# ON HOCHSCHILD COHOMOLOGY OF A CLASS OF WEAKLY SYMMETRIC ALGEBRAS WITH RADICAL CUBE ZERO

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Abstract. This paper is based on my talk given at the Symposium on Ring Theory and Representation Theory held at Shinsyu University, Japan, 7–9 September 2012. In this paper, we provide an explicit minimal projective bimodule resolution for some weakly symmetric algebras with radical cube zero. Then by using this resolution we compute the dimension of its Hochschild cohomology groups and determine the Hochschild cohomology ring modulo nilpotence.

## 1. Introduction

We consider the bound quiver algebra $A = k\Gamma/I$ where $\Gamma$ is the quiver with $m$ vertices and $2m$ arrows as follows:

$$\begin{tikzcd} e_0 \arrow[loop left, "a_0"] \arrow[r, shift left, "a_1"] & e_1 \arrow[l, shift left, "\overline{a}_1"] \arrow[r, shift left, "a_2"] & \cdots \arrow[l, shift left, "\overline{a}_2"] \arrow[r, shift left, "a_{m-1}"] & e_{m-1} \arrow[l, shift left, "\overline{a}_{m-1}"] \arrow[loop right, "a_m"] \end{tikzcd}$$

for an integer $m \geq 3$, and $I$ is the ideal of $k\Gamma$ generated by the following elements:

$$a_1\overline{a}_1 - a_0^2, \quad a_m^2 - \overline{a}_{m-1}a_{m-1}, \quad \overline{a}_1 a_0, \quad a_m\overline{a}_{m-1},$$
$$a_i\overline{a}_i - \overline{a}_{i-1}a_{i-1}, \quad a_j a_{j+1}, \quad \overline{a}_{l+1}\overline{a}_l,$$

for $2 \leq i \leq m-1$, $0 \leq j \leq m-1$ and $1 \leq l \leq m-2$. Then, the following elements form a $k$-basis of $A$.

$$e_i,\ a_j,\ \overline{a}_l,\ a_r\overline{a}_r,\ a_m^2$$

for $0 \leq i \leq m-1$, $0 \leq j \leq m$ and $1 \leq l, r \leq m-1$. It is known that $A$ is a Koszul weakly symmetric algebra with radical cube zero.

We denote by $A^e$ the enveloping algebra $A \otimes_k A^{op}$ of $A$, so that left $A^e$-modules correspond to $A$-bimodules. The Hochschild cohomology ring is given by $\mathrm{HH}^*(A) = \mathrm{Ext}^*_{A^e}(A, A) = \oplus_{n \geq 0} \mathrm{Ext}^n_{A^e}(A, A)$ with Yoneda product. It is well-known that $\mathrm{HH}^*(A)$ is a graded commutative ring. Let $\mathcal{N}$ denote the ideal of $\mathrm{HH}^*(A)$ which is generated by all homogeneous nilpotent elements. Then $\mathcal{N}$ is contained in every maximal ideal of $\mathrm{HH}^*(A)$, so that the maximal ideals of $\mathrm{HH}^*(A)$ are in 1-1 correspondence with those in the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$. In this paper, we describe the ring structure of $\mathrm{HH}^*(A)/\mathcal{N}$.

In [8], Snashall and Solberg defined the support varieties for finitely generated modules over a finite dimensional algebra by using the Hochschild cohomology ring modulo nilpotence. Furthermore, in [2], Erdmann, Holloway, Snashall, Solberg and Taillefer introduced some reasonable "finiteness conditions," denoted by (Fg), for any finite dimensional algebra, and they showed that if a finite dimensional algebra satisfies (Fg), then the support varieties have a lot of analogous properties of support varieties for finite group algebras.

The detailed version of this paper has been submitted for publication elsewhere.

Recently, in [3], Erdman and Solberg gave necessary and sufficient conditions for any Koszul algebra to satisfy (Fg). Consequently, they showed that $A$ satisfies (Fg). So the Hochschild cohomology ring of $A$ is finitely generated as an algebra. On the other hand, in the case where $m = 2$ and $\operatorname{char} k \neq 2$, $A$ is precisely the principal block of the tame Hecke algebra $H_q(S_5)$ for $q = -1$. In this case, a $k$-basis of the Hochschild cohomology groups of $A$ was described by Schroll and Snashall in [7]. They proved independently that $A$ satisfies (Fg), and gave some properties of the support varieties for modules over $A$.

