0(W^1,W^2) \xrightarrow[\iota_0]{} \mathcal{M}_0(W)\ ,$$

where $\kappa_0\colon \mathcal{T}_0(W^1,W^2) \to \mathcal{M}_0(W^1)\times\mathcal{M}_0(W^2)$ be the morphism defined by taking limit $\lim_{t\to 0} t\cdot[B,\alpha,\beta]$ and $\iota_0\colon \mathcal{T}_0(W^1,W^2) \hookrightarrow \mathcal{M}_0(W)$ be the closed embedding. Let $\widetilde{\mathrm{Res}} := (\kappa_0)_!\iota_0^*\colon D^b(\mathcal{M}_0(W)) \to D^b(\mathcal{M}_0(W^1)\times\mathcal{M}_0(W^2))$ be the restriction functor defined by the above morphism. It can be shown that $\widetilde{\mathrm{Res}}(\mathcal{Q}_W) \subset \mathcal{Q}_{W^1}\boxtimes\mathcal{Q}_{W^2}$. Using the restriction functor (with some shifts), we get the following definition of quantum Grothendieck ring.

Definition 10. Let $\mathcal{R}_v$ be the subring of $\prod_W \mathrm{Hom}_{\mathbb{Z}[v^{\pm 1}]}(K_W, \mathbb{Z}[v^{\pm 1}])$ which consists the $\mathbb{Z}[v^{\pm 1}]$-linear module homomorphisms satisfy

$$\langle f_W, \mathrm{IC}_W(V)\rangle = \langle f_{W-C_Q(z)V}, \mathrm{IC}_{W-C_Q(z)V}(0)\rangle$$

for arbitrary dominant (V, W).

Let $\{L_W(V)\}$ be the dual basis of $\{\mathbf{IC}(\mathcal{M}(V, W))\}$ and $L = \{L_W\}$ be the basis of $\mathcal{R}_v$ which is determined by $\{L_W(0)\}$. It is known that L has positive structure constants and there is a embedding $\mathcal{R}$ into quantum torus using the generating function with respect to the pairing with $\{\pi_W(V)\}$, where $\pi_W(V) = \pi_! \mathbb{C}_{\mathcal{M}(V,W)}[\dim \mathcal{M}(V, W)]$.

We consider the support condition $(*)_\ell$ on $W = \bigoplus_{(i,n)\in\mathcal{S}_0} W_i(n)$:

$$W_i(n) = 0 \text{ unless } n \in \{0, 2, \cdots, 2\ell\}.$$

Let $\mathcal{R}_{v,\ell}$ be the $\mathbb{Z}[v^{\pm1}]$-subalgebra which satisfies the support condition $(*)_\ell$ and $\mathcal{R}_\ell$ be the specialization at $v = 1$. It can be shown that $L\mid_{v=1} \cap \mathcal{R}_\ell$ gives a basis of $\mathcal{R}_\ell$.

3.3. **T-system quiver.** For an acyclic quiver $Q = (Q_0, Q_1)$ and non-negative integer ℓ, we consider the following ice quiver $T_{Q,\ell}$. Let $(T_{Q,\ell})_0$ be the set $Q_0 \times \{0, 1, \cdots, \ell\}$ and $(T_{Q,\ell})_1 = \{(\alpha, k)\colon (\mathrm{out}(\alpha), k) \to (\mathrm{in}(\alpha), k)\}_{\alpha\in Q_1, 0\le k\le \ell-1} \cup \{\sigma(\alpha, k)\colon (\mathrm{in}(\alpha), k) \to (\mathrm{out}(\alpha), k-1)\}_{\alpha\in Q_1, 1\le k\le\ell} \cup \{t_{i,k}\colon (i, k) \to (i, k+1)\}_{i\in Q_0, 0\le k\le\ell-1}$. We call $T_{Q,\ell}$ by T-system quiver with level ℓ and we set $(T_{Q,\ell})_0^{\mathrm{fr}} = Q_0 \times \{\ell\}$.

It is a special case of the quivers in [BFZ05] and [GLS11b, GLS11a] which are associated with the $\ell+1$ power $c_Q^{\ell+1}$ of the acyclic Coxeter element c_Q and the corresponding unipotent subgroup $N(c_Q^{\ell+1})$.

Conjecture 11. *There is a ring isomorphism $\Phi\colon \mathcal{A}(T_{Q,\ell}) \simeq \mathcal{R}_\ell$ and the image of the cluster monomials is contained in the basis $L\mid_{v=1} \cap \mathcal{R}_\ell$.*

We remark that the subring $\mathcal{R}_\ell$ is an analogue of $K_0(\mathcal{C}_\ell)$, where $\mathcal{C}_\ell$ is the tensor subcategory in [HL10] and there is a natural quantum analogue between the quantum cluster algebra [GLS11a] and the (twisted) quantum Grothendieck ring. This should yields the quantization conjecture in [Kim12].

4. Level 1 case

We prove the above conjecture holds in $\ell = 1$ case.

4.1. **Description of quiver varieties.** We consider the $\ell = 1$ case. Let $W = \bigoplus_{(i,n)\in\mathcal{S}_0} W_i(n)$ be $\mathcal{S}_0$-graded vector space such that $W_i(n) = 0$ unless $n \in \{0, 2\}$. Since the full subquiver of $\mathcal{S}$ on $Q_0 \times \{0, 2\}$ does not contain oriented cycles and the mesh relations, $\mathrm{Rep}_W(\mathcal{S})$ is an affine space. Let S_i be a simple module of Q, I_i be an injective envelop of S_i and P_i be the projective cover of S_i.

