DERIVED GABRIEL TOPOLOGY, LOCALIZATION AND COMPLETION OF DG-ALGEBRAS

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Abstract. Gabriel topology is a special class of linear topology on rings, which plays an important role in the theory of localization of (not necessary commutative) rings []. Several evidences have suggested that there should be a corresponding notion for dg-algebras. In my talk I introduced a notion of Gabriel topology on dg-algebras, derived Gabriel topology, and showed its basic properties.

In the same way as the definition of derived Gabriel topology on a dg-algebra, we gave the definition of topological dg-modules over a dg-algebra equipped with derived Gabriel topology. An important example of topology on dg-modules is the finite topology on the bi-dual module $M^{\circledast\circledast}$ of a dg-module M by another dg-module J.

We show that a derived bi-duality dg-module is quasi-isomorphic to the homotopy limit of a certain tautological functor. This is a simple observation, which seems to be true in wider context. From the view point of derived Gabriel topology, this is a derived version of results of J. Lambek about localization and completion of ordinary rings. However the important point is that we can obtain a simple formula for the bi-duality modules only when we come to the derived world from the abelian world.

We give applications. 1. we give a generalization and an intuitive proof of Efimov-Dwyer-Greenlees-Iyenger Theorem which asserts that the completion of commutative ring satisfying some conditions is obtained as a derived bi-commutator. (We can also prove Koszul duality for dg-algebras with Adams grading satisfying mild conditions.) 2. We prove that every smashing localization of dg-category is obtained as a derived bi-commutator of some pure injective module. This is a derived version of the classical results in localization theory of ordinary rings.

These applications show that our formula together with the viewpoint that a derived bi-commutator is a completion in some sense, provide us a fundamental understanding of a derived bi-duality module.

Key Words: Derived bi-duality, homotopy limit, dg-algebras, completion, localization, Koszul duality, Lambek Theorem.

1. Introduction

The following situation and its variants are ubiquitous in Algebras and Representation theory:

Let R be a ring, J an R-module and $E := \mathrm{End}_R(J)^{\mathrm{op}}$ the opposite ring of the endomorphism ring of J over R. Then we have the duality

$$(-)^* := \mathrm{Hom}_R(-, J) : \mathrm{Mod}R \rightleftarrows (\mathrm{Mod}E)^{\mathrm{op}} : \mathrm{Hom}_E(-, J) =: (-)^*$$

and the unite map $\epsilon_M : M \to M^{**}$ is given by the evaluation map:

$$\epsilon_M(m) : \mathrm{Hom}_R(M, J) \to J,\ f \mapsto f(m) \text{ for } m \in M.$$

The detailed version of this paper will has been submitted for publication elsewhere.

The bi-dual R^{**} of R is called the *bi-commutator* (or the *double centralizer*) and denoted by $\mathrm{Bic}_R(J)$. The following is more popular expression (or the usual definition) of the bi-commutator

$$\mathrm{Bic}_R(J) := \mathrm{End}_E(J)^{\mathrm{op}}.$$

The bi-commutator has a ring structure and the evaluation map $\epsilon_R : R \to \mathrm{Bic}_R(J)$ become a ring homomorphism. In particular, the case where the canonical algebra homomorphism $R \to \mathrm{Bic}_R(J)$ become an isomorphism, the module J is said to have the *double centralizer property.* Dualities together with evaluation maps, bi-commutators and double centralizer properties are one of the central topics in Algebras and Representation theory. (See e.g. [5, 8, 10, 11, 17, 26])

Recently the concern with the *derived bi-commutators* (or the *derived double centralizers*) has been growing:

Let R a ring (or more generally dg-algebra) J an (dg-)R-module and $\mathcal{E} := \mathbb{R}\mathrm{End}_R(J)^{\mathrm{op}}$ the opposite dg-algebra of the endomorphism dg-algebra of J. Then the *derived bi-commutator* is defined by

$$\mathbb{B}\mathrm{ic}_R(J) := \mathbb{R}\mathrm{End}_{\mathcal{E}}(J)^{\mathrm{op}}.$$

There also exists a canonical algebra homomorphism $R \to \mathbb{B}\mathrm{ic}_R(J)$. In particular, the case where the canonical algebra homomorphism $R \to \mathbb{B}\mathrm{ic}_R(J)$ become an isomorphism, the module J is said to have the *derived double centralizer property.* Derived double centralizer property for special modules has been extensively studied as a part of Koszul duality. (See e.g. [12, 22].)

