SELFINJECTIVE ALGEBRAS AND QUIVERS WITH POTENTIALS

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Abstract. We study silting mutations (Okuyama-Rickard complexes) for selfinjective algebras given by quivers with potential (QPs). We show that silting mutation is compatible with QP mutation. As an application, we get a family of derived equivalences of Jacobian algebras.

1. Introduction

Derived categories are nowadays considered as an essential tool in the study of many areas of mathematics. In the representation theory of algebras, derived equivalences of algebras have been one of the central themes and extensively investigated. It is well-known that endomorphism algebras of tilting complexes are derived equivalent to the original algebra [20]. Therefore it is an important problem to give concrete methods to calculate endomorphism algebras of tilting complexes. In this note, we focus on one of the fundamental tilting complexes over selfinjective algebras, known as Okuyama-Rickard complexes, which play an important role in the study of Broué's abelian defect group conjecture. From a categorical viewpoint, they are nowadays interpreted as a special case of silting mutation [3]. We provide a method to determine the quivers with relations of the endomorphism algebras of Okuyama-Rickard complexes when selfinjective algebras are given by quivers with potential (QPs for short).

The notion of QPs was introduced by [7], which plays a significant role in the study of cluster algebras (we refer to [13]). Recently it has been discovered that mutations of QPs (Definition 2) give rise to derived equivalences [5, 15, 18, 22]. The aim of this note is to give a similar (but different) type of derived equivalences by comparing QP mutation and silting mutation (Definition 4).

Conventions. Let K be an algebraically closed field and $D := \mathrm{Hom}_K(-, K)$. All modules are left modules. For a finite dimensional algebra Λ, we denote by $\mathrm{mod}\Lambda$ the category of finitely generated Λ-modules and by $\mathrm{add}M$ the subcategory of $\mathrm{mod}\Lambda$ consisting of direct summands of finite direct sums of copies of $M \in \mathrm{mod}\Lambda$. The composition fg means first f, then g. For a quiver Q, we denote by Q_0 vertices and Q_1 arrows of Q and by $a : s(a) \to e(a)$ the start and end vertices of an arrow or path a.

2. Preliminaries

2.1. **Quivers with potential.** We recall the definition of quivers with potential. We follow [7].

The detailed version of this paper will be submitted for publication elsewhere.

• Let Q be a finite connected quiver without loops. We denote by KQ_i the K-vector space with basis consisting of paths of length i in Q, and by $KQ_{i,cyc}$ the subspace of KQ_i spanned by all cycles. We denote the *complete path algebra* by

$$\widehat{KQ} = \prod_{i \geq 0} KQ_i$$

and by $J_{\widehat{KQ}}$ the Jacobson radical of $\widehat{KQ}$. A *quiver with potential* (QP) is a pair (Q, W) consisting of a finite connected quiver Q without loops and an element $W \in \prod_{i \geq 2} KQ_{i,\mathrm{cyc}}$, called a *potential*. For each arrow a in Q, the *cyclic derivative* $\partial_a : \widehat{KQ}_{cyc} \to \widehat{KQ}$ is defined as the continuous linear map satisfying $\partial_a(a_1 \cdots a_d) = \sum_{a_i = a} a_{i+1} \cdots a_d a_1 \cdots a_{i-1}$ for a cycle $a_1 \cdots a_d$. For a QP (Q, W), we define the *Jacobian algebra* by

$$\mathcal{P}(Q, W) = \widehat{KQ}/\mathcal{J}(W),$$

where $\mathcal{J}(W) = \overline{\langle \partial_a W \mid a \in Q_1 \rangle}$ is the closure of the ideal generated by $\partial_a W$ with respect to the $J_{\widehat{KQ}}$-adic topology.

• A QP (Q, W) is called *trivial* if W is a linear combination of cycles of length 2 and $\mathcal{P}(Q, W)$ is isomorphic to the semisimple algebra $\widehat{KQ_0}$. It is called *reduced* if $W \in \prod_{i \geq 3} KQ_{i,\mathrm{cyc}}$.

Following [9], we use this terminology.

Definition 1. We call a QP (Q, W) *selfinjective* if $\mathcal{P}(Q, W)$ is a finite dimensional self-injective algebra.

Next we recall the definition of mutation of QPs.

Definition 2. For each vertex k in Q not lying on a 2-cycle, we define a new QP $\widetilde{\mu}_k(Q, W) := (Q', W')$ as follows.

