A GENERALIZATION OF GOLDIE TORSION THEORY

YASUHIKO TAKEHANA

Abstract. Throughout this paper R is a ring with a unit element, every right R-module is unital and Mod-R is the category of right R-modules. Let $\mathcal{C}$ be a subclass of Mod-R. A torsion theory for $\mathcal{C}$ is a pair of $(\mathcal{T},\mathcal{F})$ of classes of objects of $\mathcal{C}$ such that (i) $\mathrm{Hom}_R(T,F)=0$ for all $T\in\mathcal{T}$, $F\in\mathcal{F}$. (ii) If $\mathrm{Hom}_R(M,F)=0$ for all $F\in\mathcal{F}$, then $M\in\mathcal{T}$. (iii) If $\mathrm{Hom}_R(T,N)=0$ for all $T\in\mathcal{T}$, then $N\in\mathcal{F}$. Let $\mathcal{B}$ be a subclass of Mod-R, $\mathcal{F}=\{M\in \text{Mod-}R|\mathrm{Hom}_R(B,M)=0$ for any $B\in\mathcal{B}\}$ and $\mathcal{T}=\{M\in\text{Mod-}R|\mathrm{Hom}_R(M,P)=0$ for any $P\in\mathcal{T}\}$. Then $(\mathcal{T},\mathcal{F})$ is called to be a torsion theory generated by $\mathcal{B}$. If $\mathcal{B}$ is the class of all modules M/N such that N is essential in M, a torsion theory generated by $\mathcal{B}$ is called the Goldie torsion theory. In this paper we generalize Goldie torsion theory by using left exact radical σ and study the dualization of this.

1. Introduction

For a subclass $\mathcal{E}$ of Mod-R and for a short exact sequence $0\to A\to B\to C\to 0$, it is said that $\mathcal{E}$ is closed under taking extensions if $A,C\in\mathcal{E}$ then $B\in\mathcal{E}$. It is well known that if $\mathcal{B}$ is closed under taking factor modules, direct sums and extensions then $(\mathcal{B},\mathcal{F})$ is a torsion theory. A torsion theory cogenerated by a subclass of Mod-R is defined dually, as follows. Let $\mathcal{B}$ be a subclass of Mod-R, $\mathcal{T}=\{M\in\text{Mod-}R|\mathrm{Hom}_R(M,B)=0$ for any $B\in\mathcal{B}\}$ and $\mathcal{F}=\{M\in\text{Mod-}R|\mathrm{Hom}_R(P,M)=0$ for any $P\in\mathcal{T}\}$. Then $(\mathcal{T},\mathcal{F})$ is called to be a torsion theory cogenerated by $\mathcal{B}$. It is well known that $\mathcal{B}$ is closed under taking submodules, direct products and extensions then $(\mathcal{T},\mathcal{B})$ is a torsion theory. A torsion theory $(\mathcal{T},\mathcal{F})$ is called to be hereditary if $\mathcal{T}$ is closed under taking submodules. It is well known that $(\mathcal{T},\mathcal{F})$ is hereditary if and only if $\mathcal{F}$ is closed under taking injective hulls.

A subfunctor of the identity functor of Mod-R is called a preradical. For preradical σ, $\mathcal{T}_\sigma:=\{M\in\text{Mod-}R|\sigma(M)=M\}$ is the class of σ-torsion right R-modules, and $\mathcal{F}_\sigma:=\{M\in\text{Mod-}R|\sigma(M)=0\}$ is the class of σ-torsion free right R-modules. A preradical t is called to be idempotent(a radical) if $t(t(M))=t(M)(t(M/t(M))=0)$. It is well known that $(\mathcal{T}_\sigma,\mathcal{F}_\sigma)$ is a torsion theory for an idempotent radical σ. For a torsion theory $(\mathcal{T},\mathcal{F})$ and a module M we put $t(M)=\sum N(N\in T)$(or equivalently $t(M)=\cap N(M/N\in\mathcal{F})$), then $\mathcal{T}=\mathcal{T}_t$, and $\mathcal{F}=\mathcal{F}_t$, and t is called an associated idempotent radical for $(\mathcal{T},\mathcal{F})$.