In [2, Section 4.16], Dwyer-Greenlees-Iyenger call a pair (R, J) *dc-complete*, in the case where J has derived double centralizer property. They proved the following surprising and impressive theorem, which we will refer as *completion theorem.*

Theorem 1 ([2],[3])**.** *Let R be a commutative Noetherian ring and $\mathfrak{a}$ an ideal such that the residue ring $R/\mathfrak{a}$ is of finite global dimension. We denote by $\widehat{R}$ the $\mathfrak{a}$-adic completion. Then we have a quasi-isomorphism*

$$\widehat{R} \simeq \mathbb{B}\mathrm{ic}_R(R/\mathfrak{a})$$

where $\mathbb{B}\mathrm{ic}_R(R/\mathfrak{a})$ is the derived bi-commutator of $R/\mathfrak{a}$ over R.

From the view point of Derived-Categorical Algebraic Geometry (DCAG), all important procedure in Algebraic Geometry should have derived-categorical interpretation. In [7] Kontsevich claimed that formal completion for a scheme is obtained as a derived bi-commutator. Following this idea, Efimov [3] introduced the derived bi-commutator of subcategory $\rfloor\mathcal{J} \subset \mathcal{D}(R)$ and proved a scheme version of completion theorem. Since formal completion plays an important role in Algebraic Geometry, completion theorem and its scheme version are expected to become important in DCAG. Therefore it is desirable to obtain better understanding of this theorem.

In the proof of completion theorem, Grothedieck vanishing theorem for local cohomology is used. Since it is special theorem for commutative Noetherian rings, it is preferable to obtain more categorical proof. Recently Porta, Shaul and Yekutieli [21] generalized completion theorem for a commutative ring R and a weakly proregular ideal $\mathfrak{a}$ based on their work [20] about the derived functors of the completion functors and the torsion functors. However it is still remain unclear that to what extent we can obtain a transcendental outcome by a homological operation with finite input. In this paper we establish a simple description of the derived bi-commutator, which enable us to give a more intuitive proof of completion theorem. Actually the description is given by a certain tautological homotopy limit, and hence seems to state that every derived bi-commutator is completion in some sense. (We can make this precise by introducing the notion of derived Gabriel topology.)

For this purpose, we study derived bi-duality:

$$(-)^{\circledast} := \mathbb{R}\mathrm{Hom}_R(-, J) : \mathcal{D}(R) \rightleftarrows \mathcal{D}(\mathcal{E})^{\mathrm{op}} : \mathbb{R}\mathrm{Hom}_{\mathcal{E}}(-, J) =: (-)^{\circledast}.$$

For a special class of modules J, derived bi-duality is already studied in the context of Gorenstein dg-algebras [4, 6, 13]. We consider general dg-modules J and establish a simple description of the derived bi-dual module $M^{\circledast\circledast}$ via a certain tautological homotopy limit. This is the main result of this paper. As an application other than completion theorem, we discuss smashing localization of dg-categories.

As mentioned above, derived bi-dualities, derived bi-commutator and derived double centralizer property are expected to play prominent roles in Algebras, Representation theory, Derived-Categorical Algebraic Geometry. Our main theorem together with the view point that derived bi-commutators are completion in some sense, would have many applications. Moreover since the main theorem is proved in a formal argument, the same formula should hold in more wider context. Bi-duality is a basic operation which is ubiquitous in mathematics. So it can be expect that our main theorem become an indispensable tool in many area of mathematics.

Below we give an outline, in which the readers see that if we omit homotopy theoretical details, things become very simple. However, we will see that it is inevitable to work with homotopy theory.

