REALIZING CLUSTER CATEGORIES OF DYNKIN TYPE A_n AS STABLE CATEGORIES OF LATTICES

XUEYU LUO

Abstract. Cluster tilting objects of the cluster category $\mathcal{C}$ of Dynkin type A_{n-3} are known to be indexed by triangulations of a regular polygon P with n vertices. Given a triangulation of P, we associate a quiver with potential with frozen vertices such that the frozen part of the associated Jacobian algebra has the structure of a $K[\![x]\!]$-order denoted as Λ_n. Then we show that $\mathcal{C}$ is equivalent to the stable category of the category of Λ_n-lattices.

Let $n \geq 3$ be an integer, K be a field and $R = K[\![x]\!]$ be the formal power series ring in one variable over K.

1. The Order

Definition 1. Let Λ be an *R-order*, i.e. an R-algebra which is a finitely generated free R-module. A left Λ-module L is called a *Λ-lattice* if it is finitely generated free as an R-module. We denote by $\mathrm{CM}(\Lambda)$ the category of Λ-lattices.

The order we use to study cluster categories of type A_{n-3} is Λ_n:

$$\Lambda_n = \begin{bmatrix} R & R & R & \cdots & R & (x^{-1}) \\ (x) & R & R & \cdots & R & R \\ (x^2) & (x) & R & \cdots & R & R \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (x^2) & (x^2) & (x^2) & \cdots & R & R \\ (x^2) & (x^2) & (x^2) & \cdots & (x) & R \end{bmatrix}_{n \times n}.$$

The detailed version of this paper will be submitted for publication elsewhere.

The category $\mathrm{CM}(\Lambda_n)$ has Auslander-Reiten sequences. We draw the Auslander-Reiten quiver of Λ_n for n even: it is similar for n odd.

$$
\begin{tikzcd}
 & L_{1,1,n-2} \arrow[dr] \arrow[rrr, dotted, no head] & & & L_{\frac{n}{2},1,\frac{n}{2}-1} \arrow[dr] & \\
L_{2,n-2,0} \arrow[ur] \arrow[dr] & & L_{1,2,n-3} \arrow[r, dotted, no head] & L_{\frac{n}{2}-1,2,\frac{n}{2}-1} \arrow[ur] \arrow[dr] & & L_{\frac{n}{2},2,\frac{n}{2}-2} \\
 & L_{3,n-3,0} \arrow[ur] \arrow[rrr, dotted, no head] \arrow[d, dotted, no head] & & & L_{\frac{n}{2}-1,3,\frac{n}{2}-2} \arrow[ur] \arrow[d, dotted, no head] & \\
 & L_{\frac{n}{2},3,\frac{n}{2}-3} \arrow[dr] \arrow[rrr, dotted, no head] & & & L_{2,n-3,1} \arrow[dr] & \\
L_{\frac{n}{2},2,\frac{n}{2}-2} \arrow[ur] \arrow[dr] & & L_{\frac{n}{2}+1,2,\frac{n}{2}-3} \arrow[r, dotted] & L_{1,n-2,1} \arrow[ur] \arrow[dr] & & L_{2,n-2,0} \\
 & L_{\frac{n}{2}+1,1,\frac{n}{2}-2} \arrow[ur] \arrow[rrr, dotted, no head] & & & L_{1,n-1,0} \arrow[ur] &
\end{tikzcd}
$$

where

$$
L_{m_1,m_2,m_3} = \begin{bmatrix} R \\ \vdots \\ R \\ (x) \\ \vdots \\ (x) \\ (x^2) \\ \vdots \\ (x^2) \end{bmatrix},
$$

with R appearing m_1 times, (x) appearing m_2 times and (x^2) appearing m_3 times.

This is a Mobius strip, both the first and last row of which consist of $\frac{n}{2}$ projective-injective Λ_n-lattices, with $n-3$ τ-orbits between them.

2. The Jacobian Algebra

Let Q be a finite quiver. The complete path algebra $\widehat{KQ}$ is the completion of the path algebra with respect to the $\mathcal{J}$-adic topology for $\mathcal{J}$ the ideal generated by all arrows of Q. A quiver with potential (QP for short) is a finite quiver with a linear combination of cycles of the quiver.