In this paper, we provide an explicit minimal projective bimodule resolution of $A$ for $m \geq 3$, and then determine the ring structure of the Hochschild cohomology ring modulo nilpotence $\mathrm{HH}^*(A)/\mathcal{N}$.

The contents of this paper are organized as follows. In Section 2, we determine sets $\mathcal{G}^n$ ($n \geq 0$), introduced in [6], for the right $A$-module $A/\operatorname{rad} A$. Then, using $\mathcal{G}^n$, we construct a minimal projective resolution $(P_\bullet, \partial_\bullet)$ of $A$ as an $A^e$-module (Theorem 1). In Section 3, we first determine the dimension of the Hochschild cohomology groups for $m \geq 3$ (Theorem 4), and then we give an explicit $k$-basis of the Hochschild cohomology groups (Propositions 2, 3) and determine the Hochschild cohomology ring modulo nilpotence (Theorem 6).

Throughout this paper, for any arrow $a$ in $\Gamma$, we denote the origin of $a$ by $o(a)$ and the terminus by $t(a)$. We write $\otimes_k$ as $\otimes$ for simplicity,

## 2. A projective bimodule resolution

In this section, we give an explicit minimal projective bimodule resolution

$$(P_\bullet, \partial_\bullet): \quad \cdots \xrightarrow{\partial_4} P_3 \xrightarrow{\partial_3} P_2 \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} A \to 0$$

of $A = k\Gamma/I$ for $m \geq 3$ by using the argument in [5].

Let $B = kQ/I'$ with a finite quiver $Q$ and an admissible ideal $I'$ in $kQ$. In [6], Green, Solberg and Zacharia introduced the following subsets $\mathcal{G}^n$ ($n \geq 0$) of $kQ$, and used the subsets to give a minimal projective resolution of the right $B$-module $B/\operatorname{rad} B$.

Let $\mathcal{G}^0$ the set of all vertices of $Q$, $\mathcal{G}^1$ the set of all arrows of $Q$ and $\mathcal{G}^2$ a minimal set of generators of $I$. In [6], the authors proved that for each $n \geq 3$ there is a subset $\mathcal{G}^n$ of $kQ$ satisfying the following two conditions:

(a) Each of the elements $x$ of $\mathcal{G}^n$ is a uniform element satisfying

$$x = \sum_{y \in \mathcal{G}^{n-1}} y r_y = \sum_{z \in \mathcal{G}^{n-2}} z s_z \quad \text{for unique } r_y, s_z \in kQ.$$

(b) There is a minimal projective $B$-resolution of $B/\operatorname{rad} B$

$$(R_\bullet, \delta_\bullet): \quad \cdots \xrightarrow{\delta_4} R_3 \xrightarrow{\delta_3} R_2 \xrightarrow{\delta_2} R_1 \xrightarrow{\delta_1} R_0 \xrightarrow{\delta_0} B/J \to 0,$$

satisfying the following conditions:

(i) For each $j \geq 0$, $R_j = \bigoplus_{x \in \mathcal{G}^j} t(x)B$.

(ii) For each $j \geq 1$, the differential $\delta_j : R_j \to R_{j-1}$ is defined by

$$t(x)\lambda \longmapsto \sum_{y \in \mathcal{G}^{j-1}} r_y t(x)\lambda \quad \text{for } x \in \mathcal{G}^j \text{ and } \lambda \in B,$$

where $r_y$ are elements in the expression (a).

In [5], Green, Hartman, Marcos and Solberg used the subsets $\mathcal{G}^n$ $(n \geq 0)$ of $kQ$ to give a minimal projective bimodule resolution for any finite dimensional Koszul algebra. This set also appears in the papers [3], [7] and [9] in constructing minimal projective bimodule resolutions.

In order to give sets $\mathcal{G}^n$ $(n \geq 0)$ for $A = k\Gamma/I$, we first define the following quiver $\Delta$ and morphisms of quivers $\phi^i = (\phi_0^i, \phi_1^i) : \Delta \to \Gamma$ for $i = 0, 1, \ldots, m-1$.