2. Derived bi-duality via homotopy limit

Let $\mathcal{A}$ be a dg-algebra and J a dg $\mathcal{A}$-module. We denote $\mathcal{E} := (\mathbb{R}\mathrm{End}_{\mathcal{A}}(J))^{\mathrm{op}}$ be the opposite dg-algebra of the endomorphism dg-algebra. Then J has a natural dg $\mathcal{E}$-module structure. We obtain the dualities

$$(-)^{\circledast} := \mathbb{R}\mathrm{Hom}_{\mathcal{A}}(-, J) : \mathcal{D}(\mathcal{A}) \rightleftarrows \mathcal{D}(\mathcal{E})^{\mathrm{op}} : \mathbb{R}\mathrm{Hom}_{\mathcal{E}}(-, J) =: (-)^{\circledast}.$$

There are natural transformations $\epsilon : 1_{\mathcal{D}(\mathcal{A})} \to (-)^{\circledast\circledast}$ induced from evaluation morphisms.

We denote by $\langle J \rangle$ the smallest thick subcategory containing J. Namely $\langle J \rangle$ is the full triangulated subcategory of $\mathcal{D}(\mathcal{A})$ consisting those objects which constructed from J by taking cones, shifts, and direct summands finitely many times.

Let M be a dg $\mathcal{A}$-module. We denote by $\langle J \rangle_{M/}$ the under category. Namely, the objects of $\langle J \rangle_{M/}$ are morphisms $k : M \to K$ with $K \in \langle J \rangle$ and the morphisms from $k : M \to K$ to $\ell : M \to L$ are the morphisms $\psi : K \to L$ in $\langle J \rangle$ such that $\ell = \psi \circ k$. This category $\langle J \rangle_{M/}$ comes naturally equipped with the co-domain functor $\Gamma : \langle J \rangle_{M/} \to \mathcal{D}(\mathcal{A})$ which sends an object $k : M \to K$ to its co-domain K.

$$\Gamma : \langle J \rangle_{M/} \to \mathcal{D}(\mathcal{A}), \quad [k : M \to K] \mapsto K.$$

The following simple formula is the main theorem.

Theorem 2. *We have the following quasi-isomorphism*

$$M^{\circledast\circledast} \simeq \operatorname*{holim}_{\langle J\rangle_{M/}} \Gamma$$

Remark 3. In the above Theorem 2, Theorem 4 and Corollary 5, we omit homotopy theoretical details. For the rigorous statements see [14].

To explain an idea of a proof, we give the following heuristic arguments. First we claim that if K belongs to $\langle J\rangle$, then the evaluation map $\epsilon_K : K \to K^{\circledast\circledast}$ is an isomorphism. Indeed the case $K = J$ is clear. Since the bi-dual $(-)^{\circledast\circledast}$ is an exact functor, we can check the claim for general $K \in \langle J\rangle$.

It follows from the above claim that every morphism $k : M \to K$ with $K \in \langle J\rangle$ factors though $\epsilon_M : M \to M^{\circledast\circledast}$.

$$\begin{tikzcd}
M \arrow[r, "\epsilon_M"] \arrow[d, "k"'] & M^{\circledast\circledast} \arrow[d, "k^{\circledast\circledast}"] \\
K & K^{\circledast\circledast} \arrow[l, "\epsilon_K^{-1}"', "\cong"]
\end{tikzcd}$$

It seems that the derived bi-dual module $M^{\circledast\circledast}$ satisfies one of the two conditions of the limit of the family $M \to K$ of morphisms. In the following way, we can catch a glimpse of the other condition that we can reach from $K \in \langle J\rangle$ to $M^{\circledast\circledast}$:

It is well-known that a dg-module is obtained as a filtered homotopy colimit perfect modules. Hence the dg $\mathcal{E}$-module $M^{\circledast}$ is quasi-isomorphic to the homotopy colimit of some family $\{P_\lambda\}_\Lambda$ of perfect $\mathcal{E}$-modules.