(a) Q' is a quiver obtained from Q by the following changes.
- Replace each arrow $a : k \to v$ in Q by a new arrow $a^* : v \to k$.
- Replace each arrow $b : u \to k$ in Q by a new arrow $b^* : k \to u$.
- For each pair of arrows $u \xrightarrow{b} k \xrightarrow{a} v$, add a new arrow $[ba] : u \to v$

(b) $W' = [W] + \Delta$ is defined as follows.
- $[W]$ is obtained from the potential W by replacing all compositions ba by the new arrows $[ba]$ for each pair of arrows $u \xrightarrow{b} k \xrightarrow{a} v$.
- $\Delta = \displaystyle\sum_{\substack{a,b \in Q_1 \\ e(b)=k=s(a)}} [ba]a^*b^*.$

Then *mutation* $\mu_k(Q, W)$ is defined as a *reduced part* of $\widetilde{\mu}_k(Q, W)$ (we refer to [7]).

2.2. **Silting mutation.** The notion of silting objects was introduced by [14], which is a generalization of tilting objects. Recently its theory has been rapidly developed and many connections have been discovered, for example [6, 3, 8, 16]. In this subsection, we briefly recall their definitions and properties.

Now let Λ be a finite dimensional algebra and $\mathcal{T} := \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ be the homotopy category of bounded complexes of finitely generated projective Λ-modules.

Definition 3. Let T be an object of $\mathcal{T}$. We call T *silting* (respectively, *tilting*) if $\mathrm{Hom}_{\mathcal{T}}(T, T[i]) = 0$ for any positive integer $i > 0$ (for any integer $i \neq 0$) and satisfies $\mathcal{T} = \mathsf{thick}T$, where $\mathsf{thick}T$ denote by the smallest thick subcategory of $\mathcal{T}$ containing T.

We call a morphism $f : X \to Y$ *left minimal* if any morphism $g : Y \to Y$ satisfying $fg = f$ is an isomorphism. For an object $M \in \mathcal{T}$, we call a morphism $f : X \to M'$ *left* $(\mathrm{add}M)$*-approximation* of X if M' belongs to $\mathrm{add}M$ and $\mathrm{Hom}_{\mathcal{T}}(f, M'')$ is surjective for any object M'' in $\mathrm{add}M$. Dually we define a *right minimal* morphism and a *right* $(\mathrm{add}M)$*-approximation*.

Definition 4. Let T be a basic silting object in $\mathcal{T}$ and take an arbitrary decomposition $T = X \oplus M$. We take a minimal left $(\mathrm{add}M)$-approximation $f : X \to M'$ of X and a triangle

$$X \xrightarrow{f} M' \longrightarrow Y \longrightarrow X[1].$$

We put $\mu_X(T) := Y \oplus M$ and call it a *left silting mutation* of T with respect to X. Dually we define a *right silting mutation*.

We recall an important result of silting mutation.

Theorem 5. [3, Theorem 2.31] *Any mutation of a silting object is again a silting object.*

Next we give some notations for our setting.

Let Q be a finite connected quiver and $\Lambda := KQ/(R)$ be a finite dimensional algebra. We denote by $\{e_k \mid k \in Q_0\}$ a complete set of primitive orthogonal idempotents of Λ. Take a set of vertices $I := \{k_1, \ldots, k_n\} \subset Q_0$ and we denote by $e_I := e_{k_1} + \cdots + e_{k_n}$ and $\mu_I(\Lambda) := \mu_{\Lambda e_I}(\Lambda)$. We remark that an Okuyama-Rickard complex is nothing but a silting object of $\mathcal{T}$ [3, Theorem 2.50].

By Theorem 5, $\mu_I(\Lambda)$ is always a silting object of $\mathcal{T}$, but it is not necessarily a tilting object. However, for selfinjective algebras, it is a tilting object if it satisfies a condition given by Nakayama permutations.

Definition 6. Let Λ be a selfinjective algebra above. Then there exists a permutation $\sigma : Q_0 \to Q_0$ satisfying $D(e_k\Lambda) \cong \Lambda e_{\sigma(k)}$ for any $k \in Q_0$, where $\nu := D\mathrm{Hom}_{\Lambda}(-, \Lambda) : \mathrm{mod}\ \Lambda \to \mathrm{mod}\ \Lambda$ is the Nakayama functor. We call σ the *Nakayama permutation* of Λ.

Note that $\Lambda e_I \cong \nu(\Lambda e_I)$ if and only if $I = \sigma I$. The following result is useful. We refer to [1, 3] for the proof.

Proposition 7. *Let Λ be a selfinjective algebra above. Then $\mu_I(\Lambda)$ is a tilting object in $\mathcal{T}$ if and only if $I = \sigma I$.*

3. Main results

For a set of vertices $I := \{k_1, \ldots, k_n\} \subset Q_0$, we assume the following conditions.

(a1) Any vertex in I is not contained in 2-cycles in Q.
(a2) There are no arrows between vertices in I.