A preradical t is called to be left exact if $t(N)=N\cap t(M)$ holds for any module M and its submodule N. For a preradical σ and a module M and its submodule N, N is called to be σ-dense submodule of M if $M/N\in\mathcal{T}_\sigma$. For a preradical σ, t is called σ-left exact preradical if $t(N)=N\cap t(M)$ holds for any module M and its σ-dense submodule N. If N is an essential and σ-dense submodule of M, then N is called to be a σ-essential submodule of $M(M$ is a σ-essential extension of $N)$. For an idempotent radical σ a

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module M is called to be σ-injective if the functor $\mathrm{Hom}_R(\_, M)$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $C \in \mathcal{T}_\sigma$.

We denote $E(M)$ the injective hull of a module M. For an idempotent radical σ, $E_\sigma(M)$ is called the σ-injective hull of a module M, where $E_\sigma(M)$ is defined by $E_\sigma(M)/M := \sigma(E(M)/M)$. Then even if σ is not left exact, $E_\sigma(M)$ is σ-injective and a σ-essential extension of M, is a maximal σ-essential extension of M and is a minimal σ-injective extension of M.

2. σ-HEREDITARY TORSION THEORIES AND σ-STABLE TORSION THEORIES

Let σ be an idempotent radical. We call a torsion theory $(\mathcal{T}_t, \mathcal{F}_t)$ σ-hereditary if $\mathcal{T}_t$ is closed under taking σ-dense submodules. A σ-hereditary torsion theory is characterized in [1], as follows. A torsion theory $(\mathcal{T}_t, \mathcal{F}_t)$ is σ-hereditary if and only if t is σ-left exact, moreover if σ is left exact then $(\mathcal{T}_t, \mathcal{F}_t)$ is σ-hereditary if and only if $\mathcal{F}$ is closed under taking σ-injective hulls. A torsion theory $(\mathcal{T}, \mathcal{F})$ is called to be stable if $\mathcal{T}$ is closed under taking injective hulls. For a preradical σ, we call a torsion theory $(\mathcal{T}, \mathcal{F})$ σ-stable if $\mathcal{T}$ is closed under taking σ-injective hulls. σ-stable torsion theory is characterized in [3] as follows. A torsion theory $(\mathcal{T}_t, \mathcal{F}_t)$ is σ-stable if and only if for any σ-injective module E, $t(E)$ is also σ-injective(if and only if, for any module M it holds that $E_\sigma(t(M)) \subseteq t(E_\sigma(M))$).

It is well known that a torsion theory generated by a class of Mod-R closed under taking submodules and quotient modules is hereditary. We generalize this as follows.

Proposition 1. *Let σ be a left exact radical and $\mathcal{B}$ a class of modules closed under taking σ-dense submodules and quotient modules. Then a torsion theory generated by $\mathcal{B}$ is σ-hereditary.*

Proof. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory generated by $\mathcal{B}$. We show that $\mathcal{F}$ is closed under taking σ-injective hulls. Let $M \in \mathcal{F}$. We will show that $E_\sigma(M) \in \mathcal{F}$. Suppose that $E_\sigma(M) \notin \mathcal{F}$, then there exists some $B \in \mathcal{B}$ such that $\mathrm{Hom}_R(B, E_\sigma(M)) \neq 0$, and so there exists $0 \neq f : B \to E_\sigma(M)$, and so $f(B) \neq 0$. As M is essential in $E_\sigma(M)$, it follows that $M \cap f(B) \neq 0$. Since $\mathcal{B}$ is closed under taking factor modules, $f(B) \in \mathcal{B}$. Since $f(B)/(M \cap f(B)) \cong (M + f(B))/M \subseteq E_\sigma(M)/M \in \mathcal{T}_\sigma$, $M \cap f(B)$ is a σ-dense in $f(B)$. Thus by the assumption of $\mathcal{B}$ it follows that $M \cap f(B) \in \mathcal{B}$. Since $M \in \mathcal{F}$, $M \cap f(B) \in \mathcal{F}$. Thus $M \cap f(B) \in \mathcal{B} \cap \mathcal{F} \subseteq \mathcal{T} \cap \mathcal{F} = \{0\}$. This is a contradiction to $M \cap f(B) \neq 0$, and so $E_\sigma(M) \in \mathcal{F}$. □