$$M^{\circledast} \simeq \operatorname*{hocolim}_{\Lambda} P_\lambda \tag{2.1}$$

Applying the dual functor $(-)^{\circledast}$ to this quasi-isomorphism, we obtain the following (quasi-)isomorphisms

$$M^{\circledast\circledast} \simeq (\operatorname*{hocolim}_{\Lambda} P_\lambda)^{\circledast} \simeq \operatorname*{holim}_{\Lambda}(P_\lambda^{\circledast}).$$

It is clear that $\mathcal{E}^{\circledast} \simeq J$. Therefore, since P_λ is a perfect $\mathcal{E}$-module, the dual $P_\lambda^{\circledast}$ belongs to $\langle J\rangle$. This shows that we can reach from $K \in \langle J\rangle$ to $M^{\circledast\circledast}$. Actually the following Theorem 4 which is a version of the quasi-isomorphism (2.1) is a key of the proof of the main theorem.

Theorem 4. *Let X be a dg $\mathcal{E}$-module. We denote by $\operatorname{Perf}\mathcal{E}$ the category of perfect $\mathcal{E}$-modules. Then the over category $(\operatorname{Perf}\mathcal{E})_{/X}$ comes naturally equipped with the domain functor*

$$\Upsilon : \operatorname{Perf}\mathcal{E} \to \mathcal{D}(\mathcal{E}),\ [p : P \to X] \mapsto P.$$

Then the canonical morphism

$$\operatorname*{hocolim}_{(\operatorname{Perf}\mathcal{E})_{/X}} \Upsilon \to X$$

is a quasi-isomorphism.

Since the bi-dual $\mathcal{A}^{\circledast\circledast}$ of $\mathcal{A}$ is naturally isomorphic to the derived bi-commutator $\mathbb{B}\mathrm{ic}_{\mathcal{A}}(J)$,

$$\mathcal{A}^{\circledast\circledast} = \mathbb{R}\mathrm{Hom}_{\mathcal{E}}(\mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}, J), J) \cong \mathbb{R}\mathrm{Hom}_{\mathcal{E}}(J, J) = \mathbb{B}\mathrm{ic}_{\mathcal{A}}(J)$$

in particular, we have the following corollary.

Corollary 5.

$$\mathbb{B}\mathrm{ic}_{\mathcal{A}}(J) \simeq \operatorname*{holim}_{\langle J\rangle_{\mathcal{A}/}} \Gamma.$$

These theorem and corollary provide us a fundamental understanding of derived bi-duality functors.

3. Completion via derived bi-commutator

As the first application, we generalize the completion theorem and give an intuitive proof.

Let R be a ring and $\mathfrak{a}$ a two-sided ideal. An (right) R-module M is called $\mathfrak{a}$-*torsion* if for any $m \in M$ there exists $n \in \mathbb{Z}_{\geq 1}$ such that $m\mathfrak{a}^n = 0$. We denote by $\mathfrak{a}$-tor the full subcategory of Mod$R$ consisting of $\mathfrak{a}$-torsion modules. We denote by $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ the full subcategory of $\mathcal{D}(R)$ consisting of complexes with $\mathfrak{a}$-torsion cohomology groups. We denote by $\mathcal{D}(\mathfrak{a}\text{-tor})$ the full subcategory of $\mathcal{D}(R)$ consisting of complexes each term of which is $\mathfrak{a}$-torsion module.

Theorem 6. *Assume that the canonical inclusion functor $\mathcal{D}(\mathfrak{a}\text{-tor}) \to \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence and that $R/\mathfrak{a}^n$ belongs to $\langle R/\mathfrak{a}\rangle$ for $n \geq 0$. We denote by $\widehat{R}$ the $\mathfrak{a}$-adic completion. Then we have a quasi-isomorphism*

$$\mathbb{B}\mathrm{ic}_R(R/\mathfrak{a}) \simeq \widehat{R}.$$

"*Proof*".

Assumption. In this "Proof" we assume that holim $=$ lim. We identify quasi-isomorphisms with isomorphisms.

