CYCLOTOMIC KLR ALGEBRAS OF CYCLIC QUIVERS

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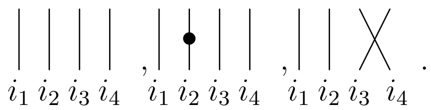
Abstract. For cyclic quiver, cyclotomic KLR algebras are defined by fixing α and Γ, two weights on vertices. We fix α and Γ in a special (but essential) case, and then show that there are systematic changes of structures.

1. Introduction

Khovanov-Lauda-Rouquier algebra (KLR algebra for short) is defined by Khovanov and Lauda, and independently Rouquier in 2008. Generators and Relations are obtained from a quiver Γ and a weight α on its vertices. We can regard generators as concatenation of such diagrams :



An another weight Λ on vertices of Γ defines a cyclotomic ideal. We call a quotient of the KLR algebra by the cyclotomic ideal a cyclotomic KLR algebra. After here, we fix quiver Γ as its vertices are $\{0, 1, 2, \cdots, n-1\}$, and its arrows are from i to $i+1$ (also $n-1$ to 0), and set $\alpha = \sum_{i:vertex} \alpha_i$, $\Lambda = \Lambda_0$.

Our aim is to describe changes of structures of cyclotomic KLR algebras for n.

2. Preliminaries

After here, K is a field and I_n is a set consisting all of permutations of $(0, 1, \cdots, n-1)$.

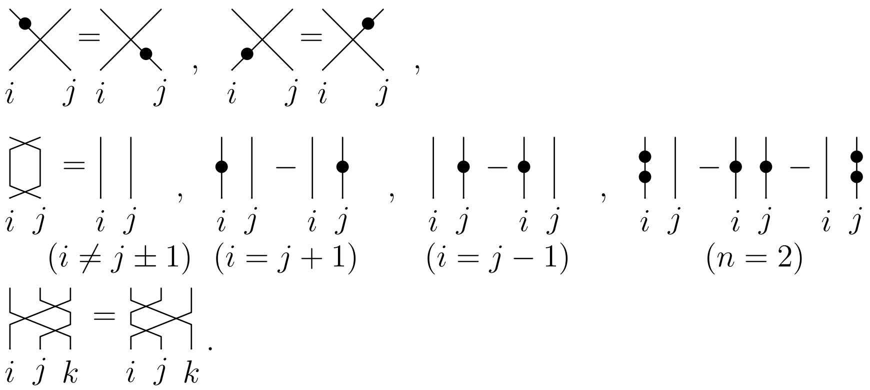
Definition 1. A KLR algebra $H_{\Gamma,\alpha}$ is an algebra obtained by following generators and relations.

- generators: $\{\mathbf{e}(\mathbf{i}) \,|\, \mathbf{i} \in I_n\} \cup \{y_1, \cdots, y_n\} \cup \{\psi_1, \cdots, \psi_{n-1}\}$
- relations:
  $\mathbf{e}(\mathbf{i})\mathbf{e}(\mathbf{j}) = \delta_{\mathbf{i},\mathbf{j}}$,
  $\sum_{\mathbf{i}\in \mathrm{Seq}(\alpha)} \mathbf{e}(\mathbf{i}) = 1$,
  $y_k\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{i})y_k$,
  $\psi_k\mathbf{e}(\mathbf{i}) = \mathbf{e}(\mathbf{s_k} \cdot \mathbf{i})\psi_k$,
  $y_k y_l = y_l y_k$,
  $\psi_k y_l = y_l \psi_k$ $(l \neq k, k+1)$,
  $\psi_k \psi_l = \psi_l \psi_k$ $(|k-l| > 1)$,
  $\psi_k y_{k+1}\mathbf{e}(\mathbf{i}) = y_k \psi_k \mathbf{e}(\mathbf{i})$,

The detailed version of this paper will be submitted for publication elsewhere.