Let $\Delta$ be the following locally finite quiver with vertices $(x, y)$ and arrows $b^{(x,y)}$ : $(x, y) \to (x+1, y)$ and $c^{(x,y)} : (x, y) \to (x, y+1)$ for integers $x, y \geq 0$ as follows:

$$
\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
{\scriptstyle c^{(0,2)}}\Big\uparrow & & {\scriptstyle c^{(1,2)}}\Big\uparrow & & {\scriptstyle c^{(2,2)}}\Big\uparrow & & \\
(0,2) & \xrightarrow{b^{(0,2)}} & (1,2) & \xrightarrow{b^{(1,2)}} & (2,2) & \xrightarrow{b^{(2,2)}} & \cdots \\
{\scriptstyle c^{(0,1)}}\Big\uparrow & & {\scriptstyle c^{(1,1)}}\Big\uparrow & & {\scriptstyle c^{(2,1)}}\Big\uparrow & & \\
(0,1) & \xrightarrow{b^{(0,1)}} & (1,1) & \xrightarrow{b^{(1,1)}} & (2,1) & \xrightarrow{b^{(2,1)}} & \cdots \\
{\scriptstyle c^{(0,0)}}\Big\uparrow & & {\scriptstyle c^{(1,0)}}\Big\uparrow & & {\scriptstyle c^{(2,0)}}\Big\uparrow & & \\
(0,0) & \xrightarrow{b^{(0,0)}} & (1,0) & \xrightarrow{b^{(1,0)}} & (2,0) & \xrightarrow{b^{(2,0)}} & \cdots
\end{array}
$$

For any integer $z$, let $Q(z)$ be the quotient and $\overline{z}$ the remainder when we divide $z$ by $m$. Then we have $0 \leq \overline{z} \leq m-1$. We denote the sets of vertices of $\Delta$ and $\Gamma$ by $\Delta_0$ and $\Gamma_0$, respectively. Also, we denote the sets of arrows of $\Delta$ and $\Gamma$ by $\Delta_1$ and $\Gamma_1$, respectively. For each $i = 0, 1, \ldots, m-1$, we define the maps $\phi_0^i : \Delta_0 \to \Gamma_0$ and $\phi_1^i : \Delta_1 \to \Gamma_1$ by

(1) For $(x, y) \in \Delta_0$

$$\phi_0^i(x, y) := \begin{cases} e_{\overline{x-y+i}} & \text{if } Q(x-y+i) \in 2\mathbb{Z}, \\ e_{m-1-\overline{x-y+i}} & \text{if } Q(x-y+i) \notin 2\mathbb{Z}. \end{cases}$$

(2) For $b^{(x,y)}, c^{(x,y)} \in \Delta_1$

$$\phi_1^i(b^{(x,y)}) := \begin{cases} a_{\overline{x-y+i}+1} & \text{if } Q(x-y+i) \in 2\mathbb{Z}, \\ \overline{a}_{m-1-\overline{x-y+i}} & \text{if } Q(x-y+i) \notin 2\mathbb{Z}, \end{cases}$$

$$\phi_1^i(c^{(x,y)}) := \begin{cases} \overline{a}_{\overline{x-y+i}} & \text{if } Q(x-y+i) \in 2\mathbb{Z}, \\ a_{m-\overline{x-y+i}} & \text{if } Q(x-y+i) \notin 2\mathbb{Z}. \end{cases}$$

where we put $\overline{a}_0 := a_0$ for our convenience.

Then, for all $i = 0, 1, \ldots, m-1$ and arrows $b^{(x,y)}$ and $c^{(x,y)}$ in $\Delta$, we have

$$\begin{aligned} o(\phi_1^i(b^{(x,y)})) &= o(\phi_1^i(c^{(x,y)})) = \phi_0^i(x, y), \\ t(\phi_1^i(b^{(x,y)})) &= \phi_0^i(x+1, y), \\ t(\phi_1^i(c^{(x,y)})) &= \phi_0^i(x, y+1). \end{aligned}$$

Thus $\phi_1^i$ is a morphism of quivers. Note that $\phi_1^i$ naturally induces the map between the set of paths of $\Delta$ and that of $\Gamma$ as follows:

$$\phi_1^i(p_1 \cdots p_r) = \phi_1^i(p_1) \cdots \phi_1^i(p_r),$$

for a path $p_1 \cdots p_r (r \geq 1)$ of $\Delta$ where $p_j$ is an arrow for $1 \leq j \leq r$.

Now, we can define the sets $\mathcal{G}^n$ $(n \geq 0)$ for $A$ in the similar way in [9]. For integers $n \geq 0$, $x, y \geq 0$ with $x + y = n$ and $i = 0, 1, \ldots, m-1$, we define the element $g^n_{x,y,i}$ in $k\Gamma$ by

$$g^n_{x,y,i} := \sum_p (-1)^{s_p} \phi_1^i(p),$$

where

- $p$ ranges over all paths in $\Delta$ starting at $(0,0)$ and ending with $(x,y)$; and
- $s_p$ is an integer determined as follows: If we write $p = p_1 p_2 \ldots p_n$ with $p_j$ arrows in $\Delta$ for $1 \leq j \leq n$, then $s_p = \sum_{p_j = c^{(x',y')}} j$ where $x'$ and $y'$ are positive integers with $x' + y' = j - 1$.

