CHARACTERIZATION OF GORENSTEIN STRONGLY KOSZUL HIBI RINGS BY F-INVARIANTS

KAZUNORI MATSUDA

Abstract. Hibi rings are a kind of graded toric ring on a finite distributive lattice $D = J(P)$, where P is a partially ordered set. In this article, we compute diagonal F-thresholds and F-pure thresholds of Hibi rings and give a characterization of Hibi rings which satisfy the equality between these invariants in terms of its trivialness in the sense of Herzog-Hibi-Restuccia.

1. Introduction

This is a partially joint work with T. Chiba.

Firstly, we recall the definition of Hibi rings(see[Hib]).

Let $P = \{p_1, p_2, \ldots, p_N\}$ be a finite partially ordered set(poset for short), and let$J(P)$ be the set of all poset ideals of P, where a poset ideal of P is a subset I of P such that if $x \in I$, $y \in P$ and $y \leq x$ then $y \in I$.

A *chain* X of P is a totally ordered subset of P. The *length* of a chain X of P is $\#X - 1$, where $\#X$ is the cardinality of X. The *rank* of P, denoted by rank$P$, is the maximum of the lengths of chains in P. A poset is called *pure* if its all maximal chains have the same length. For $x, y \in P$, we say that *y covers x*, denoted by $x \lessdot y$, if $x < y$ and there is no $z \in P$ such that $x < z < y$.

Definition 1. ([Hib]) Let the notation be as above. Let φ be the following map:

$$\varphi : J(P) \longrightarrow k[T, X_1, \ldots, X_N], \qquad I \longmapsto T \prod_{p_i \in I} X_i$$

Then the *Hibi ring* $R(P)$ is defined as follows:

$$R(P) = k[\varphi(I) \mid I \in J(P)].$$

Remark 2. (1) ([Hib]) Hibi rings are graded toric rings.
(2) $\dim R(P) = \#P + 1$.
(3) ([Hib]) $R(P)$ is Gorenstein if and only if P is pure.

Finally, we define $\mathrm{rank}^* P$ and $\mathrm{rank}_* P$ for a poset P in order to state our main theorem.

A sequence $C = (q_1, \ldots, q_t)$ is called a *path* of P if C satisfies the following conditions:

(1) $q_1, \ldots, q_t$ are distinct elements of P,
(2) q_1 is a minimal element of P and $q_{t-1} \lessdot q_t$,
(3) $q_i \lessdot q_{i+1}$ or $q_{i+1} \lessdot q_i$.

The detailed version of this paper will be submitted for publication elsewhere.

In short, we regard the Hasse diagram of P as a graph, and consider paths on it. In particular, if q_t is a maximal element of P, then we call C *maximal* path. For a path $C = (q_1, \ldots, q_t)$, we denote $C = q_1 \to q_t$.

For a path $C = (q_1, \ldots, q_t)$, q_i is said to be a *locally maximal element* of C if $q_{i-1} \lessdot q_i$ and $q_{i+1} \lessdot q_i$, and a *locally minimal element* of C if $q_i \lessdot q_{i-1}$ and $q_i \lessdot q_{i+1}$. For convenience, we consider that q_1 is a locally minimal element and q_t is a locally maximal element of C.

For a path $C = (q_1, \ldots, q_t)$, if $q_1 \leq \cdots \leq q_t$ then we call C an *ascending chain* and if $q_1 \geq \cdots \geq q_t$ then we call C a *descending chain*. We denote a ascending chain by a symbol A and a descending chain by a symbol D. For a ascending chain $A = (q_1, \ldots, q_t)$, we put $t(A) = q_t$ and $< A >= \{q \in P \mid q \leq t(A)\}$. Since $< A >$ is a poset ideal of P generated by A, we note that $< A >\in J(P)$.