$$y_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = \psi_k y_k\mathbf{e}(\mathbf{i}),$$
$$\psi_k^2\mathbf{e}(\mathbf{i}) = \begin{cases} \mathbf{e}(\mathbf{i}) & (i_k \not\leftrightarrow i_{k+1}) \\ (y_{k+1} - y_k)\mathbf{e}(\mathbf{i}) & (i_k \to i_{k+1}) \\ (y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftarrow i_{k+1}) \\ (y_{k+1} - y_k)(y_k - y_{k+1})\mathbf{e}(\mathbf{i}) & (i_k \leftrightarrow i_{k+1}) \end{cases},$$
$$\psi_k\psi_{k+1}\psi_k\mathbf{e}(\mathbf{i}) = \psi_{k+1}\psi_k\psi_{k+1}\mathbf{e}(\mathbf{i}).$$

The three generators are respectively coresponding to the three diagrams in section 1. A multiplication of two generators are obtained as a concatenation of two diagrams (but if the colors of connecting part are different, it becomes 0). Each relations are also given by following diagrams :



A cyclotomic ideal and a cyclotomic KLR algebra are defined from Λ as follows.

Definition 2. Generators of cyclotomic ideal are as follows :

$$\{y_1\mathbf{e}(\mathbf{i}) \,|\, \mathbf{i} \in I_n, i_1 = 0\} \cup \{\mathbf{e}(\mathbf{i}) \,|\, \mathbf{i} \in I_n, i_1 \neq 0\}.$$

Denote H_n for corresponding cyclotomic KLR algebra, a quotient of $H_{\Gamma,\alpha}$ by the ideal.

3. Properties

In this section, we describe four properties of H_n. We need some representation theoretical facts written in next section for proof.

Theorem 3. *The number of* $\mathbf{i} \in I_n$ *satisfying* $\mathbf{e}(\mathbf{i}) \neq 0$ *is exactly* 2^{n-2}. *Moreover, the set consisting all of such* $\mathbf{e}(\mathbf{i})$*s is complete set of primitive orthogonal idempotents.*

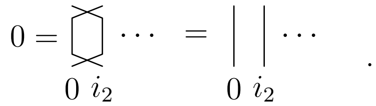
Proof. Fix n. We show there are at most 2^{n-2} $\mathbf{i}$s satisfying $\mathbf{e}(\mathbf{i}) \neq 0$ by constructing $\mathbf{i}$ from i_1 to i_b avoiding $\mathbf{e}(\mathbf{i}) = 0$. The rest part is proved in next section.

In the case of $n = 2$, there is only $(0, 1)$.

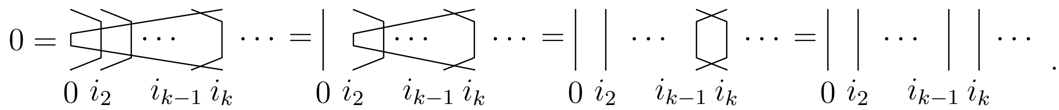
In the case of $n > 2$, at first i_1 must be 0 from the definition of the cyclotomic ideal. Next, i_2 must be 1 or $n - 1$ which are neighborhood of 0 in the quiver. If not, we obtain

$$\begin{aligned} \mathbf{e}((0, i_2, \cdots)) &= \psi_1^2\mathbf{e}((0, i_2, \cdots)) \\ &= \psi_1\mathbf{e}((i_2, 0, \cdots))\psi_1 \\ &= 0. \end{aligned}$$

We can write this equation by using diagrams as follows :



We must keep taking one of the two neighborhoods for $i_k (2 < k < n-1)$. If not, $\mathbf{e}(\mathbf{i}) = 0$ from following equation :



At last, we can set the rest number for i_n. Then we can obtain 2^{n-2} **i** is constructed by using above method. $\square$

Proposition 4. *Let* $\mathbf{e}(\mathbf{i}) \neq 0$ *in* H_n. *Then these properties hold :*

(a) $y_k \mathbf{e}(\mathbf{i}) = 0 \ (1 \leq k < n)$,
(b) $y_n^2 \mathbf{e}(\mathbf{i}) = 0$,
(c) $y_n \mathbf{e}(\mathbf{i}) \neq 0$.