For each $n \geq 0$, we put

$$\mathcal{G}^n := \{g^n_{x,n-x,i} | \, 0 \leq x \leq n \text{ and } 0 \leq i \leq m-1\}.$$

Then, for $n = 0, 1, 2$, $\mathcal{G}^n$ can be described as follows:

$$\begin{aligned}
&\mathcal{G}^0 = \{e_0, e_1, \ldots, e_{m-1}\},\\
&\mathcal{G}^1 = \{a_1, \ldots, , a_m, -a_0 - \overline{a}_1, -\overline{a}_2, \ldots, -\overline{a}_{m-1}\},\\
&\mathcal{G}^2 =\\
&\{-\phi_1^i(c^{(0,0)}c^{(0,1)}), \phi_1^i(b^{(0,0)}c^{(1,0)}) - \phi_1^i(c^{(0,0)}b^{(0,1)}), \phi_1^i(b^{(0,0)}b^{(1,0)}) \mid 0 \leq i \leq m-1\}\\
&\ = \{-a_0 a_1, -\overline{a}_1 a_0, -\overline{a}_i \overline{a}_{i-1}, a_1 \overline{a}_1 - a_0^2, a_{j+1}\overline{a}_{j+1} - \overline{a}_j a_j, a_m^2 - \overline{a}_{m-1} a_{m-1},\\
&\quad a_{l+1} a_{l+2}, a_m \overline{a}_{m-1} \mid 2 \leq i \leq m-1, 1 \leq j \leq m-2 \text{ and } 0 \leq l \leq m-2\}.
\end{aligned}$$

And it is easily seen that $\mathcal{G}^n$ satisfies the conditions (a) and (b) for $m \geq 3$ in the beginning of this section.

Now, for any integer $n \geq 0$, we define a left $A^e$-module

$$P_n := \coprod_{g \in \mathcal{G}^n} Ao(g) \otimes t(g)A.$$

Using the argument of [5], we have the following minimal projective resolution of $A$.

**Theorem 1.** [4, Theorem 2.3] *The following sequence is a minimal projective resolution of the left $A^e$-module $A$.*

$$(P_\bullet, \partial_\bullet) : \cdots \to P_n \xrightarrow{\partial_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\pi} A \to 0,$$

*where $\pi$ is the multiplication map and left $A^e$-homomorphisms $\partial_n$ are defined by*

(1) *In the case where* $i = 0$,

$$
\partial_n(o(g^n_{x,n-x,0}) \otimes t(g^n_{x,n-x,0})) =
\begin{cases}
(-1)^n o(g^{n-1}_{0,n-1,0}) \otimes \phi^0_1(c^{(0,n-1)}) \\
\quad + \begin{cases} \phi^0_1(c^{(0,0)}) \otimes t(g^{n-1}_{n-1,0,0}) & \text{if } n \equiv 0,1 (\mathrm{mod}\, 4), \\ -\phi^0_1(c^{(0,0)}) \otimes t(g^{n-1}_{n-1,0,0}) & \text{if } n \equiv 2,3 (\mathrm{mod}\, 4), \end{cases} \quad \text{if } x = 0, \\
o(g^{n-1}_{x-1,n-x,0}) \otimes \phi^0_1(b^{(x-1,n-x)}) + (-1)^n o(g^{n-1}_{x,n-1-x,0}) \otimes \phi^0_1(c^{(x,n-1-x)}) \\
\quad +(-1)^x \phi^0_1(b^{(0,0)}) \otimes t(g^{n-1}_{x-1,n-x,1}) \\
\qquad + \begin{cases} (-1)^x \phi^0_1(c^{(0,0)}) \otimes t(g^{n-1}_{n-1-x,x,0}) & \text{if } n \equiv 0,1 (\mathrm{mod}\, 4), \\ -(-1)^x \phi^0_1(c^{(0,0)}) \otimes t(g^{n-1}_{n-1-x,x,0}) & \text{if } n \equiv 2,3 (\mathrm{mod}\, 4), \end{cases} \\
\qquad \text{if } 1 \le x \le n-1, \\
o(g^{n-1}_{n-1,0,0}) \otimes \phi^0_1(b^{(n-1,0)}) + (-1)^n \phi^0_1(b^{(0,0)}) \otimes t(g^{n-1}_{n-1,0,1}) \quad \text{if } x = n.
\end{cases}
$$