Let $C = (q_1, \ldots, q_t)$ be a path and $V(C)$ the vertices of C. We now introduce the notion of the *decomposition* of C. We decompose $V(C)$ as follows:

$$V(C) = V(A_1) \coprod V(D_1) \coprod V(A_2) \coprod \cdots \coprod V(D_{n-1}) \coprod V(A_n)$$

such that

$$\begin{aligned} V(A_1) &= \{q_1, \ldots, q_{a(1)}\}, \\ V(D_1) &= \{q'_1, \ldots, q'_{d(1)}\}, \\ V(A_2) &= \{q_{a(1)+1}, \ldots, q_{a(2)}\}, \\ &\vdots \\ V(D_{n-1}) &= \{q'_{d(n-2)+1}, \ldots, q'_{d(n-1)}\}, \\ V(A_n) &= \{q_{a(n-1)+1}, \ldots, q_{a(n)} = q_t\}, \end{aligned}$$

where $\{q_{a(1)}, \ldots, q_{a(n)}\}$ is the set of locally maximal elements and $\{q_1, q'_{d(1)}, \ldots, q'_{d(n-1)}\}$ is the set of locally minimal elements of C. Then A_i are ascending chains and D_j are descending chains. This decomposition is denoted by $C = A_1 + D_1 + A_2 + \cdots + D_{n-1} + A_n$.

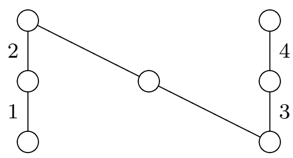
For a path $C = (q_1, \ldots, q_t)$, we define *the upper length* by

$$\text{length}^* C = \#\{(q_i, q_{i+1}) \in E(C) \mid q_i \lessdot q_{i+1}\},$$

where $E(C)$ is the set of edges of C.

Example 3. (1) If C is a chain, then $\text{length}^* C = \text{length} C$.

(2) Consider the following path C:

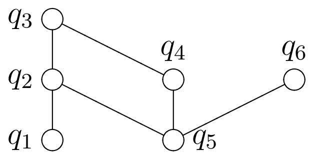


Then $\text{length}^* C = 4$.

Next, we introduce the condition (*).

Definition 4. For a path $C = (q_1, \ldots, q_t)$, we say that C *satisfies a condition* (*) if C satisfies the following conditions: for all q_r which is locally maximal element or locally minimal element of C, $q_{s'} \not\leq q_s$ for all $s' > r$ and $r > s$.

Example 5. Consider the following poset P:



Then, $C_1 = (q_1, q_2, q_5, q_6)$ satisfies the condition (*), but $C_2 = (q_1, q_2, q_3, q_4, q_5, q_6)$ does not satisfy the condition (*) because $q_2 \geq q_5$.

Remark 6. (1) For a path $C = (q_1, \ldots, q_t)$ such that C satisfies a condition (*) and q_t is a locally maximal element, we can extend C to a path $\tilde{C} = (q_1, \ldots, q_t, \ldots, q_{t'})$ such that $\tilde{C}$ is a maximal path which satisfies a condition (*). Indeed, if q_t is not a maximal element of P, then there exists q_{t+1} such that $q_t \lessdot q_{t+1}$. We decompose $C = A_1 + D_1 + \ldots + D_{n-1} + A_n$. If $q_{t+1} \in < A_i >$ for some i, then so is q_t. This means that C does not satisfy a condition (*), a contradiction. Hence a path $C' = (q_1, \ldots, q_t, q_{t+1})$ also satisfies a condition (*). Therefore, by repeating this operation, we can extend C to a path $\tilde{C} = (q_1, \ldots, q_t, \ldots, q_{t'})$ such that $\tilde{C}$ is a maximal path which satisfies a condition (*).

(2) Let $C = (q_1, \ldots, q_t)$ be a path of P. If C is a unique path such that its starting point is q_1 and its end point is q_t, then C satisfies a condition (*). Indeed, if C does not so, there exists a locally maximal(or minimal) element q_r such that $q_{s'} \leq q_s$ for some $s < r < s'$. Then, $C' = (q_1, \ldots, q_s, q_{s'}, \ldots, q_t)$ is also a path, but this is a contradiction.

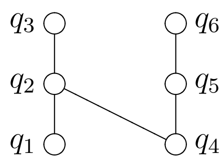
Now, we can define the upper rank $\mathrm{rank}^* P$ and the lower rank $\mathrm{rank}_* P$ for a poset P.