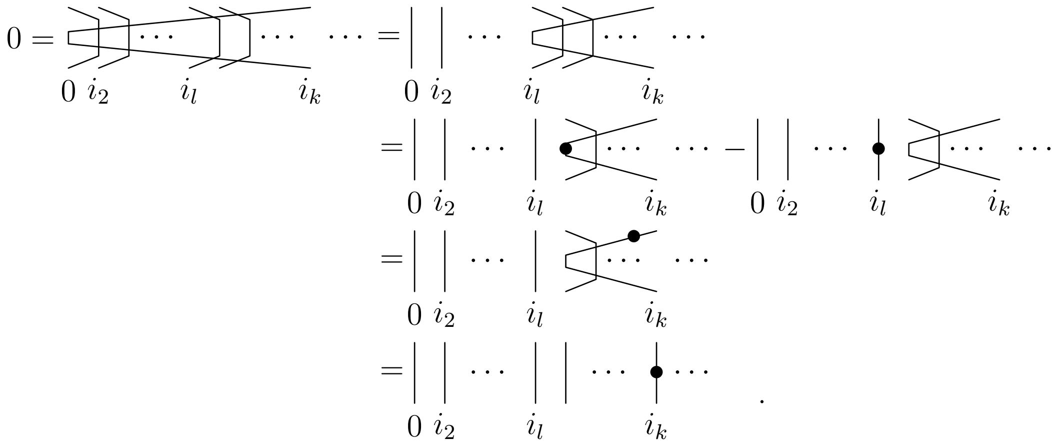
Proof. (c) will be proved in next section.

In the case of $n = 2$, (a) is by definition, (b) follows by expanding $\psi \mathbf{e}(0,1)\psi$.

In the case of $n > 2$, we prove (a) by induction for k.

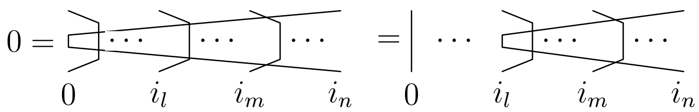
For $k = 1$, $y_k \mathbf{e}(\mathbf{i}) = 0$ from definition.

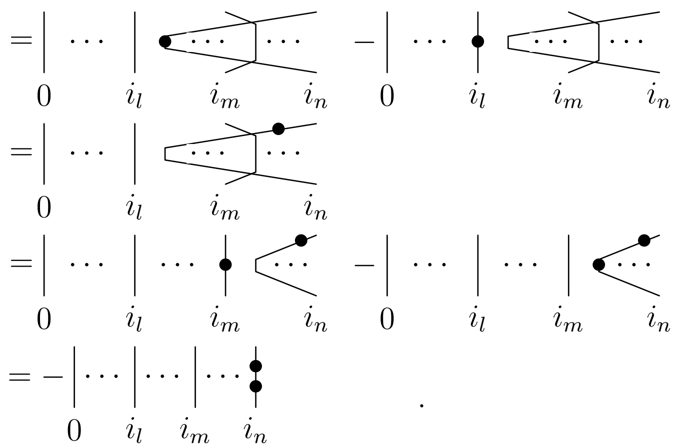
We show $y_k \mathbf{e}(\mathbf{i}) = 0$ for $k < n$. By Thm.3, there is unique $1 \leq l < k$ such that i_k and i_l are neighborhoods. Using $y_l \mathbf{e}(\mathbf{i}) = 0$ by assumption of induction, we obtain $y_k \mathbf{e}(\mathbf{i}) = 0$ from following equation :



We assume $i_l \to i_k$ in this equation, but if $i_l \leftarrow i_k$ the difference is only signs. Therefore (a) follows.