(2) *In the case where* $1 \le i \le m-2$,

$$
\partial_n(o(g^n_{x,n-x,i}) \otimes t(g^n_{x,n-x,i})) =
\begin{cases}
(-1)^n o(g^{n-1}_{0,n-1,i}) \otimes \phi^i_1(c^{(0,n-1)}) + \phi^i_1(c^{(0,0)}) \otimes t(g^{n-1}_{0,n-1,i-1}) \quad \text{if } x = 0, \\
o(g^{n-1}_{x-1,n-x,i}) \otimes \phi^i_1(b^{(x-1,n-x)}) + (-1)^n o(g^{n-1}_{x,n-1-x,i}) \otimes \phi^i_1(c^{(x,n-1-x)}) \\
\quad +(-1)^x \phi^i_1(b^{(0,0)}) \otimes t(g^{n-1}_{x-1,n-x,i+1}) \\
\qquad +(-1)^x \phi^i_1(c^{(0,0)}) \otimes t(g^{n-1}_{x,n-1-x,i-1}) \quad \text{if } 1 \le x \le n-1, \\
o(g^{n-1}_{n-1,0,i}) \otimes \phi^i_1(b^{(n-1,0)}) + (-1)^n \phi^i_1(b^{(0,0)}) \otimes t(g^{n-1}_{n-1,0,i+1}) \quad \text{if } x = n.
\end{cases}
$$

(3) *In the case where* $i = m-1$,

$$
\partial_n(o(g^n_{x,n-x,m-1}) \otimes t(g^n_{x,n-x,m-1})) =
\begin{cases}
(-1)^n o(g^{n-1}_{0,n-1,m-1}) \otimes \phi^{m-1}_1(c^{(0,n-1)}) + \phi^{m-1}_1(c^{(0,0)}) \otimes t(g^{n-1}_{0,n-1,m-2}) \\
\quad \text{if } x = 0, \\
o(g^{n-1}_{x-1,n-x,m-1}) \otimes \phi^{m-1}_1(b^{(x-1,n-x)}) \\
\quad +(-1)^n o(g^{n-1}_{x,n-1-x,m-1}) \otimes \phi^{m-1}_1(c^{(x,n-1-x)}) \\
\qquad + \begin{cases} (-1)^x \phi^{m-1}_1(b^{(0,0)}) \otimes t(g^{n-1}_{n-x,x-1,m-1}) & \text{if } n \equiv 0,1 (\mathrm{mod}\, 4), \\ -(-1)^x \phi^{m-1}_1(b^{(0,0)}) \otimes t(g^{n-1}_{n-x,x-1,m-1}) & \text{if } n \equiv 2,3 (\mathrm{mod}\, 4), \end{cases} \\
\qquad +(-1)^x \phi^{m-1}_1(c^{(0,0)}) \otimes t(g^{n-1}_{x,n-1-x,m-2}), \\
\qquad\quad \text{if } 1 \le x \le n-1, \\
o(g^{n-1}_{n-1,0,m-1}) \otimes \phi^{m-1}_1(b^{(n-1,0)}) \\
\quad + \begin{cases} (-1)^n \phi^{m-1}_1(b^{(0,0)}) \otimes t(g^{n-1}_{0,n-1,m-1}) & \text{if } n \equiv 0,1 (\mathrm{mod}\, 4), \\ -(-1)^n \phi^{m-1}_1(b^{(0,0)}) \otimes t(g^{n-1}_{0,n-1,m-1}) & \text{if } n \equiv 2,3 (\mathrm{mod}\, 4), \end{cases} \\
\qquad \text{if } x = n.
\end{cases}
$$

## 3. Hochschild cohomology of $A$

In this section, we give a $k$-basis of the Hochschild cohomology groups of $A$ and determine the ring structure of the Hochschild cohomology ring modulo nilpotence by using the minimal projective $A^e$-resolution given in Theorem 1.