Quivers of triangulations of surfaces are defined in [2] and [3] before. And QPs arising from such triangulations are defined in [4]. We extend their definitions in the case of regular polygons.

Let us fix a postive integer $n \geq 3$ and a triangulation $\triangle$ of a regular polygon with n vertices (n-gon for short).

Definition 2. The quiver $Q_\triangle$ of the triangulation $\triangle$ is the quiver the vertices of which are the (internal and external) edges of the triangulation. Whenever two edges a and b share a joint vertex, the quiver $Q_\triangle$ contains a normal arrow $a \to b$ if a is a predecessor of b with respect to clockwise orientation inside a triangle at the joint vertex of a and b. Moreover, for every vertex of the polygon with at least one internal incident edge in the triangulation, there is a dashed arrow $a \dashrightarrow b$ where a and b are its two incident external edges, a being a predecessor of b with respect to clockwise orientation.

In the following we denote $Q = Q_\triangle$. A cycle in Q is called a *cyclic triangle* if it consists of three normal arrows, and a minimal cycle in Q is called a *big cycle* if it contains exactly one dashed arrow.

Definition 3. We define the set of frozen vertices F of Q as the subset of Q_0 consisting of the n external edges of the n-gon, and the potential as

$$W = \sum \textit{cyclic triangles} - \sum \textit{big cycles}.$$

According to [1], the associated Jacobian algebra is defined by

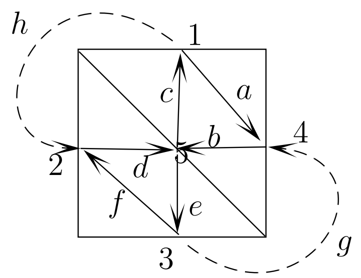
$$\mathcal{P}(Q, W, F) = \widehat{KQ}/\mathcal{J}(W, F),$$

where $\mathcal{J}(W, F)$ is the closure

$$\mathcal{J}(W, F) = \overline{\langle \partial_a W \mid a \in Q_1,\ s(a) \notin F \quad \text{or} \quad e(a) \notin F \rangle}$$

with respect to the $\mathcal{J}_{\widehat{KQ}}$-adic topology and $s(a)$ (resp. $e(a)$) is the starting vertex (resp. ending vertex) of the arrow a. Notice that cyclic derivatives associated with arrows between frozen vertices are excluded.

Example 4. We illustrate the construction of $Q_\triangle$ and W when $\triangle$ is a triangulation of a square:



In this case $W = abc + def - beg - dch$, $F = \{1, 2, 3, 4\}$ and

$$\mathcal{J}(W, F) = \overline{\langle ca - eg, ab - hd, fd - gb, ef - ch \rangle}.$$

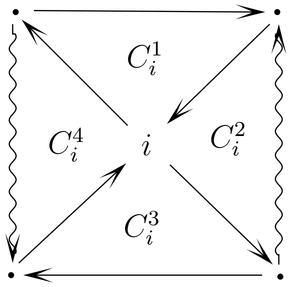
Remark 5. Since each arrow which is not between frozen vertices is shared by a big cycle and a cyclic triangle, it follows that all relations in $\mathcal{P}(Q, W, F)$ are commutativity relations.

3. A basis

Let $\triangle$ be a triangulation of the n-gon, (Q, W, F) be the associated QP with frozen vertices and $\mathcal{P}(Q, W, F)$ be its Jacobian algebra.