Proposition 2. *Let E be a σ-injective module, $\mathcal{B} =: \{M | Hom_R(M, E) = 0\}$ and $\mathcal{F} =: \{M \in$ Mod-$R | Hom_R(B,M) = 0$ for any $B \in \mathcal{B}\}$. Then $(\mathcal{B},\mathcal{F})$ is a σ-hereditary torsion theory.*

Proof. It is easily verified that $\mathcal{B}$ is closed under taking quotient modules, direct sums and extensions. Then $(\mathcal{B},\mathcal{F})$ is a torsion theory. We only show that $\mathcal{B}$ is closed under taking σ-dense submodules. Let $M \in \mathcal{B}$ and N be a σ-dense submodule of M. Suppose that $N \notin \mathcal{B}$, then there exists a nonzero $f \in \mathrm{Hom}_R(N, E)$. By the σ-injectivity of E, f extends $f' \in \mathrm{Hom}_R(M, E)$, but this is a contradiction to the fact that $M \in \mathcal{B}$. □

Proposition 3. *Let σ be a preradical and E be a σ-torsionfree module. We put $\mathcal{T} = \{M \in$ Mod-$R | Hom_R(M, E) = 0\}$ and $\mathcal{F} = \{M \in$ Mod-$R | Hom_R(T, M) = 0$ for any $T \in \mathcal{T}\}$. Then $(\mathcal{T},\mathcal{F})$ is σ-stable torsion theory.*

Proof. Since σ is a preradical, $\mathcal{T}_\sigma$ is closed under taking factor modules and $\mathcal{F}_\sigma$ is closed under taking submodules. Since it is easily verified that $\mathcal{T}$ is closed under taking factor modules, direct sums and extensions, it holds that ($\mathcal{T}$,$\mathcal{F}$) is a torsion theory. We only show that $\mathcal{T}$ is closed under taking σ-injective hulls. Suppose that $M \in \mathcal{T}$ and $E_\sigma(M) \notin \mathcal{T}$. Then it holds that $\mathrm{Hom}_R(M, E) = 0$ and $\mathrm{Hom}_R(E_\sigma(M), E) \neq 0$. Thus there exists $f \in \mathrm{Hom}_R(E_\sigma(M), E)$ such that $\mathrm{Im}\, f \neq 0$. Since $f|_M \in \mathrm{Hom}_R(M, E) = 0$, it folows that $\ker f \supseteq M$. Since $\mathcal{T}_\sigma \ni E_\sigma(M)/M \twoheadrightarrow E_\sigma(M)/\ker f \cong \mathrm{Im}\, f \subseteq E \in \mathcal{F}_\sigma$. This is a contradiction to the fact that $\mathrm{Im}\, f \neq 0$. Thus it follows that $E_\sigma(M) \in \mathcal{T}$. □

3. A generalization of Goldie torsion theory

A torsion theory($\mathcal{T}, \mathcal{F}$) generated by $\{M/N | N$ is essential in $M\}$ is called to be Goldie torsion theory. Goldie torsion theory is hereditary and stable. Let $Z(M)$ denote the singular submodule of a module M. For a module M, $Z_2(M)$ is defined by $Z_2(M)/Z(M) := Z(M/Z(M))$. It is well known that $\mathcal{T} = \mathcal{T}_{Z_2}$ and $\mathcal{F} = \mathcal{F}_{Z_2}$.

For a left exact radical σ we call a torsion theory generated by $\{M/N | N$ is σ-essential in $M\}$ σ-Goldie torsion theory.