Definition 7. For a poset P, we define

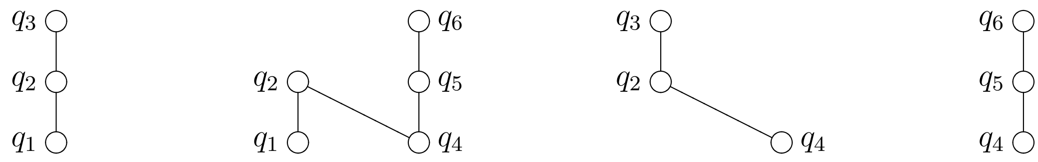
$$\mathrm{rank}^* P = \max\{\mathrm{length}^* C \mid C \text{ is a maximal path which satisfies a condition}(*)\},$$
$$\mathrm{rank}_* P = \min\{\mathrm{length}^* C \mid C \text{ is a maximal path which satisfies a condition}(*)\}.$$

We call $\mathrm{rank}^* P$ *upper rank* and $\mathrm{rank}_* P$ *lower rank* of P. We note that $\#P - 1 \geq \mathrm{rank}^* P \geq \mathrm{rank} P \geq \mathrm{rank}_* P$.

Example 8. Consider the following poset P:



Then, the following paths satisfy the condition (*):



Hence we have $\mathrm{rank}^*P = 3$ and $\mathrm{rank}_*P = \mathrm{rank}P = 2$.

2. Diagonal F-thresholds of Hibi rings

In this section, we recall the definition and several basic results of F-threshold and give a formula of the F-thresholds of Hibi rings.

2.1. **Definition and basic properties.** Let R be a Noetherian ring of characteristic $p > 0$ with $\dim R = d \geq 1$. Let $\mathfrak{m}$ be a maximal ideal of R. Suppose that $\mathfrak{a}$ and J are $\mathfrak{m}$-primary ideals of R such that $\mathfrak{a} \subseteq \sqrt{J}$ and $\mathfrak{a} \cap R^\circ \neq \emptyset$, where R° is the set of elements of R that are not contained in any minimal prime ideal of R.

Definition 9 (see [HMTW]). Let $R, \mathfrak{a}, J$ be as above. For each nonnegative integer e, put $\nu_{\mathfrak{a}}^J(p^e) = \max\{r \in \mathbb{N} \mid \mathfrak{a}^r \not\subseteq J^{[p^e]}\}$, where $J^{[p^e]} = (a^{p^e} \mid a \in J)$. Then we define

$$c^J(\mathfrak{a}) = \lim_{e \to \infty} \frac{\nu_{\mathfrak{a}}^J(p^e)}{p^e}$$

if it exists, and call it the *F-threshold* of the pair $(R, \mathfrak{a})$ with respect to J. Moreover, we call $c^{\mathfrak{a}}(\mathfrak{a})$ the *diagonal F-threshold* of R with respect to $\mathfrak{a}$.

About basic properties and examples of F-thresholds, see [HMTW]. In this section, we summarize basic properties of the diagonal F-thresholds $c^{\mathfrak{m}}(\mathfrak{m})$.

Example 10.
(1) Let $(R, \mathfrak{m})$ be a regular local ring of positive characteristic. Then $c^{\mathfrak{m}}(\mathfrak{m}) = \dim R$.
(2) Let $k[X_1, \ldots, X_d]^{(r)}$ be the r-th Veronese subring of a polynomial ring $S = k[X_1, \ldots, X_d]$. Put $\mathfrak{m} = (X_1, \ldots, X_d)^r R$. Then $c^{\mathfrak{m}}(\mathfrak{m}) = \frac{r+d-1}{r}$.
(3) ([MOY, Corollary 2.4]) If $(R, \mathfrak{m})$ is a local ring with $\dim R = 1$, then $c^{\mathfrak{m}}(\mathfrak{m}) = 1$.