Proposition 12. *For $W = W(0) \oplus W(2)$ be $\mathcal{S}_0$-graded vector space such that $W_i(n) = 0$ unless $n \in \{0, 2\}$. We set $P^{W(2)} = \bigoplus_{i\in Q_0} W_i(2) \otimes P_i$ and $I^{W(0)} = \bigoplus_{i\in Q_0} W_i(0) \otimes I_i$. Then we have an isomorphism:*

$$\Phi_0\colon \mathcal{M}_0(W) \simeq \mathbf{E}_W := \mathrm{Hom}_Q(P^{W(2)}, I^{W(0)}).$$

We also assume $\hat{Q}_0$-graded vector space V satisfies $V_i(n) = 0$ unless $n \in \{1\}$. Let $\mathcal{F}(V, W)$ be the quiver Grassmann of $I^{W(0)}$ with dimension vector $V(1)$. Let $\widetilde{\mathcal{F}}(V, W)$ be the variety of pairs (z, S) with $z \in \mathrm{Hom}_Q(P^{W(2)}, I^{W(0)})$ and $S \in \mathcal{F}(V, W)$ which satisfy $\mathrm{Im}(z) \subset S$. Let $\pi\colon \widetilde{\mathcal{F}}(V, W) \to \mathbf{E}_W$ be the first projection.

Proposition 13. *We have a G_W-equivariant isomorphism $\Phi\colon \mathcal{M}(V,W) \simeq \widetilde{\mathcal{F}}(V,W)$ which satisfies the following commutative diagram:*

$$\begin{array}{ccc} \mathcal{M}(V,W) & \xrightarrow{\Phi} & \widetilde{\mathcal{F}}(V,W)\ . \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{M}_0(W) & \xrightarrow[\Phi_0]{} & \mathbf{E}_W \end{array}$$

4.2. **Fourier-Deligne-Sato transform.** Since $\widetilde{\mathcal{F}}(V,W)$ is a vector subbundle over $\mathcal{F}(V,W)$ of the trivial bundle $\mathbf{E}_W \times \mathcal{F}(V,W)$, we consider its annihilator bundle $\widetilde{\mathcal{F}}^{\perp}(V,W) \subset \mathbf{E}^*{}_W \times \mathcal{F}(V,W)$. By the Nakayama duality, we have $\mathbf{E}_W^* \simeq \mathrm{Hom}_Q(I^{W(0)}, I^{W(2)})$ and

$$\widetilde{\mathcal{F}}^{\perp}(V,W) = \{(z^*, S) \in \mathbf{E}_W^* \times \mathcal{F}(V,W) | S \subset \mathrm{Ker}(z^*)\}\,.$$

Let $\pi^{\perp}\colon \widetilde{\mathcal{F}}^{\perp}(V,W) \to \mathbf{E}_W^*$ be the first projection. Then the fiber $(\pi^{\perp})^{-1}(z^*)$ is the quiver Grassmannian of $\mathrm{Ker}(z^*)$ with dimension vector $V(1)$. Let $\Psi\colon D^b(\mathbf{E}_W) \simeq D^b(\mathbf{E}_W^*)$ be the Fourier-Deligne-Sato transform and $\mathcal{L}_W$ be the subset of $\mathcal{P}_W$ which consists the Fourier transform $\Psi(\mathbf{IC}_W(V))$ has entire support $\mathbf{E}_W^*$. We note that $\mathbf{IC}_W(0) \in \mathcal{L}_W$.

We consider the following alternating sum of L_W:

$$\mathbb{L}_W = \sum_{\mathbf{IC}_W(V) \in \mathcal{L}_W} (-1)^{\dim \mathcal{M}(V,W)} \mathrm{rank}\Psi(\mathbf{IC}_W(V)) L_{W - C_Q(z)},$$

where $\mathrm{rank}\Psi(\mathbf{IC}_W(V))$ is the generic rank of $\Psi(\mathbf{IC}_W(V))$. It can be shown that $\mathbb{L}_W$ yields the quantum cluster character in [Qin12a].

Let $A_W = \mathrm{Aut}_Q(I^{W(0)}) \times \mathrm{Aut}_Q(I^{W(2)})$ be the automorphism group. We have natural projection of groups $A_W \to G_W$. By construction $\pi^{\perp}$ is equivariant with respect to the A_W-action, so the simple perverse sheaves which can be obtained by $\pi^{\perp}$ are A_W-equivariant perverse sheaves. By considering A_W-action, we get the following characterization.

Theorem 14. *If $\mathbf{E}_W^*$ has an open A_W-orbit, we have $\mathcal{L}_W = \{\mathbf{IC}_W(0)\}$.*

The sufficient condition for which $\mathbf{E}_W^*$ contains an open A_W-orbit can be characterized by the canonical decomposition of injective presentation by Derksen-Fei[DF09]. In particular, it can be shown that the set of quantum cluster monomials is contained in the "dual canonical basis" $\{L_W\}$. So we get the proof of the conjecture for $\ell = 1$ case.

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