Theorem 4. *For a left exact radical σ, σ-Goldie torsion theory is hereditary and σ-stable.*

Proof. Let $\mathcal{B}$ be $\{M/N | N$ is an σ-essential submodule of a module $M\}$. It is easily verified that $\mathcal{B}$ is closed under taking submodules and factor modules. A torsion theory ($\mathcal{T}$,$\mathcal{F}$) generated by $\mathcal{B}$ is hereditary by Proposition 1($\sigma = 1$). We show that $\mathcal{T}$ is closed under taking σ-injective hulls. Let M be in $\mathcal{T}$. Suppose that $E_\sigma(M) \notin \mathcal{T}$. Then there exists a module F in $\mathcal{F}$ such that $\mathrm{Hom}_R(E_\sigma(M), F) \neq 0$. Since $E_\sigma(M)/M \in \mathcal{B}$, $\mathrm{Hom}_R(E_\sigma(M)/M, F) = 0$. Since $M \in \mathcal{T}$ and $F \in \mathcal{F}$, $\mathrm{Hom}_R(M, F) = 0$. Then there exists a short exact sequence $0 \to \mathrm{Hom}_R(E_\sigma(M)/M, F) \to \mathrm{Hom}_R(E_\sigma(M), F) \to \mathrm{Hom}_R(M, F)$. This is a contradiction, and so it follows that $E_\sigma(M) \in \mathcal{T}$. □

For a module M, we denote $Z_\sigma(M) := \{m \in M | mI = 0$ for some σ-essential right ideal I of $R\}$. If $m \in Z_\sigma(M)$, there exists some σ-essential right ideal I of R such that $mI = 0$, and then $(0 : m) \supseteq I$. Since $mR \cong R/(0 : m) \twoheadleftarrow R/I \in \mathcal{T}_\sigma \cap \mathcal{T}_Z$, $mR \in \mathcal{T}_\sigma \cap \mathcal{T}_Z$, and so $mR \subseteq \sigma(M) \cap Z(M)$. Thus $Z_\sigma(M) \subseteq \sum mR_{(m \in Z_\sigma(M))} \subseteq Z(M) \cap \sigma(M)$. $Z_\sigma(M) \subseteq Z(M) \cap \sigma(M)$. Since Z and σ are left exact, $\sigma(Z(M)) = Z(M) \cap \sigma(M) = Z(\sigma(M))$, and so $Z_\sigma(M) \subseteq Z(\sigma(M)) = \sigma(Z(M))$. Conversely if $m \in Z(\sigma(M)) = \sigma(Z(M))$, then $R/(0 : m) \cong mR \in \mathcal{T}_Z \cap \mathcal{T}_\sigma$, and so $(0 : m)$ is σ-essential in R. Thus $m \in Z_\sigma(M)$, and so $Z(\sigma(M)) = \sigma(Z(M)) \subseteq Z_\sigma(M)$. Therefore $Z_\sigma(M) = Z(\sigma(M)) = \sigma(Z(M))$.

For a preradical r and a module M, put $r_1(M) := r(M)$. If β is not a limit ordinal, $r_\beta(M)/r_{\beta-1}(M) := r(M/r_{\beta-1}(M))$. If β is a limit ordinal, $r_\beta(M) := \sum_{\beta > \alpha} r_\alpha(M)$. This gives rise to an increasing sequence of preradicals. We put $\overline{r}(M) = \sum_\beta r_\beta(M)$, then $\overline{r}$ is a smallest radical larger than r. If r is idempotent, then $\overline{r}$ is also idempotent. Thus for an idempotent preradical r, a torsion theory ($\mathcal{T}$,$\mathcal{F}$) generated by $\mathcal{T}_r$ is given that $\mathcal{T} = \mathcal{T}_{\overline{r}}$ and $\mathcal{F} = \mathcal{F}_{\overline{r}}$.