In the same way, since $y_k \mathbf{e}(\mathbf{i}) = 0$ for $k < n$ and there are two neighborhoods i_l,i_m $(1 \leq l < m < n)$ of i_n, we obtain $y_n^2 \mathbf{e}(\mathbf{i}) = 0$ as follows :





Also we assume there $i_l \to i_n \to i_m$, but the difference with the case $i_l \leftarrow i_n \leftarrow i_m$ is only signs. Therefore (b) follows. □

For H_n, set two subsets I_n^e, I_n^1 of I_n as follows :

$$\begin{aligned} I_n^e &= \{\mathbf{i} \in I_n \,|\, \mathbf{e}(\mathbf{i}) \neq 0\} \\ I_n^1 &= \{\mathbf{i} \in I_n^e \,|\, i_2 = 1\} \end{aligned}$$

And set an idempotent $\mathbf{e}$ of H_n as follows :

$$\mathbf{e} = \sum_{\mathbf{i} \in I_n^1} \mathbf{e}(\mathbf{i})$$

At last, set two maps $\hat{}: I_{n-1}^e(\alpha) \to I_n^1(\alpha)$, $\bar{}: I_n^1(\alpha) \to I_{n-1}^e(\alpha)$ as follows :

$$\begin{aligned} \hat{\mathbf{i}} &= (0, 1, i_2+1, \cdots, i_{n-1}+1) \quad \mathit{for} \quad \mathbf{i} = (0, i_2, \cdots, i_{n-1}), \\ \bar{\mathbf{i}} &= (0, i_3-1, \cdots, i_n-1) \quad \mathit{for} \quad \mathbf{i} = (0, 1, i_3, \cdots, i_n). \end{aligned}$$

In other word, $\hat{}$ increments i_k except i_1 and inserts 1 at second, $\bar{}$ decrements i_k except i_1 and remove i_2. Both maps are bijection and inversion of the other.

Proposition 5. *For each $n > 2$, an isomorphism of algebras*

$$H_{n-1} \xrightarrow{\sim} \mathbf{e}H_n\mathbf{e}$$

is obtained as follows :

$$\mathbf{e}(\mathbf{i}) \mapsto \mathbf{e}(\hat{\mathbf{i}}) \ , \ y_{n-1} \mapsto y_n \ , \ \psi_k \mapsto \psi_{k+1} \ .$$

Proof. For $\mathbf{e}(\mathbf{i})$, $\mathbf{e}(\mathbf{i}) = 0$ and $\mathbf{e}(\hat{\mathbf{i}}) = 0$ are equivalent. For y_k, what we check is only $y_{n-1} \in H_{n-1}$ and $y_n \in H_n$ by Prop.4. It is easy to check each relations is preserved. Since elements in $\mathbf{e}H_n\mathbf{e}$ can be presented without ψ_1, we can make the inversion map $\mathbf{e}H_n\mathbf{e} \to H_{n-1}$ as follows :

$$\mathbf{e}(\mathbf{i}) \mapsto \mathbf{e}(\bar{\mathbf{i}}) \ , \ y_n \mapsto y_{n-1} \ , \ \psi_k \mapsto \psi_{k-1} \ .$$

□

Proposition 6. *For each H_n, the two indecomposable projective modules corresponding to two primitive idempotents $\mathbf{e}(\mathbf{i})$ and $\mathbf{e}(\mathbf{j})$ are isomorphic if and only if $i_n = j_n$.*

In particular, the isomorphic class of indecomposable projective modules has $(n-1)$ elements.

4. Appendix : Representation Theoretical Facts

Using isomorphism given in [BK], each H_n is replaced by well-known object in representation theory. Using the facts in it, we complete the proofs of previous section.

Theorem 7 (Brundan-Kleshchev,Rouquier)**.**

(a) $\displaystyle\bigoplus_{|\alpha|=n} H_{\Gamma,\alpha,\Lambda} \cong \mathcal{H}_q^{\Lambda}(n)$

The right side is Ariki-Koike algebra determined by Λ *and* $n, q = \sqrt[n]{1} \in \mathbb{C}$.

(b) H_{C_n}, α, Λ *is a block. That is, an indecomposable two-sided ideal.*

We set $\Lambda = \Lambda_0$. In this case, Ariki-Koike algebra is Hecke algebra $H_q(\mathcal{S}_r)$ of type A. The following theorem holds. For notations in the theorem, see Mathas([4] p.50 Ex.18).