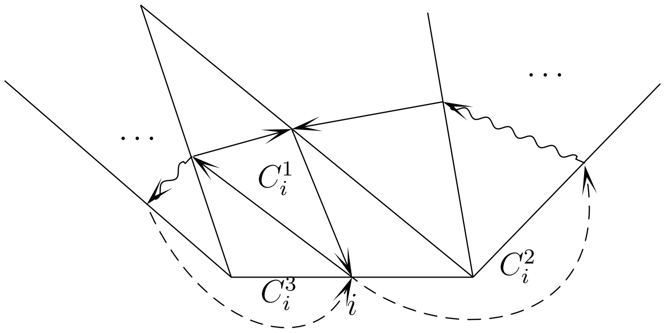
Let $i \in Q_0$. We consider all minimal cycles $C_i^1, \ldots, C_i^k$ passing through i. It is easy to check that in general there are only three cases:

1: i is one of the internal edges of the triangulation, the only case is the following:



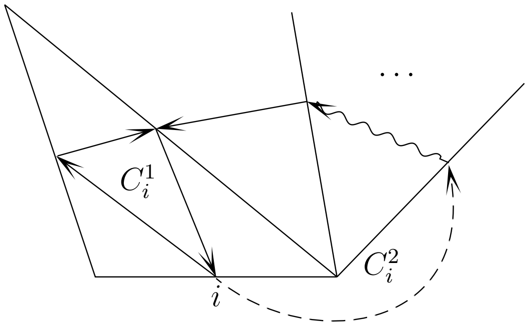
$$C_i := C_i^1 = C_i^2 = C_i^3 = C_i^4 \text{ holds in } \mathcal{P}(Q, W, F).$$

2: i is an external edge with four adjacent edges,



$$C_i := C_i^1 = C_i^2 = C_i^3 \text{ holds in } \mathcal{P}(Q, W, F).$$

3: i is an external edge with three adjacent edges,



$$C_i := C_i^1 = C_i^2 \text{ holds in } \mathcal{P}(Q,W,F).$$

By a countable basis of a complete topological vector space, we mean a linearly independent set of elements which spans a dense vector subspace. It is known that $\widehat{KQ}$ has a countable basis P_Q which is the set of all paths on Q. We say that two paths w_1 and w_2 are *equivalent* ($w_1 \sim w_2$) if $w_1 = w_2$ in $\mathcal{P}(Q,W,F)$. This gives an equivalence relation on P_Q and $P_Q/\sim$ is a countable basis of the Jacoabian algebra $\mathcal{P}(Q,W,F)$.

Consider the element $C := \sum_{i \in Q_0} C_i$ in the associated Jacobian algebra $\mathcal{P}(Q,W,F)$, then C is in the center of $\mathcal{P}(Q,W,F)$.

Definition 6. For any vertices $i, j \in Q_0$, a path w from i to j is called *C-free* if there is no path w' from i to j satisfying $w \sim w'C$.

Considering all the C-free paths, we have the following proposition.

Proposition 7. For any vertices $i, j \in Q_0$,

(1) there exists a unique C-free path w_0 from i to j up to $\sim$.
(2) $\{w_0, w_0C, w_0C^2, w_0C^3, \dots\}$ is a countable basis of $e_i\mathcal{P}(Q,W,F)e_j$.

The Jacobian algebra $\mathcal{P}(Q,W,F)$ is an R-algebra through $x \mapsto C$. According to this proposition, it is an R-order whose set of generators consists of C-free paths.

4. Main Results

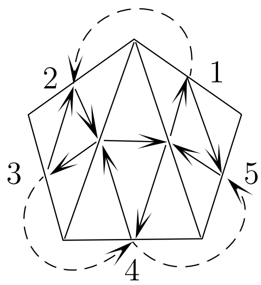
Let $\triangle$ be a triangulation of the n-gon ($n \geq 3$), (Q,W,F) be the associated QP with frozen vertices and $\mathcal{P}(Q,W,F)$ be its Jacobian algebra. Let Λ_n be the order defined in Section 1.