By setting $P_n^* := \mathrm{Hom}_{A^e}(P_n, A)$ and $\partial_n^* = \mathrm{Hom}_{A^e}(\partial_n, A)$ for $n \geq 0$, we get the following complex.

$$(P_\bullet^*, \partial_\bullet^*): \quad 0 \to P_0^* \xrightarrow{\partial_1^*} P_1^* \xrightarrow{\partial_2^*} \cdots \xrightarrow{\partial_{n-1}^*} P_{n-1}^* \xrightarrow{\partial_n^*} P_n^* \xrightarrow{\partial_{n+1}^*} \cdots .$$

Then, for $n \geq 0$, the $n$-th Hochschild cohomology group $\mathrm{HH}^n(A)$ of $A$ is given by $\mathrm{HH}^n(A) := \mathrm{Ext}^n_{A^e}(A, A) = \mathrm{Ker}\,\partial_{n+1}^*/\mathrm{Im}\,\partial_n^*$.

In the rest of the paper, for an integer $n \geq 0$, we set $p := Q(n)$ and $t := \overline{n}$, that is, $p$ and $t$ are unique integers such that $n = pm + t$ with $p \geq 0$ and $0 \leq t \leq m-1$.

Using the complex $(P_\bullet^*, \partial_\bullet^*)$, we compute a $k$-basis of $\mathrm{HH}^n(A)$ for $n \geq 0$. Now we consider the case where $m$ is even. In the case where $m$ is odd, we have the similar results.

**Proposition 2.** [4, Proposition 3.7] *Suppose that $m \geq 3$. Then the following elements form a $k$-basis of the center $Z(A) = \mathrm{HH}^0(A) = \mathrm{Ker}\,\partial_1^*$ of $A$.*

$$\sum_{i=0}^{m-1} e_i,\ a_0,\ a_m,\ a_j\overline{a}_j \quad \textit{for } 1 \leq j \leq m.$$

**Proposition 3.** [4, Proposition 3.8] *Suppose $m \geq 3$ and $m$ is even. For each $n = pm+t \geq 1$, the following elements form a $k$-basis of $\mathrm{HH}^{pm+t}(A)$.*

(1) *In the case where $p$ and $t$ are even, we have a $k$-basis of $\mathrm{HH}^{pm+t}(A)$ as follows:*
  (a) *If $x_1 = (p-\alpha)m + t/2$, $x_2 = \alpha m + t/2$,*

$$\chi_{n,\alpha}: \begin{cases} e_i \otimes \phi_0^i(x_1, n-x_1) \mapsto \begin{cases} e_i & \text{if } i \text{ is even,} \\ (-1)^{t/2} e_i & \text{if } i \text{ is odd,} \end{cases} \\ e_i \otimes \phi_0^i(x_2, n-x_2) \mapsto \begin{cases} e_i & \text{if } i \text{ is even,} \\ (-1)^{t/2} e_i & \text{if } i \text{ is odd,} \end{cases} \end{cases}$$

  *for $0 \leq i \leq m-1$, $0 \leq \alpha \leq p/2$.*
  (b) *If $x = pm/2 + t/2$, $\pi_{n,1} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_0$.*
  (c) *If $x = pm/2 + t/2$, $\pi_{n,2} : e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m$.*
  (d) *If $x = (p-\alpha)m + t/2$,*
  $F_{n,\alpha} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$ *for $0 \leq \alpha \leq p/2 - 1$.*
  (e) *If $x = pm/2 + t/2$, $\mathrm{char}\,k = 2$, $F_{n,p/2} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$.*

(2) *In the case where $p$ is even and $t$ is odd, we have a $k$-basis of $\mathrm{HH}^{pm+t}(A)$ as follows:*

(a) *If* $x_1=(p-\alpha)m+(t-1)/2$ *and* $x_2=\alpha m+(t-1)/2$,

$$\mu_{n,\alpha}:\begin{cases} e_i\otimes\phi_0^i(x_1,n-x_1)\mapsto\begin{cases}\overline{a}_i & \text{if } i \text{ is even},\\ (-1)^{(t-1)/2}\overline{a}_i & \text{if } i \text{ is odd},\end{cases}\\ e_i\otimes\phi_0^i(x_2,n-x_2)\mapsto\begin{cases}\overline{a}_i & \text{if } i \text{ is even}(\neq 0),\\ (-1)^{(t-1)/2}\overline{a}_i & \text{if } i \text{ is odd},\end{cases}\\ e_{m-1}\otimes\phi_0^{m-1}(x_2+1,n-x_2-1)\mapsto(-1)^{(t+1)/2}a_m,\end{cases}$$

*for* $0\le i\le m-1$, $0\le\alpha\le p/2-1$.