In this case, since the mutation is independent of the choice of order of mutations, we can define the successive mutation

$$\mu_I(Q,W) := \mu_{k_1} \circ \cdots \circ \mu_{k_n}(Q,W).$$

Then our main result is the following.

Theorem 8. *Let (Q,W) be a selfinjective QP and $\Lambda := \mathcal{P}(Q,W)$. Let I be a set of vertices of Q_0 satisfying the conditions* (a1) *and* (a2)*. Then we have a K-algebra isomorphism*

$$\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)}(\mu_I(\Lambda)) \cong \mathcal{P}(\mu_I(Q,W)).$$

We will give the proof in the next section. Combining with Theorem 7, we have the following result.

Corollary 9. *Let I be a set of vertices of Q_0 satisfying $\sigma I = I$ and the conditions* (a1) *and* (a2)*. Then $\mathcal{P}(Q,W)$ and $\mathcal{P}(\mu_I(Q,W))$ are derived equivalent.*

Proof. By Theorem 7, $\mu_I(\Lambda)$ is a tilting object of $\mathcal{T}$. Then Λ and $\mathrm{End}_{\mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)}(\mu_I(\Lambda))$ are derive equivalent [20] and the result follows from Theorem 8 □

Moreover, since selfinjectivity is preserved by derived equivalence [4], we have the following result, which is given in [9, Theorem 4.2].

Corollary 10. *Let I be a set of vertices of Q_0 satisfying $\sigma I = I$ and the conditions* (a1) *and* (a2)*. Then $\mu_I(Q,W)$ is a selfinjective QP.*

We note that the Nakayama permutation of $\mu_I(Q,W)$ is again given by the same permutation [9, Proposition 4.4.(b)]. By this corollary, we can apply Corollary 9 to new QPs repeatedly and, consequently, obtain a lot of derived equivalences.

Example 11. Let (Q,W) be the QP given as follows

$$\begin{tikzcd}
 & & 1 \arrow[dl, "a_1"'] & & \\
 & 2 \arrow[dl, "a_2"'] & & 6 \arrow[ul, "a_6"'] & \\
3 \arrow[rr, "a_3"] & & 4 \arrow[rr, "a_4"] & & 5, \arrow[ul, "a_5"']
\end{tikzcd} \qquad W = a_1a_2a_3a_4a_5a_6.$$

Then (Q,W) is a selfinjective QP with a Nakayama permutation $(153)(264)$. Let $\Lambda := \mathcal{P}(Q,W)$ and $\mathcal{T} := \mathsf{K}^{\mathrm{b}}(\mathrm{proj}\Lambda)$ and take a silting object in $\mathcal{T}$

$$\mu_1(\Lambda) = \left\{ \begin{array}{ccc} \overset{-1}{\Lambda e_1} & \xrightarrow{a_1} & \overset{0}{\Lambda e_2} \\ & & \oplus \\ & & \Lambda(1-e_1). \end{array} \right.$$

Then by Theorem 8, we have an isomorphism

$$\mathrm{End}_{\mathcal{T}}(\mu_1(\Lambda)) \cong \mathcal{P}(\mu_1(Q,W)),$$

where $\mu_1(Q,W)$ is the QP given as follows

$$
\begin{tikzcd}
 & & 1 \arrow[dr, "a_6^*"] & & \\
 & 2 \arrow[ur, "a_1^*"] \arrow[dl, "a_2"'] & & 6 \arrow[ll, "{[a_6a_1]}"] & \\
3 \arrow[rr, "a_3"] & & 4 \arrow[rr, "a_4"] & & 5, \arrow[ul, "a_5"']
\end{tikzcd}
\qquad [a_6a_1]a_1^*a_6^* + [a_6a_1]a_2a_3a_4a_5.
$$

Next we consider the σ-orbit of the vertex 1 and let $I = \{1, 3, 5\}$. Then we have a tilting object

$$
\mu_I(\Lambda) = \left\{ \begin{array}{ccc} \Lambda e_1 \oplus \overset{-1}{\Lambda e_3} \oplus \Lambda e_5 & \xrightarrow{\begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_3 & 0 \\ 0 & 0 & a_5 \end{pmatrix}} & \Lambda e_2 \oplus \overset{0}{\Lambda e_4} \oplus \Lambda e_6 \\ & & \oplus \\ & & \Lambda(1 - e_I). \end{array} \right.
$$

Then we have an isomorphism

$$
\mathrm{End}_{\mathcal{T}}(\mu_I(\Lambda)) \cong \mathcal{P}(\mu_I(Q, W)),
$$

where $\mu_I(Q, W)$ is the QP given as follows

$$
\begin{tikzcd}
 & & 1 \arrow[dr, "a_6^*"] & & \\
 & 2 \arrow[ur, "a_1^*"] \arrow[dr, "{[a_2a_3]}"'] & & 6 \arrow[ll, "{[a_6a_1]}"'] \arrow[dr, "a_5^*"] & \\
3 \arrow[ur, "a_2^*"] & & 4 \arrow[ll, "a_3^*"] \arrow[ur, "{[a_4a_5]}"'] & & 5, \arrow[ll, "a_4^*"]
\end{tikzcd}
$$

$[a_6a_1]a_1^*a_6^* + [a_2a_3]a_3^*a_2^* + [a_4a_5]a_5^*a_4^* + [a_6a_1][a_2a_3][a_4a_5]$.