We denote by $\mathcal{I}$ the (non-full) subcategory of $\langle R/\mathfrak{a}\rangle_{R/}$ which consists of objects $\pi^n : R \to R/\mathfrak{a}^n$ for $n \geq 1$ and of morphisms $\pi^m \to \pi^n$ induced from the canonical projections $\varphi^{m,n} : R/\mathfrak{a}^m \to R/\mathfrak{a}^n$ for $m \geq n$. In other words, $\mathcal{I}$ is the image of the functor $(\mathbb{Z}_{\geq 1})^{\mathrm{op}} \to \mathcal{D}(\mathcal{A})$ which sends an object n to π^n and a morphism $m \to n$ to $\pi^m \to \pi^n$ where we consider the ordered set $\mathbb{Z}_{\geq 1}$ as a category in the standard way. Therefore we have

$$\lim_{\mathcal{I}} \Gamma|_{\mathcal{I}} \cong \lim_{n\to\infty} R/\mathfrak{a}^n \cong \widehat{R}.$$

Thanks to Corollary 5 the problem is reduced to show that $\lim_{\mathcal{I}} \Gamma|_{\mathcal{I}} \cong \lim_{\langle R/\mathfrak{a}\rangle_{R/}} \Gamma$. Therefore it is enough to prove that $\mathcal{I}$ is a left cofinal subcategory of $\langle R/\mathfrak{a}\rangle_{R/}$. Namely only we have to show that the over category $\mathcal{I}_{/k}$ is non-empty and connected for each $k \in \langle R/\mathfrak{a}\rangle_{R/}$.

Let $k : R \to K$ be an object of $\langle R/\mathfrak{a}\rangle_{R/}$. It is clear that $\langle R/\mathfrak{a}\rangle$ contained in $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$. Since we assume that $\mathcal{D}(\mathfrak{a}\text{-tor}) \xrightarrow{\sim} \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$, K belongs to $\mathcal{D}(\mathfrak{a}\text{-tor})$. It follows that K is (quasi-)isomorphic to a complex each term of which is an $\mathfrak{a}$-torsion modules. Therefore a morphism $k : R \to K$ canonically factors through some cyclic $\mathfrak{a}$-torsion module $R/\mathfrak{a}^n$.

$$\begin{tikzcd} R \arrow[d, "\pi^n"'] \arrow[dr, "k"] & \\ R/\mathfrak{a}^n \arrow[r, "\psi"'] & K \end{tikzcd}$$

In other words, there exists a morphism $\psi : \pi^n \to k$ in $\langle R/\mathfrak{a}\rangle_{R/}$. This proves the non-emptiness of $\mathcal{I}_{/k}$. Since the factorization $k = \psi \circ \pi^n$ is canonical, we see that $\mathcal{I}_{/k}$ is connected. This shows that $\mathcal{I}$ is left co-final in $\langle R/\mathfrak{a}\rangle_{R/}$ and completes the "proof". "$\square$"

In [21] Porta, Shaul and Yekutieli generalized completion theorem (Theorem 1) by using a compact generator of $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$.

Theorem 7 ([21, Theorem 4.2])**.** *Let R be a commutative ring and $\mathfrak{a}$ a weakly pro-regular ideal. Let K be a compact generator of $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$. Then we have the following quasi-isomorphism of dg-algebras under R.*

$$\mathbb{B}\mathrm{ic}_R(K) \simeq \widehat{R}.$$

By our method, we give a generalization of this theorem.

Theorem 8. *Let R be a ring and $\mathfrak{a}$ an two-sided ideal such that the canonical functor $\mathcal{D}(\mathfrak{a}\text{-tor}) \to \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence. Let K be a compact generator of $\mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$. Then we have a quasi-isomorphism*

$$\mathbb{B}\mathrm{ic}_R(K) \simeq \widehat{R}.$$

It is proved by [20, Corollary 3.31] that if a ring R is commutative and an ideal $\mathfrak{a}$ is weakly pro-regular, then the canonical functor $\mathcal{D}(\mathfrak{a}\text{-tor}) \to \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence. Therefore Theorem 8 implies Theorem 7.

Remark 9. The conditions on Theorem 6 and Theorem 8 are not practical. The reason why we put these artificial conditions is not to obtain generality but to clarify to what extent the derived bi-commutator gives the completion.