(b) *If* $x=pm/2+(t-1)/2$ *and* $\operatorname{char}k\neq 2$,

$$\mu_{n,p/2}:\begin{cases} e_i\otimes\phi_0^i(x,n-x)\mapsto\begin{cases}\overline{a}_i & \text{if } i \text{ is even},\\ (-1)^{(t-1)/2}\overline{a}_i & \text{if } i \text{ is odd},\end{cases}\\ e_i\otimes\phi_0^i(x+1,n-1-x)\mapsto\begin{cases}-a_{i+1} & \text{if } i \text{ is even},\\ (-1)^{(t+1)/2}a_{i+1} & \text{if } i \text{ is odd},\end{cases}\end{cases}$$

*for* $0\le i\le m-1$.

(c) *If* $x=pm/2+(t-1)/2$ *and* $\operatorname{char}k=2$, $\mu_{n,p/2}:e_0\otimes\phi_0^0(x,n-x)\mapsto a_0$.

(d) *If* $x=pm/2+(t+1)/2$ *and* $\operatorname{char}k=2$,

$\mu'_{n,p/2}:e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m$.

(e) *If* $x=(p-\alpha)m+(t-1)/2$,

$$\nu_{n,\alpha}:\begin{cases} e_0\otimes\phi_0^0(x+1,n-1-x)\mapsto a_1,\\ e_1\otimes\phi_0^1(x,n-x)\mapsto(-1)^{(t-1)/2}\overline{a}_1,\end{cases}$$

*for* $0\le\alpha\le p/2-1$.

(f) *If* $x=pm/2+(t-1)/2$, $E_{n,1}:e_0\otimes\phi_0^0(x,n-x)\mapsto a_1\overline{a}_1$.

(g) *If* $x=pm/2+(t+1)/2$, $E_{n,2}:e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m^2$.

(3) *In the case where* $p$ *is odd and* $t$ *is even, we have a* $k$*-basis of* $\mathrm{HH}^{pm+t}(A)$ *as follows*:

(a) *If* $x_1=(p-\alpha-1)m+(m+t)/2$ *and* $x_2=\alpha m+(m+t)/2$,

$$\chi_{n,\alpha}:\begin{cases} e_i\otimes\phi_0^i(x_1,n-x_1)\mapsto\begin{cases}e_i & \text{if } i \text{ is even},\\ (-1)^{(m+t)/2}e_i & \text{if } i \text{ is odd},\end{cases}\\ e_i\otimes\phi_0^i(x_2,n-x_2)\mapsto\begin{cases}e_i & \text{if } i \text{ is even},\\ (-1)^{(m+t)/2}e_i & \text{if } i \text{ is odd},\end{cases}\end{cases}$$

*for* $0\le i\le m-1$, $0\le\alpha\le(p-1)/2$.

(b) *If* $x=(p-1)m/2+(m+t)/2$, $\pi_{n,1}:e_0\otimes\phi_0^0(x,n-x)\mapsto a_0$.

(c) *If* $x=(p-1)m/2+(m+t)/2$, $\pi_{n,2}:e_{m-1}\otimes\phi_0^{m-1}(x,n-x)\mapsto a_m$.

(d) *If* $x=(p-\alpha-1)m+(m+t)/2$,

$F_{n,\alpha}:e_0\otimes\phi_0^0(x,n-x)\mapsto a_1\overline{a}_1$ *for* $0\le\alpha\le(p-1)/2-1$.

(e) *If* $x=(p-1)m/2+(m+t)/2$ *and* $\operatorname{char}k=2$,

$F_{n,(p-1)/2}:e_0\otimes\phi_0^0(x,n-x)\mapsto a_1\overline{a}_1$.

(4) *In the case where* $p$ *and* $t$ *are odd, we have a* $k$*-basis of* $\mathrm{HH}^{pm+t}(A)$ *as follows*:

(a) *If* $x_1 = (p-\alpha-1)m + (m+t-1)/2$ *and* $x_2 = \alpha m + (m+t-1)/2$,

$$\mu_{n,\alpha} : \begin{cases} e_i \otimes \phi_0^i(x_1, n-x_1) \mapsto \begin{cases} \overline{a}_i & \text{if } i \text{ is even,} \\ (-1)^{(m+t-1)/2}\overline{a}_i & \text{if } i \text{ is odd,} \end{cases} \\ e_i \otimes \phi_0^i(x_2, n-x_2) \mapsto \begin{cases} \overline{a}_i & \text{if } i \text{ is even}(\neq 0), \\ (-1)^{(m+t-1)/2}\overline{a}_i & \text{if } i \text{ is odd,} \end{cases} \\ e_{m-1} \otimes \phi_0^{m-1}(x_2+1, n-x_2-1) \mapsto (-1)^{(m+t+1)/2} a_m, \end{cases}$$

*for* $0 \le i \le m-1$, $0 \le \alpha \le (p-1)/2 - 1$.