Example 11. ([MOY, Theorem 2]) Let $S = k[X_1, \ldots, X_m, Y_1, \ldots, Y_n]$ be a polynomial ring over k in $m + n$ variables, and put $\mathfrak{n} = (X_1, \ldots, X_m, Y_1, \ldots, Y_n)S$. Take a binomial $f = X_1^{a_1} \cdots X_m^{a_m} - Y_1^{b_1} \cdots Y_n^{b_n} \in S$, where $a_1 \geq \cdots \geq a_m, b_1 \geq \cdots \geq b_n$. Let $R = S_{\mathfrak{n}}/(f)$ be a binomial hypersurface local ring with the unique maximal ideal $\mathfrak{m}$. Then

$$c^{\mathfrak{m}}(\mathfrak{m}) = m + n - 2 + \frac{\max\{a_1 + b_1 - \min\{\sum_{i=1}^m a_i, \sum_{j=1}^n b_j\}, 0\}}{\max\{a_1, b_1\}}.$$

In [CM], we gave a formula of $c^{\mathfrak{m}}(\mathfrak{m})$ of Hibi rings.

Theorem 12 (see [CM]). *Let P be a finite poset, and $R = R(P)$ the Hibi ring made from P. Let $\mathfrak{m} = R_+$ be the graded maximal ideal of R. Then*

$$c^{\mathfrak{m}}(\mathfrak{m}) = \mathrm{rank}^*P + 2.$$

3. F-pure thresholds of Hibi rings

In this section, we recall the definition of the F-pure threshold and give a formula of the F-pure thresholds of Hibi rings. This formula is given by Chiba.

The F-pure threshold, which was introduced by [TW], is an invariant of an ideal of an F-finite F-pure ring. F-pure threshold can be calculated by computing generalized test ideals (see [HY]), and [Bl] showed how to compute generalized test ideals in the case of toric rings and its monomial ideals. Since Hibi rings are toric rings, we can compute F-pure thresholds of the homogeneous maximal ideal of arbitrary Hibi rings, and will be described in terms of poset.

Definition 13 (see [TW])**.** Let R be an F-finite F-pure ring of characteristic $p > 0$, $\mathfrak{a}$ a nonzero ideal of R, and t a non-negative real number. The pair $(R, \mathfrak{a}^t)$ is said to be F-pure if for all large $q = p^e$, there exists an element $d \in \mathfrak{a}^{\lceil t(q-1) \rceil}$ such that the map $R \longrightarrow R^{1/q}$ $(1 \mapsto d^{1/q})$ splits as an R-linear map. Then the F-pure threshold $\mathrm{fpt}(\mathfrak{a})$ is defined as follows:

$$\mathrm{fpt}(\mathfrak{a}) = \sup\{t \in \mathbb{R}_{\geq 0} \mid (R, \mathfrak{a}^t) \text{ is } F\text{-pure}\}.$$

Hara and Yoshida [HY] introduced the generalized test ideal $\tau(\mathfrak{a}^t)$ (t is a non negative real number). Then $\mathrm{fpt}(\mathfrak{a})$ can be calculated as the minimum jumping number of $\tau(\mathfrak{a}^c)$, that is,

$$\mathrm{fpt}(\mathfrak{a}) = \sup\{t \in R_{\geq 0} \mid \tau(\mathfrak{a}^t) = R\}.$$

Chiba gave a formula of $\mathrm{fpt}(\mathfrak{m})$ of Hibi ring $R = R(P)$.

Theorem 14 (see [CM])**.** *Let P be a finite poset, and $R = R(P)$ the Hibi ring made from P. Let $\mathfrak{m} = R_+$ be the graded maximal ideal of R. Then*

$$\mathrm{fpt}(\mathfrak{m}) = \mathrm{rank}_* P + 2.$$

4. $-a(R)$ of Hibi rings and Characterization of Hibi rings which satisfy $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R) = \mathrm{fpt}(\mathfrak{m})$

The first main theorem of this article is the following:

Theorem 15 (see [CM], [BH])**.** *Let P be a poset, and $R = R(P)$ the Hibi ring made from P. Let $\mathfrak{m} = R_+$ the unique graded maximal ideal of R. Then*

$$\begin{aligned} c^{\mathfrak{m}}(\mathfrak{m}) &= \mathrm{rank}^* P + 2, \\ -a(R) &= \mathrm{rank} P + 2, \\ \mathrm{fpt}(\mathfrak{m}) &= \mathrm{rank}_* P + 2, \end{aligned}$$

where $a(R)$ is a-invariant of R(see [GW]*). In particular, $c^{\mathfrak{m}}(\mathfrak{m}) \geq -a(R) \geq \mathrm{fpt}(\mathfrak{m})$.*

In this section, we give a characterization of Hibi rings which satisfy $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R) = \mathrm{fpt}(\mathfrak{m})$, that is, we consider the following question:

Question: When does $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R) = \mathrm{fpt}(\mathfrak{m})$ hold for Hibi rings?

Hirose, Watanabe and Yoshida [HWY] showed that for any homogeneous affine toric ring R with the unique graded maximal ideal $\mathfrak{m}$, R is Gorenstein if and only if $\mathrm{fpt}(\mathfrak{m}) = -a(R)$. Hence we need to study Hibi rings which satisfy $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R)$.

Let P_1, P_2 be posets and let $R_1 = R(P_1)$, $R_2 = R(P_2)$ be Hibi rings made from P_1, P_2 respectively. In order to give an answer of the above question, we observe the tensor products and Segre products of R_1 and R_2(see [Hib], [HeHiR]).

Firstly, we define some notions.

Definition 16. A ring R is *trivial* if R can be made by the following operations : starting from polynomial rings, repeated applications of tensor products and Segre products.

Definition 17. (see [HeHiR]) A poset P is *simple* if there is no element of P which is comparable with any other element of P.

Tensor Products:

Let P be a *not* simple poset. Then there exists $p \in P$ such that p is comparable with any other element of P. Put $P_1 = \{q \in P \mid q < p\}$ and $P_2 = \{q \in P \mid q > p\}$. Then

$$R(P) \simeq R_1 \otimes R_2$$

holds. Moreover, it is easy to see that

$$\begin{aligned} \mathrm{rank}^* P &= \mathrm{rank}^* P_1 + \mathrm{rank}^* P_2 + 2, \\ \mathrm{rank} P &= \mathrm{rank} P_1 + \mathrm{rank} P_2 + 2, \\ \mathrm{rank}_* P &= \mathrm{rank}_* P_1 + \mathrm{rank}_* P_2 + 2. \end{aligned}$$

Hence we have

$$\mathrm{rank}^* P = \mathrm{rank} P = \mathrm{rank}_* P$$

$$\Updownarrow$$

$$\mathrm{rank}^* P_1 = \mathrm{rank} P_1 = \mathrm{rank}_* P_1 \quad \text{and} \quad \mathrm{rank}^* P_2 = \mathrm{rank} P_2 = \mathrm{rank}_* P_2.$$

Segre Products:

Let P be a *not* connected (that is, its Hasse diagram is not connected) poset. Then there exist two non-empty subposets P_1 and P_2 of P such that the elements of P_1 and P_2 are incomparable. Then

$$R(P) \simeq R_1 \# R_2$$

holds. Moreover, it is easy to see that

$$\begin{aligned} \mathrm{rank}^* P &= \max\{\mathrm{rank}^* P_1, \mathrm{rank}^* P_2\}, \\ \mathrm{rank} P &= \max\{\mathrm{rank} P_1, \mathrm{rank} P_2\}, \\ \mathrm{rank}_* P &= \min\{\mathrm{rank}_* P_1, \mathrm{rank}_* P_2\}. \end{aligned}$$

Hence we have

$$\mathrm{rank}^* P_1 = \mathrm{rank} P_1 \ \text{ and } \ \mathrm{rank}^* P_2 = \mathrm{rank} P_2 \quad \Rightarrow \quad \mathrm{rank}^* P = \mathrm{rank} P$$

and

$$\mathrm{rank} P = \mathrm{rank}_* P \quad \Rightarrow \quad \mathrm{rank} P_1 = \mathrm{rank}_* P_1 \ \text{ and } \ \mathrm{rank} P_2 = \mathrm{rank}_* P_2$$

holds. If P is pure, then the converses of the above assertion are also true, that is

$$\mathrm{rank}^* P = \mathrm{rank} P = \mathrm{rank}_* P$$

$$\Updownarrow$$

$$\mathrm{rank}^* P_1 = \mathrm{rank} P_1 = \mathrm{rank}_* P_1 \quad \text{and} \quad \mathrm{rank}^* P_2 = \mathrm{rank} P_2 = \mathrm{rank}_* P_2$$

holds since $\mathrm{rank} P = \mathrm{rank} P_1 = \mathrm{rank} P_2$.