The condition that the canonical functor $\mathbf{can}_{\mathfrak{a}} : \mathcal{D}(\mathfrak{a}\text{-tor}) \to \mathcal{D}_{\mathfrak{a}\text{-tor}}(R)$ gives an equivalence is satisfied if the subcategory $\mathfrak{a}$-tor is closed under taking injective hull. This condition is satisfied if a right ideal $\mathfrak{a}$ has the Artin-Rees property. In particular, in the case where a ring R is commutative Noetherian, for any ideal $\mathfrak{a}$ the functor $\mathbf{can}_{\mathfrak{a}}$ is an equivalence. As we mentioned before, if a ring R is commutative and an ideal $\mathfrak{a}$ is weakly pro-regular, then the canonical functor $\mathbf{can}_{\mathfrak{a}}$ is an equivalence. It should be noted that if R is commutative Noetherian, any ideal $\mathfrak{a}$ is weakly pro-regular (See [1, 20, 23]). It is showed in [20, Example 3.35] that a weakly pro-regular ideal in non-Noetherian ring naturally appears.

The following question arises: find a necessary and sufficient condition on rings R and ideals $\mathfrak{a}$ such that the canonical functor $\mathbf{can}_{\mathfrak{a}}$ is an equivalence.

4. Smashing localization via derived bi-commutator

First we recall the following classical fact.

Theorem 10 ([10, Corollary 3.4.1], [17, Theorem 7.1])**.** *Let $f : R \to S$ be a (right) Gabriel localization of a ring R, that is, f is an epimorphism in the category of rings and S is left flat over R. Let J be a co-generator of the torsion theory which corresponds to the Gabriel localization f. If we take a product $J' := J^{\kappa}$ of copies of J over large enough cardinal κ, then we have an isomorphism*

$$\mathrm{Bic}_R(J') \cong S.$$

In this section we prove a derived version. A morphisms $f : \mathcal{A} \to \mathcal{B}$ of dg-algebras is called *smashing localization* (or *homological epimorphism*) if the restriction functor $f_* : \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ is fully faithful. Recall that a ring homomorphism $R \to S$ is an epimorphism in the category of rings if and only if the restriction functor $f_* : \mathrm{Mod}S \to \mathrm{Mod}R$ is fully faithful. Therefore smashing localization can be considered as a dg-version of epimorphisms of rings.

Theorem 11. *Let $\mathcal{A} \to \mathcal{B}$ be a smashing localization of dg-algebras and J be a pure injective co-generator of $\mathcal{D}(\mathcal{B})$. Then we have a quasi-isomorphism over $\mathcal{A}$*

$$\mathbb{B}\mathrm{ic}_{\mathcal{A}}(f_* J') \simeq \mathcal{B}.$$

where $J' = J^{\Pi\kappa}$ is a large enough product of J.

The notion of pure injective co-generator which is introduced by Krause [9] is a dg-version of injective co-generator for the module category $\mathrm{Mod}R$ of an ordinary ring R.

Remark 12. Nicolás and Saorin [18] proved that for any smashing localization $F : \mathcal{D}(\mathcal{A}) \to \mathcal{S}$, there exists a subcategory $\mathcal{I} \subset \mathcal{D}(\mathcal{A})$ such that the functor $\mathbb{L}\iota_{\mathcal{I}}^* : \mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathbb{B}\mathrm{ic}_{\mathcal{A}}(\mathcal{I}))$ induced from the canonical morphism $\iota_{\mathcal{I}} : \mathcal{A} \to \mathbb{B}\mathrm{ic}_{\mathcal{A}}(\mathcal{I})$ is equivalent to F.

In our way of the proof, an essential point is the following theorem.

Theorem 13. *Let J a pure injective co-generator of $\mathcal{D}(\mathcal{A})$ and M a dg $\mathcal{A}$-module. If we take a product $J' = J^{\Pi\kappa}$ of copies of J over large enough cardinal κ, then the evaluation morphism is a quasi-isomorphism*

$$\epsilon_M : M \xrightarrow{\sim} M^{\circledast\circledast}$$

where the bi-dual is taken over J'.