We note that, although $\mathcal{P}(\mu_I(Q, W))$ is selfinjective and derived equivalent to $\mathcal{P}(Q, W)$, $\mathcal{P}(\mu_1(Q, W))$ is neither selfinjective nor derived equivalent to $\mathcal{P}(Q, W)$.

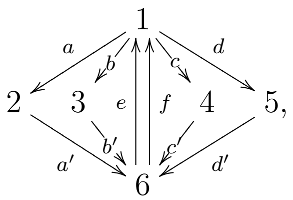
Example 12. Let (Q, W) be the QP given as follows

$$
\begin{tikzcd}
1 \arrow[r] & 2 \arrow[d] & 3 \arrow[l] \\
4 \arrow[u] \arrow[d] & 5 \arrow[l] \arrow[r] & 6 \arrow[u] \arrow[d] \\
7 \arrow[r] & 8 \arrow[u] & 9, \arrow[l]
\end{tikzcd}
$$

where the potential is the sum of each small squares. Then (Q, W) is a selfinjective QP with a Nakayama permutation $(19)(28)(37)(46)(5)$. For σ-orbits $I^1 := \{1, 9\}$ and $I^3 := \{3, 7\}$, we have selfinjective QPs $\mu_{I^1}(Q, W)$ and $\mu_{I^3} \circ \mu_{I^1}(Q, W)$ and their Jacobian algebras are derived equivalent to $\mathcal{P}(Q, W)$.

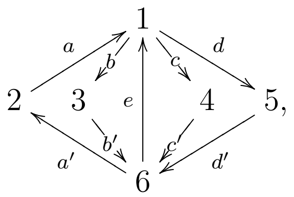
$$
\xrightarrow{\mu_{I^1}(Q,W)}
\begin{tikzcd}
1 \arrow[d] & 2 \arrow[l] \arrow[d] & 3 \arrow[l] \\
4 \arrow[ur] \arrow[d] & 5 \arrow[l] \arrow[r] & 6 \arrow[u] \arrow[dl] \\
7 \arrow[r] & 8 \arrow[u] \arrow[r] & 9, \arrow[u]
\end{tikzcd}
\qquad
\xrightarrow{\mu_{I^3}\circ\mu_{I^1}(Q,W)}
\begin{tikzcd}
1 \arrow[d] & 2 \arrow[l] \arrow[r] \arrow[d] & 3 \arrow[d] \\
4 \arrow[ur] \arrow[dr] & 5 \arrow[l] \arrow[r] & 6 \arrow[ul] \arrow[dl] \\
7 \arrow[u] & 8 \arrow[l] \arrow[u] \arrow[r] & 9. \arrow[u]
\end{tikzcd}
$$

Example 13. Let (Q, W) be the QP associated with tubular algebra of type $(2, 2, 2, 2)$



$$W = aa'e + bb'e + cc'e + aa'f + \lambda bb'f + dd'f, \lambda \in K \setminus \{0, 1\}.$$

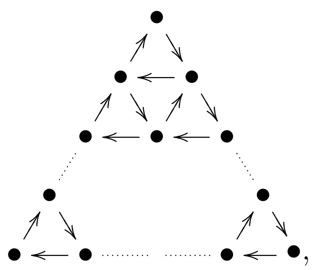
Then (Q, W) is a selfinjective QP [9] and the Nakayama permutation is the identity. Thus mutation of the QP at any vertex admits a derived equivalence in this case. For example, $\mu_2(Q, W)$ is the following QP with $\lambda' = \frac{\lambda}{\lambda - 1}$



$$bb'e + cc'e + dd'e + \lambda' bb'a'a + dd'a'a.$$

Thus $\mu_2(Q, W)$ is a selfinjective QP and $\mathcal{P}(\mu_2(Q, W))$ is derived equivalent to $\mathcal{P}(Q, W)$.

Example 14. Let (Q, W) be the QP given as follows



where the potential is the sums of each small triangles. Then (Q, W) is a selfinjective QP and one can easily get a lot of derived equivalence classes of algebras by the same procedures. See [9, Figure 4] for one of the concrete description. We refer to [12], which enables one to compute quiver mutations immediately.

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