By using these observation, we prove the following proposition.

Proposition 18. *Let P be a finite poset, and $R = R(P)$ the Hibi ring made from P. Let $\mathfrak{m} = R_+$ be the graded maximal ideal of R. Then if R is trivial, then $\mathrm{rank}^* P = \mathrm{rank} P$, that is, $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R)$. Moreover, if P is pure, the converse is also true.*

Proof. The first assertion is clear from the above observation and the fact that $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R)$ if R is a polynomial ring.

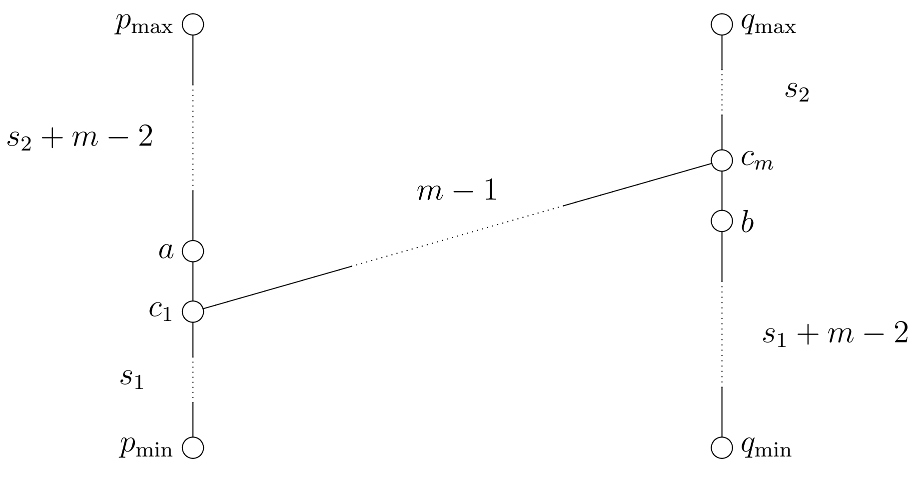
We prove that the converse is true if P is pure. Assume that R is *not* trivial. From the above observation, we may assume that P is simple and connected.

Firstly, we refer the following lemma.

Lemma 19. *([HeHiR, Lemma 3.5]) Every simple and connected poset P possesses a saturated ascending chain $A = c_1 \to c_m$ $(m \geq 2)$ together with $a,\ b \in P$ satisfying the following condition : (i) $c_m \gtrdot b$; (ii) $a \gtrdot c_1$; (iii) $c_1 \not\leq b$; (iv) $a \not\leq c_m$.*

Hence, it is enough to show that $\mathrm{rank}^* P > \mathrm{rank} P$ under the situation as in Lemma 3.5.

We consider three paths $C_1 = p_{\min} \to p_{\max}$, $C_2 = p_{\min} \to q_{\max}$ and $C_3 = q_{\min} \to q_{\max}$ as the following:



We put $\mathrm{length}(p_{\min} \to c_1) = s_1$ and $\mathrm{length}(c_m \to q_{\max}) = s_2$. Since P is pure,

$$\mathrm{rank} P = \mathrm{length} C_1 = \mathrm{length} C_2 = \mathrm{length} C_3 = s_1 + s_2 + m - 1.$$

Hence we have

$$\text{length}(a \to p_{\max}) = s_2 + m - 2, \quad \text{length}(q_{\min} \to b) = s_1 + m - 2.$$