(b) *If* $x = (p-1)m/2 + (m+t-1)/2$ *and* $\operatorname{char} k \neq 2$,

$$\mu_{n,(p-1)/2} : \begin{cases} e_i \otimes \phi_0^i(x, n-x) \mapsto \begin{cases} \overline{a}_i & \text{if } i \text{ is even,} \\ (-1)^{(m+t-1)/2}\overline{a}_i & \text{if } i \text{ is odd,} \end{cases} \\ e_i \otimes \phi_0^i(x+1, n-1-x) \mapsto \\ \quad \begin{cases} -a_{i+1} & \text{if } i \text{ is even,} \\ (-1)^{(m+t+1)/2} a_{i+1} & \text{if } i \text{ is odd,} \end{cases} \end{cases}$$

*for* $0 \le i \le m-1$.

(c) *If* $x = (p-1)m/2 + (m+t-1)/2$ *and* $\operatorname{char} k = 2$,

$\mu_{n,(p-1)/2} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_0$.

(d) *If* $x = (p-1)m/2 + (m+t+1)/2$ *and* $\operatorname{char} k = 2$,

$\mu'_{n,(p-1)/2} : e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m$.

(e) *If* $x_1 = pm + m - 1$, $x_2 = 0$, $t = m-1$,

$$\psi_n : \begin{cases} e_i \otimes \phi_0^i(pm+m-1, 0) \mapsto (-1)^i \overline{a}_i, \\ e_i \otimes \phi_0^i(0, pm+m-1) \mapsto (-1)^{i+1} a_{i+1}, \end{cases}$$

*for* $0 \le i \le m-1$.

(f) *If* $x = (p-\alpha-1)m + (m+t-1)/2$,

$$\nu_{n,\alpha} : \begin{cases} e_0 \otimes \phi_0^0(x+1, n-1-x) \mapsto a_1, \\ e_1 \otimes \phi_0^1(x, n-x) \mapsto (-1)^{(m+t-1)/2}\overline{a}_1, \end{cases}$$

*for* $0 \le \alpha \le (p-1)/2 - 1$.

(g) *If* $x = (p-1)m/2 + (m+t-1)/2$, $E_{n,1} : e_0 \otimes \phi_0^0(x, n-x) \mapsto a_1\overline{a}_1$.

(h) *If* $x = (p-1)m/2 + (m+t+1)/2$, $E_{n,2} : e_{m-1} \otimes \phi_0^{m-1}(x, n-x) \mapsto a_m^2$.

By Propositions 2 and 3, we have the dimension of $\mathrm{HH}^n(A)$.

**Theorem 4.** [4, Theorem 3.5] *In the case* $m \ge 3$, *we have* $\dim_k \mathrm{HH}^0(A) = m + 3$ *and, for* $pm + t \ge 1$,

$$\dim_k \mathrm{HH}^{pm+t}(A) = p + \begin{cases} 3 & \text{if } p \text{ is even and } \operatorname{char} k \neq 2, \\ 2 & \text{if } p \text{ is odd, } t \neq m-1 \text{ and } \operatorname{char} k \neq 2, \\ 3 & \text{if } p \text{ is odd, } t = m-1 \text{ and } \operatorname{char} k \neq 2, \\ 4 & \text{if } p \text{ is even and } \operatorname{char} k = 2, \\ 3 & \text{if } p \text{ is odd, } t \neq m-1 \text{ and } \operatorname{char} k = 2, \\ 4 & \text{if } p \text{ is odd, } t = m-1 \text{ and } \operatorname{char} k = 2. \end{cases}$$

*Remark* 5. In the case $m = 2$, by Theorem 4, we have the dimension of the Hochschild cohomology groups of $A$ given in [7].

By corresponding Yoneda product of the basis elements of $\mathrm{HH}^*(A)$ given in Propositions 2 and 3, we have the generators of $\mathrm{HH}^*(A)$ and the following results.

**Theorem 6.** *In the case where $m$ is even with $m \geq 3$, and* $\operatorname{char} k \neq 2$*, The Hochschild cohomology ring modulo nilpotence* $\mathrm{HH}^*(A)/\mathcal{N}$ *of $A$ is isomorphic to the polynomial ring of two variables* $k[\chi_{2,0}, \chi_{2m,0}]$.

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