GOLDIE EXTENDING MODULES

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Abstract. Let R be a ring. A right R-module M is said to be *Goldie extending* (*u-Goldie extending*) if, for any (uniform) submodule X of M, there exist an essential submodule Y of X and a direct summand N of M such that Y is essential in N. A Goldie extending module is introduced by Akalan-Birkenmeier-Tercan [1]. Note that Goldie extending modules are dual to H-supplemented modules (cf. [7]).

In this paper, we show some characterizations of Goldie extending and consider generalizations of relative injectivity. And we apply them to the study of the open problems "When is a direct sum of Goldie extending (uniform) modules Goldie extending ?" and "Is the property Goldie extending inherited by direct summands ?" in Akalan-Birkenmeier-Tercan [1].

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1. Introduction

Throughout this paper R is a ring with identity and all modules considered are unitary right R-modules. A submodule X of a module M is said to be *essential* in M or an *essential submodule* of M, if $X \cap Y \neq 0$ for any non-zero submodule Y of M and we write $X \subseteq_e M$ in this case. Y is called a *closed* in M or a *closed submodule* of M if Y has no proper essential extensions inside M. Let $A \subseteq B \subseteq M$. B is said to be *closure* of A in M if B is closed in M and $A \subseteq_e B$. $K <_\oplus N$ means that K is a direct summand of N.

Let $M = M_1 \oplus M_2$ and let $\varphi : M_1 \to M_2$ be a homomorphism. Put $\langle M_1 \overset{\varphi}{\to} M_2 \rangle = \{m_1 - \varphi(m_1) \mid m_1 \in M_1\}$. Then this is a submodule of M which is called *the graph* with respect to $M_1 \overset{\varphi}{\to} M_2$. Note that $M = M_1 \oplus M_2 = \langle M_1 \overset{\varphi}{\to} M_2 \rangle \oplus M_2$.

Let $\{M_i \mid i \in I\}$ be a family of modules. The direct sum decomposition $M = \oplus_I M_i$ is said to be *exchangeable* if, for any direct summand X of M, there exists $\overline{M_i} \subseteq M_i$ $(i \in I)$ such that $M = X \oplus (\oplus_I \overline{M_i})$. A module M is said to have the (*finite*) *internal exchange property* if, any (finite) direct sum decomposition $M = \oplus_I M_i$ is exchangeable.

A module M is said to be *extending* (*u-extending*) if, for any (uniform) submodule X of M, there exists a direct summand N of M such that X is essential in N. An indecomposable extending module is called *uniform*. A module M is said to be *semi-continuous* if M is extending with the finite internal exchange property. A module M is said to be *quasi-continuous* if M is extending with the following condition (C_3):

(C_3) If A and B are direct summands of M such that $A \cap B = 0$, then $A \oplus B$ is a direct summand of M.

The detailed version of this paper will be submitted for publication elsewhere.

A module M is said to be *G-extending* or *Goldie extending* (*u-G-extending* or *u-Goldie extending*) if, for any (uniform) submodule X of M, there exist an essential submodule Y of X and a direct summand N of M such that Y is essential in N. A module M is said to be *G^+-extending* if any direct summand of M is G-extending (cf. [1]). Let $\{M_i \mid i \in I\}$ be a family of modules and put $M = \oplus_I M_i$. Then M is said to be *(u-)G-extending for the decomposition* $M = \oplus_I M_i$ if, for any (uniform) submodule X of M, there exist an essential submodule Y of X, a direct summand N of M and a submodule M_i' of M_i ($i \in I$) such that $M = N \oplus (\oplus_I M_i')$ and Y is essential in N.

We see that the following implications hold:

quasi-continuous $\Rightarrow$ semi-continuous $\Rightarrow$ extending $\Rightarrow$ G^+-extending.

In general, the converse is not ture. For example, $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ is semi-continuous but not quasi-continuous. $\mathbb{Z} \oplus \mathbb{Z}$ is extending but not semi-continuous. And $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/8\mathbb{Z}$ is G^+-extending but not extending.

A module A is said to be *B-ejective* if, for any submodule X of B and any homomorphism $f : X \to A$, there exist an essential submodule X' of X and a homomorphism $g : B \to A$ such that $g|_{X'} = f|_{X'}$ (cf. [1]).

For undefined terminologies, the reader is referred to [2], [3], [7] and [9].

2. G-extending modules and generalizations of relative injectivities

Firstly, we show a connection between extending modules and G-extending modules.

Proposition 1. *Let M be a module and consider the following conditions:*

(1) *M is G-extending and B is essentially A-injective for any decomposition $M = A \oplus B$,*

(2) *M is extending.*

Then (1) $\Rightarrow$ (2) holds. In particular, if M has the finite internal exchange property, then the converse holds.

Proposition 2. *Let A and B be modules. Then A is B-injective if and only if A is B-ejective and essentially B-injective.*

Let M be a module with the decomposition $M = A \oplus B$. If M is G-extending for the decomposition $M = A \oplus B$, then A is G-extending. Thus we obtain the following:

Theorem 3. *Let M be a module with the finite internal exchange property. Then M is G^+-extending if and only if M is G-extending.*

A module A is said to be *weakly (weakly mono-)B-ojective* if, for any submodule X of B and any homomorphism (monomorphism) $f : X \to A$, there exist an essential submodule X' of X, decompositions $A = A_1 \oplus A_2$, $B = B_1 \oplus B_2$, a homomorphism (monomorphism) $g_1 : B_1 \to A_1$ and a monomorphism $g_2 : A_2 \to B_2$ satisfying the following condition ($*$):

($*$) For any $x' \in X'$, we express x' and $f(x')$ in $B = B_1 \oplus B_2$ and $A = A_1 \oplus A_2$ as $x' = b_1 + b_2$ and $f(x') = a_1 + a_2$, respectively. Then $g_1(b_1) = a_1$ and $g_2(a_2) = b_2$ (cf. [4], [6]). Now we consider some properties of weakly ojectivities.