From the view point that a derived bi-commutator is a completion, we can give an intuitive proof of Theorem 13 by using Theorem 2. (In the case where $\mathcal{A}$ is an ordinary ring and M is a module, the same results is already proved by Shamir [24] in a different way.) In the rest of this subsection, we use the same assumption with that of "Proof" of Theorem 6.

For the sake of simplicity we deal with the case where $\mathcal{A}$ is an ordinary ring, M is an $\mathcal{A}$-module and J is an injective co-generator of $\mathrm{Mod}\mathcal{A}$. Then the module M has an injective resolution by the products of J

$$0 \to M \to J^{\Pi\kappa_0} \to J^{\Pi\kappa_1} \to J^{\Pi\kappa_2} \to \cdots .$$

We can reduce the problem to the following theorem by setting $\kappa := \sup\{\kappa_i \mid i \in \mathbb{Z}\}$.

Theorem 14. *Let $M \xrightarrow{\sim} J^\bullet$ be an injective resolution of M.*

$$0 \to M \to J^0 \to J^1 \to J^2 \to \cdots . \tag{4.1}$$

Assume that J^i is a direct summand of J. Then the evaluation map $\epsilon_M : M \to M^{\circledast\circledast}$ is a quasi-isomorphism.

We denote by I^n the totalization of the n-th truncated resolution.

$$I^n := \mathrm{tot}[J^0 \to J^1 \to \cdots \to J^n].$$

Then by assumption the complex I^n belongs to the thick subcategory $\langle J\rangle$ generated by J. Therefore the canonical morphism $\pi^n : M \to I^n$ belongs to the under category $\langle J\rangle_{M/}$. Moreover we have a canonical morphism $\varphi^{n+1,n} : I^{n+1} \to I^n$ for $n \geq 0$ which is compatible with π^n.

$$\begin{tikzcd} & M \arrow[d, "\pi^n"'] \arrow[dr, "\pi^{n-1}"] & & \\ \cdots \arrow[r, "\varphi^{n+1,n}"'] & I^n \arrow[r, "\varphi^{n,n-1}"'] & I^{n-1} \arrow[r, "\varphi^{n-1,n-2}"'] & \cdots \end{tikzcd} \qquad .$$

Note that since the limit $\lim_{n\to\infty} I^n$ is the totalization of the injective resolution (4.1), the morphisms $\{\pi^n\}$ induces a (quasi-)isomorphism $M \to \lim_{n\to\infty} I^n$. We will see that the family $\{\pi^n : M \to I^n\}_{n\geq 0}$ is an "approximation" for the morphisms $k : M \to K$ with $K \in \langle J\rangle$.

We denote by $\mathcal{I}$ the subcategory of $\langle J\rangle_{M/}$ consisting of objects $\pi^n : M \to I^n$ and of morphisms $\phi^{m,n} : \pi^m \to \pi^n$ so that $\mathcal{I}$ is isomorphic to $(\mathbb{Z}_{\geq 0})^{\mathrm{op}}$. Then it is clear that

$$\lim_{\mathcal{I}} \Gamma|_{\mathcal{I}} \cong \lim_{n\to\infty} I^n \simeq M.$$

Therefore by "Theorem" 2 it is enough to prove that the subcategory $\mathcal{I} \subset \langle J\rangle_{M/}$ is left co-final. Namely for each $k \in \langle J\rangle_{M/}$ the over category $\mathcal{I}_{/k}$ is non-empty and connected.

We recall the following elementary fact from Homological algebra: Let M' be another $\mathcal{A}$-module and $M \xrightarrow{\sim} J'^{\bullet}$ an injective resolution. Assume that an $\mathcal{A}$-homomorphism $f : M \to M'$ is given. Then (1) there exists a morphism $\psi : J^{\bullet} \to J'^{\bullet}$ of complexes which completes the commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "f"'] & J^{\bullet} \arrow[d, "\psi"] \\ M' \arrow[r] & J'^{\bullet}. \end{tikzcd}$$

(2) This morphism ψ is not uniquely determined. (3) However it is uniquely determined up to homotopy.