Theorem 8 (Dipper-James)**.** *Let* λ *be a partition of* r.
There exists $H_q(\mathcal{S}_r)$*-module* S^λ *with following properties :*
Let n *be minimum integer satisfying* $1 + q + q^2 + \cdots + q^{n-1} = 0$.

(a) *If* λ *is n-regular (the same number doesn't continue n times), then* top *of* S^λ *is uniquely determined. In this case, we denote* D^λ *for* topS^λ.

(b) $\left\{D^\lambda \mid \lambda : n\text{-regular}\right\}$ *is complete list of simple* $H_q(\mathcal{S}_r)$*-modules.*

The following lemma holds in general.

Lemma 9. *Let* P^λ *a indecomposable projective module corresponding to* D^λ.
As a left module,

$$H_q(\mathcal{S}_r) \cong \bigoplus_{\lambda} (\mathrm{dim} D^\lambda) P^\lambda$$

The following property holds in this time [5].

Theorem 10. *As an element of Grothendieck group,*

- $\left[D^{(n)}\right] = \left[S^{(n)}\right]$
- $\left[D^{(n-k,1^k)}\right] = -\left[D^{(n-k+1,1^{k-1})}\right] + \left[S^{(n-k,1^k)}\right]$

By using hook length formula, the following property holds.

Proposition 11.

$$\mathrm{dim} S^{(n-k,1^k)} = \binom{n-1}{k}$$

Proof. The Young diagram corresponding to $(n-k, 1^k)$ is as follows :

n	$n-k-1$	$\cdots$	2	1
k				
$\cdots$				
1				

$$\begin{aligned} \mathrm{dim} S^{(n-k,1^k)} &= \frac{n!}{n \cdot k!(n-k-1)!} \\ &= \frac{(n-1)!}{((n-1)-k)!k!} \\ &= \binom{n-1}{k} \end{aligned}$$

□

By using Thm.10 and Prop.11, the following property holds.

Proposition 12. *For $0 \leq k \leq n-1$, denote $\lambda_k = (n-k, 1^k)$.*

$$\sum_{k=0}^{n-1} \dim D^{\lambda_k} = 2^{n-2}$$

Proof. Since $\dim D^{\lambda_k} = -\dim D^{\lambda_{k-1}} + \dim S^{\lambda_k}$,
we obtain $\dim D^{\lambda_k} + \dim D^{\lambda_{k-1}} = \dim S^{\lambda_k} = \binom{n-1}{k}$.
Therefore if n is odd,

$$\sum_{k=0}^{n-1} \dim D^{\lambda_k} = 1 + \binom{n-1}{2} + \binom{n-1}{4} + \cdots + \binom{n-1}{n-1} = 2^{n-2}$$

if even,

$$\sum_{k=0}^{n-1} \dim D^{\lambda_k} = \binom{n-1}{1} + \binom{n-1}{3} + \cdots + \binom{n-1}{n-1} = 2^{n-2}$$

□

Therefore we obtain the following corollary.

Corollary 13. *Every 2^{n-2} $\mathbf{e}(\mathbf{i})$s obtained in Thm.3 is primitive idempotent.*

The folloing preperty holds.

Proposition 14. *If $\mathbf{e}(\mathbf{i}) \neq 0$ then $y_n\mathbf{e}(\mathbf{i}) \neq 0$.*

Proof. There are no elements except for $y_n\mathbf{e}(\mathbf{i})$ in $\mathbf{e}(\mathbf{i})H_n\mathbf{e}(\mathbf{i})$ such that linearly independent to $\mathbf{e}(\mathbf{i})$. On the other hand, since there are no indecomposable simple projective modules by Thm.10, $\dim(\mathrm{End}(\mathbf{e}(\mathbf{i})H_n)) \geq 2$. Hence $y_n\mathbf{e}(\mathbf{i}) \neq 0$ from $\mathrm{End}(\mathbf{e}(\mathbf{i})H_n) \cong \mathbf{e}(\mathbf{i})H_n\mathbf{e}(\mathbf{i})$. □

About Prop.6, if part follows from [1] and only if part follows from the fact ;
H_n is Morita equivalent to Brauer tree algebra of A_n type.

References

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