Theorem 8. *The cluster category $\mathcal{C}_{A_{n-3}}$ of type A_{n-3} is equivalent to the stable category of the category $\mathrm{CM}(\Lambda_n)$ of Λ_n-lattices.*

Theorem 9. *Let e_F be the sum of the idempotents at frozen vertices. Then*

(1) *$e_F\mathcal{P}(Q,W,F)e_F$ is isomorphic to Λ_n as an R-order.*
(2) *the Λ_n-module $e_F\mathcal{P}(Q,W,F)$ is a cluster tilting object of $\mathrm{CM}(\Lambda_n)$, i.e. a Λ_n-lattice X satisfies $\mathrm{Ext}^1(e_F\mathcal{P}(Q,W,F), X) = 0$ precisely when it is a direct summand of direct sum of finite copies of $e_F\mathcal{P}(Q,W,F)$.*
(3) *$\mathrm{End}_{\Lambda_n}(e_F\mathcal{P}(Q,W,F))$ is isomorphic to $\mathcal{P}(Q,W,F)$ as an R-order.*

Example 10. The Jacobian algebra associated with the following triangulation of the pentagon:



is isomorphic to the following R-order:

$$\mathcal{P}(Q,W,F) \cong \begin{bmatrix} R & R & R & R & (x^{-1}) & R & R \\ (x) & R & R & R & R & R & R \\ (x^2) & (x) & R & R & R & (x) & (x) \\ (x^2) & (x^2) & (x) & R & R & (x) & (x) \\ (x^2) & (x^2) & (x^2) & (x) & R & (x^2) & (x) \\ (x) & (x) & R & R & R & R & R \\ (x) & (x) & (x) & R & R & (x) & R \end{bmatrix}$$

It is clear that $e_F\mathcal{P}(Q,W,F)e_F \cong \Lambda_5$ holds in this case. The Auslander-Reiten quiver of Λ_5 is the following:

$$\begin{tikzcd}
\begin{bmatrix} R \\ (x) \\ (x^2) \\ (x^2) \\ (x^2) \end{bmatrix} \arrow[dr] & & \begin{bmatrix} R \\ R \\ (x) \\ (x^2) \\ (x^2) \end{bmatrix} \arrow[dr] & & \begin{bmatrix} R \\ R \\ R \\ (x) \\ (x^2) \end{bmatrix} \arrow[dr] & \\
& \begin{bmatrix} R \\ (x) \\ (x) \\ (x^2) \\ (x^2) \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x^2) \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} R \\ R \\ R \\ (x) \\ (x) \end{bmatrix} \\
\begin{bmatrix} R \\ R \\ R \\ (x) \\ (x) \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} R \\ (x) \\ (x) \\ (x) \\ (x^2) \end{bmatrix} \arrow[ur] \arrow[dr] & & \begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x) \end{bmatrix} \arrow[ur] \arrow[dr] & \\
& \begin{bmatrix} R \\ R \\ R \\ R \\ (x) \end{bmatrix} \arrow[ur] & & \begin{bmatrix} R \\ (x) \\ (x) \\ (x) \\ (x) \end{bmatrix} \arrow[ur] & & \begin{bmatrix} R \\ (x) \\ (x^2) \\ (x^2) \\ (x^2) \end{bmatrix}
\end{tikzcd}$$

As a Λ_5-module, $e_F\mathcal{P}(Q,W,F)$ is isomorphic to

$$\begin{bmatrix} R \\ (x) \\ (x^2) \\ (x^2) \\ (x^2) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ (x) \\ (x^2) \\ (x^2) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ R \\ (x) \\ (x^2) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ R \\ R \\ (x) \end{bmatrix} \oplus \begin{bmatrix} R \\ (x) \\ (x) \\ (x) \\ (x) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x^2) \end{bmatrix}$$

which is cluster tilting in $\mathrm{CM}(\Lambda_5)$. As expected, its summands correspond bijectively to the (internal and external) edges of the triangulation, or equivalently to the vertices of the corresponding quiver.

The stable category $\underline{\mathrm{CM}}(\Lambda_5)$ is equivalent to $\mathcal{C}_{A_2}$ and has a cluster tilting object

$$\begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x) \end{bmatrix} \oplus \begin{bmatrix} R \\ R \\ (x) \\ (x) \\ (x^2) \end{bmatrix}.$$

References

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Graduate School of Mathematics
Nagoya University
Furocho, Chikusaku, Nagoya 464-8602 JAPAN
E-mail address: d1005p@math.nagoya-u.ac.jp