Proposition 4. *Let A be a module and let B be a extending module with the finite internal exchange property. Then*

(1) *If A is weakly B-ojective, then A is weakly B'-ojective for any $B' <_{\oplus} B$.*
(2) *If A is weakly B-ojective, then A is weakly mono-B-ojective.*

By a quite similar proof of [8, Theorem 2.1], we get the following:

Proposition 5. *Let A be an extending module with the finite internal exchange property and let B be a G^+-extending module. If A is weakly B-ojective, then A' is weakly B-ojective for any $A' <_{\oplus} A$.*

Theorem 6. *Let M_1 and M_2 be G^+-extending modules and put $M = M_1 \oplus M_2$. If M_1' is weakly mono-M_2'-ojective for any $M_i' <_{\oplus} M_i$ $(i = 1, 2)$, then M is G-extending for the decomposition $M = M_1 \oplus M_2$.*

The following is a main result in this section:

Theorem 7. *Let M_1 and M_2 be G-extending modules with the finite internal exchange property and put $M = M_1 \oplus M_2$. Then the following conditions are equivalent:*
(1) *M is G-extending for $M = M_1 \oplus M_2$,*
(2) *$N = M_1' \oplus M_2'$ is G-extending for $N = M_1' \oplus M_2'$, for any $M_i' <_{\oplus} M_i$ $(i = 1, 2)$,*
(3) *M_1' is weakly M_2'-ojective for any $M_i' <_{\oplus} M_i$ $(i = 1, 2)$.*

Let A and B be modules and let $f : A \to B$ be a monomorphism. f is called a *proper monomorphism* if f is not an isomorphism. If there exists a proper monomorphism from A to B, we write $A \prec B$ or $A \overset{f}{\prec} B$. If there is no proper monomorphism from A to B, we write $A \not\prec B$. By Theorem 7, we obtain the following:

Theorem 8. *Let M_1 and M_2 be G^+-extending and put $M = M_1 \oplus M_2$. Suppose that $M \not\prec M$. Then the following conditions are equivalent:*
(1) *M is G^+-extending and the decomposition $M = M_1 \oplus M_2$ is exchangeable,*
(2) *M is G^+-extending for $M = M_1 \oplus M_2$,*
(3) *M_1' is weakly mono-M_2'-ojective for any $M_i' <_{\oplus} M_i$ $(i = 1, 2)$.*

3. Direct sums of uniform modules

In this section, we consider the problem "When is a direct sum of uniform modules (G-)extending ?". Firstly we show the following:

Proposition 9. *Let $\{U_i \mid i \in I\}$ be a family of uniform modules and put $M = \oplus_I U_i$. Then the following conditions are equivalent:*
(1) *M is u-G-extending for $M = \oplus_I U_i$,*
(2) *For any $J \subseteq I$, $N = \oplus_J U_j$ is u-G-extending for $N = \oplus_J U_j$,*
(3) *U_i is weakly mono-U_j-ojective for any $i \neq j$.*

The following theorem is obtained by a quite similar proof of [5, Theorem 2.3].

Theorem 10. *(cf.* [5, Theorem 2.3]*) Let $\{U_i \mid i \in I\}$ be a family of uniform modules and put $M = \oplus_I U_i$. We consider the following condition:*
(1) *U_i is weakly mono-U_j-ojective for any $i \neq j \in I$,*
(2) *There is no infinite sequence f_1, f_2, f_3, f_4, $\cdots$ of proper monomorphisms $f_k : U_{i_k} \to U_{i_{k+1}}$ with all $i_k \in I$ distinct.*

If M satisfies the conditions (a) and (b), then M is G-extending for $M = \oplus_I U_i$.

Let $\{U_i \mid i \in I\}$ be a family of uniform modules and put $M = \oplus_I U_i$. If M is G-extending for $M = \oplus_I U_i$ and U_i is essentially U_j-injective $(i \neq j)$, then the condition (b) in Theorem 10 holds. Thus we obtain the following result:

Theorem 11. *Let $\{U_i \mid i \in I\}$ be a family of uniform modules and put $M = \oplus_I U_i$. Then the following conditions are equivalent:*

(1) *M is extending with the (finite) internal exchange property,*
(2) *M is extending and the decomposition $M = \oplus_I U_i$ is exchangeable,*
(3) (a) *M is u-extending for the decomposition $M = \oplus_I U_i$,*
  (b) *M satisfies the condition (b) in Theorem 10,*
(4) (a) *M is G-extending for the decomposition $M = \oplus_I U_i$,*
  (b) *U_i is essentially U_j-injective for any $i \neq j$,*
  (c) *(A_2') holds for all U_i and $\{U_j \mid i \neq j \in I\}$,*
(5) (a) *M is u-G-extending for the decomposition $M = \oplus_I U_i$,*
  (b) *U_i is essentially U_j-injective for any $i \neq j$,*
  (c) *(A_2') holds for all U_i and $\{U_j \mid i \neq j \in I\}$,*
  (d) *M satisfies the condition (b) in Theorem 10,*
(6) (a) *U_i is essentially U_j-injective and weakly mono-U_j-ojective for any $i \neq j$,*
  (b) *(A_2') holds for all U_i and $\{U_j \mid i \neq j \in I\}$,*
  (c) *M satisfies the condition (b) in Theorem 10.*

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