We define $(Z_\sigma)_2(M)$ by $(Z_\sigma)_2(M)/Z_\sigma(M) := Z_\sigma(M/Z_\sigma(M))$.

Lemma 5. *Let σ be a radical and $\sigma(M) \supseteq N$ for a module M and its submodule N. Then it holds that $\sigma(M/N) = \sigma(M)/N$.*

Theorem 6. *For a left exact radical σ, it holds that $\mathbf{Z}_2\sigma = (Z_\sigma)_2 = \overline{Z_\sigma}$.*

Proof. By Lemma 5, $\sigma(M/Z(\sigma(M))) = \sigma(M)/Z(\sigma(M))$.
Thus $(Z_\sigma)_2(M)/Z_\sigma(M) = Z_\sigma(M/Z_\sigma(M)) = Z\{\sigma(M/Z(\sigma(M)))\}$
$= Z(\sigma(M)/Z(\sigma(M))) = Z_2(\sigma(M))/Z(\sigma(M))$. Thus $\mathbf{Z}_2\sigma = (Z_\sigma)_2$. Since $\mathbf{Z}_2$ and σ are left exact radicals, $(Z_\sigma)_2$ is a left exact radical. Since $\overline{Z_\sigma}$ is the smallest radical containing Z_σ, $(Z_\sigma)_2 \supseteq \overline{Z_\sigma}$. By construction of $\overline{Z_\sigma}$, it holds that $(Z_\sigma)_2 \subseteq \overline{Z_\sigma}$, and so $(Z_\sigma)_2 = \overline{Z_\sigma}$, as desired. □

Let G be a Goldie torsion functor. The followings are well known. (1) $G(M) = M$ if and only if $Z(M)$ is essential in M. (2) $G(M) = 0$ if and only if $Z(M) = 0$. (3) If $Z(R) = 0$, then $G = Z$. We can generalize this as follows.

Corollary 7. *Let G_σ be a σ-Goldie torsion functor. Then the following facts hold.*

(1) *$G_\sigma(M) = M$ if and only if $\sigma(M) = M$ and $Z(M)$ is essential in M.*
(2) *$G_\sigma(M) = 0$ if and only if $Z(\sigma(M)) = 0$.*
(3) *If $Z(\sigma(R)) = 0$, then $G_\sigma = Z_\sigma$*

4. σ-costable torsion theory and σ-cohereditary torsion theory

From now on, assume that R be a right perfect ring. A right R-module M is called σ-projective if the functor $\mathrm{Hom}_R(M, \quad)$ preserves the exactness for any exact sequence $0 \to A \to B \to C \to 0$ with $A \in \mathcal{F}_\sigma$. For an idempotent radical σ, a short exact sequence $[0 \to K_\sigma(M) \to P_\sigma(M) \stackrel{\pi_M^\sigma}{\to} M \to 0]$ is called σ-projective cover of a module M when $P_\sigma(M)$ is σ-projective, $K_\sigma(M)$ is σ-torsion free and $K_\sigma(M)$ is small in $P_\sigma(M)$. Since R is a right perfect ring and σ is an idempotent radical, σ-projective cover of a module always exists. A torsion theory $(\mathcal{T},\mathcal{F})$ is called costable if $\mathcal{F}$ is closed under taking projective covers. We generalize this in [4]. We call a torsion theory $(\mathcal{T},\mathcal{F})$ σ-costable if $\mathcal{F}$ is closed under taking σ-projective covers.

Proposition 8. *Let E be a module such that for any module X and any $f \in Hom_R(E, X)$, $f(E)$ is σ-torsionfree and not small in X. We denote $\mathcal{F} = \{M | Hom_R(E, M) = 0\}$ and $\mathcal{T} = \{M | Hom_R(M, F) = 0$ for any $F \in \mathcal{F}\}$. Then $(\mathcal{T}, \mathcal{F})$ is a σ-costable torsion theory.*