Using the same methods of the proof of (1), we can check that $\mathcal{I}_{/k}$ is non-empty. By the same reason with (2), the category $\mathcal{I}_{/k}$ is *not* connected. However in the same way of the proof of (3), we can verify that $\mathcal{I}_{/k}$ is "*homotopically connected*". We explain detail in the special case where the co-domain K of $k : M \to K$ is an injective module:

Since the canonical morphism $\pi^0 : M \to I^0 = J^0$ is injective, there exists an extension $\psi : I^0 \to K$ of π^0. This shows that $\mathcal{I}_{/k} \neq \emptyset$. However there is no canonical choice of an extension. Moreover since the degree 0-part of the canonical morphism $\varphi^{n,0} : I^n \to I^0$ is the identity map $1_{J^0} : J^0 \to J^0$, two extensions ψ and ψ' are not connected to each other in $\mathcal{I}_{/k}$, unless $\psi = \psi'$. Nevertheless we can see that for any pair (ψ, ψ') of extensions, there exists a homotopy commutative diagram

$$\begin{tikzcd}
\pi^1 \arrow[r, "\varphi^{1,0}"] \arrow[d, "\varphi^{1,0}"'] & \pi^0 \arrow[d, "\psi"] \\
\pi^0 \arrow[r, "\psi'"'] & k.
\end{tikzcd}$$

Hence the objects ψ and ψ' of $\mathcal{I}_{/k}$ is homotopically connected to each other in $\mathcal{I}_{/k}$. This shows that it is inevitable to work with homotopy theory.

5. Koszul duality for Adams graded dg-algebras (a part of joint work with A. Takahashi)

The following theorem will be proved and applied in [16].

Theorem 15. *Let $\mathcal{A} := \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \mathcal{A}_2 \oplus \cdots$ be an $\mathbb{N}$-Adams graded dg-algebra. If the $\mathcal{A}_0$-modules $\mathcal{A}_n$ satisfies a mild condition. Then we have a quasi-isomorphism*

$$\mathbb{B}\mathrm{ic}_{\mathcal{A}}(\mathcal{A}/\mathcal{A}_{\geq 1}) \simeq \mathcal{A}$$

The proof is given in the same way of the proof of Theorem 6. Here we consider the Adams grading as a "linear topology" on $\mathcal{A}$. The condition is that $\mathcal{A}$ is "complete" with respect to this topology.

6. From the view point of Derived Gabriel topology

Gabriel topology is a special class of linear topology on rings, which plays an important role in the theory of localization of rings [25]. The notion of derived Gabriel topology, which is a derived version of Gabriel topology, is introduced in [15]. From the view point of derived Gabriel topology, Theorem 2 says that the derived bi-dual $M^{\circledast\circledast}$ equipped with "the finite topology" is the "J-adic completion" of M. In this sense Theorem 2 is inspired by the following results of J. Lambek.

Theorem 16 ([10, Theorem 4.2],(See also [11, Theorem 3.7]))**.** *Let R be a ring and J an injective R-module. For an R-module M, we denote by $Q(M)$ the module of quotients with respect to J. Assume that every torsionfree factor module of $Q(M)$ is J-divisible. Then the (ordinary) bi-duality $\mathrm{Hom}_{\mathrm{End}_R(J)}(\mathrm{Hom}_R(M,J),J)$ equipped with the finite topology is the J-adic completion of $Q(M)$.*

Recently many results in ring theory have been becoming to have their derived analogue ([19, 27]). However it can be said that the statements of these derived versions are parallel to that of the original versions. Contrary to this, our derived version of Lambek theorem is definitely improved from the original version. The assumptions and conditions in the original version is removed in the derived version. So the point is that we can obtain a simple formula for the bi-duality modules only when we come to the derived world from the abelian world.

At the first sight, three theorems below concerning on derived bi-dualities

- Completion theorem
- Localization theorem

- Koszul duality

seem to be theorems of different kind. However in the present paper we will see that these are consequences of a simple formula, which is the main theorem 2. From the view point of derived Gabriel topology, these theorems are consequences of completeness of each algebras with respect to appropriate topologies.

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