Let $C = q_{\min} \to c_m \to c_1 \to p_{\max}$ be a path. Then it is easy to show that C satisfies a condition (*). Moreover,

$$\begin{aligned} \text{length}^* C &= (s_2 + m - 1) + (s_1 + m - 1) \\ &= s_1 + s_2 + 2m - 2 \\ &> s_1 + s_2 + m - 1 \\ &= \text{rank} P \end{aligned}$$

since $m \geq 2$. Therefore we have $\text{rank}^* P > \text{rank} P$. □

In [HeHiR], Herzog, Hibi and Restuccia introduced the notion of strongly Koszulness for homogeneous k-algebra, and they proved that a Hibi ring is strongly Koszul if and only if it is trivial(see [HeHiR, Theorem 3.2]). Moreover, from [HWY], we can see that for any Hibi ring $R = R(P)$ with the unique graded maximal ideal $\mathfrak{m}$, $\text{rank} P = \text{rank}_* P$ if and only if P is pure. Therefore, we get the following theorem:

Theorem 20 (see [CM], [HeHiR]). *Let P be a finite poset, and $R = R(P)$ the Hibi ring made from P. Let $\mathfrak{m} = R_+$ be the graded maximal ideal of R. The the following assertions are equivalent:*

(1) *R is trivial and Gorenstein.*
(2) *R is strongly Koszul and Gorenstein.*
(3) *R satisfies $c^{\mathfrak{m}}(\mathfrak{m}) = -a(R) = \text{fpt}(\mathfrak{m})$.*
(4) *P satisfies $\text{rank}^* P = \text{rank} P = \text{rank}_* P$.*

References

[BH] W. Bruns and J. Herzog, *On the computation of a-invariants*, manuscripta math., **77** (1992), 201–213.

[Bl] M. Blickle, *Multiplier ideals and modules on toric varieties*, Math. Z, **248** (2004), 113–121.

[CM] T. Chiba and K. Matsuda, *Characterization of Gorenstein strongly Koszul Hibi rings by F-invariants*, arXiv:1201.5691.

[GW] S. Goto and K.-i. Watanabe, *On graded rings*, I, J. Math. Soc. Japan **30**(2) (1978), 179–213.

[HeHiR] J. Herzog, T. Hibi and G. Restuccia, *Strongly Koszul algebras*, Math. Scand., **86** (2000), 161–178.

[Hib] T. Hibi, *Distributive lattices, affine semigroup rings and algebras with straightening laws*, in "Commutative Algebra and Combinatorics" (M. Nagata and H. Matsumura, eds.) Adv. Stud. Pure Math. 11, North Holland, Amsterdam, (1987), 93–109.

[Hir] D. Hirose, *Formulas of F-thresholds and F-jumping coefficients on toric rings*, Kodai Math. J., **32** (2009), 238–255.

[HMTW] C. Huneke, M. Mustaţă, S. Takagi and K.-i. Watanabe, *F-thresholds, tight closure, integral closure, and multiplicity bounds*, Michigan Math. J., **57** (2008), 461–480.

[HY] N. Hara and K. Yoshida, *A generalization of tight closure and multiplier ideals*, Trans. Amer. Math, **355** (2003), 3143–3174.

[HWY] D. Hirose, K.-i. Watanabe and K. Yoshida, *F-thresholds vs. a-invariants for homogeneous toric rings*, preprint.

[MOY] K. Matsuda, M. Ohtani and K. Yoshida, *Diagonal F-thresholds on binomial hypersurfaces*, Communications in Algebra, **38** (2010), 2992–3013.

[MTW] M. Mustaţă, S. Takagi and K.-i. Watanabe, *F-thresholds and Bernstein-Sato polynomials*, European Congress of Mathematics, 341–364, Eur. Math. Soc., Zürich.
[St] R. Stanley, *Two Poset Polytopes*, Discrete & Computational Geometry, **1** (1986), 9–23.
[TW] S. Takagi and K.-i. Watanabe, *On F-pure thresholds*, J. Algebra, **282** (2004), 278–297.

Graduate School of Mathematics
Nagoya University
Furocho, Chikusaku 464-8602 JAPAN
E-mail address: d09003p@math.nagoya-u.ac.jp