Proof. Let M be in $\mathcal{F}$. Then $\mathrm{Hom}_R(E, M) = 0$. Suppose that $P_\sigma(M) \notin \mathcal{F}$. Then $\mathrm{Hom}_R(E, P_\sigma(M)) \neq 0$. Thus there exists an $f \in \mathrm{Hom}_R(E, P_\sigma(M))$ such that $f(E) \neq 0$. Since $\pi_M^\sigma f \in \mathrm{Hom}_R(E, M) = 0$, $f(E) \subseteq K_\sigma(M)$ is small in $P_\sigma(M)$. Thus $f(E)$ is σ-torsionfree and small in $P_\sigma(M)$. This is a contradiction. Thus $\mathcal{F}$ is closed under taking σ-projective covers. □

Next we state the dual of Proposition 1. A preradical σ is called epi-preserving if $\sigma(M/N) = (\sigma(M) + N)/N$ holds for any module M and any submodule N of M. If σ is an epipreserving idempotent radical, $\mathcal{F}_\sigma$ is closed under taking factor modules and then $(\mathcal{T},\mathcal{F})$ is called cohereditary. We say that a subclass $\mathcal{C}$ of Mod-R is closed under taking σ-factor modules if : if $M \in \mathcal{C}$ and N is a σ-torsionfree submodule of M then $M/N \in \mathcal{C}$. We call a torsion theory $(\mathcal{T},\mathcal{F})$ σ-cohereditary if $\mathcal{F}$ is closed under taking σ-factor modules.

Proposition 9. *Let σ be an epi-preserving idempotent radical. Let $\mathcal{B}$ be a class of modules closed under taking σ-factor modules and submodules. Then the torsion theory cogenerated by $\mathcal{B}$ is σ-cohereditary.*

Proof. Let $(\mathcal{T},\ \mathcal{F})$ be a torsion theory cogenerated by $\mathcal{B}$. We show that $\mathcal{T}$ is closed under taking σ-projective covers. Let $M \in \mathcal{T}$. We show that $P_\sigma(M) \in \mathcal{T}$. Suppose that $P_\sigma(M) \notin \mathcal{T}$. Then there exists $B \in \mathcal{B}$ such that $\mathrm{Hom}_R(P_\sigma(M), \mathcal{B}) \neq 0$. Then there exists a nonzero homomorphism $\alpha : P_\sigma(M) \to B$. $\mathcal{B} \ni B \supseteq \alpha(P_\sigma(M)) \supseteq \alpha(K_\sigma(M)) \in \mathcal{F}_\sigma$. Since $\mathcal{B}$ is closed under taking σ-factor modules and submodules, $\alpha(P_\sigma(M))/\alpha(K_\sigma(M)) \in \mathcal{B}$. Then α induces $\widetilde{\alpha} : M \simeq P_\sigma(M)/K_\sigma(M) \twoheadrightarrow \alpha(P_\sigma(M))/\alpha(K_\sigma(M)) \in \mathcal{B}$. Since $M \in \mathcal{T}$, it follows that $\alpha(P_\sigma(M))/\alpha(K_\sigma(M)) = 0$, and so $\alpha(P_\sigma(M)) = \alpha(K_\sigma(M))$. Therefore it follows that $\alpha^{-1}\{\alpha(P_\sigma(M))\} = \alpha^{-1}\{\alpha(K_\sigma(M))\}$, and so $P_\sigma(M) = \alpha^{-1}(0) + K_\sigma(M)$. Since $K_\sigma(M)$ is small in $P_\sigma(M)$, it follows that $P_\sigma(M) = \alpha^{-1}(0)$. But this is a contradiction to the fact that $\alpha \neq 0$. Thus $P_\sigma(M) \in \mathcal{T}$. □

Proposition 10. *Let P be a σ-projective module and $\mathcal{B}$ be $\{M | Hom_R(P, M) = 0\}$. Then a torsion theory generated by $\mathcal{B}$ is σ-cohereditary.*

Proof. It is easily verified that $\mathcal{B}$ is closed under taking submodules, direct products and extensions. We only show that $\mathcal{B}$ is closed under taking σ-factor modules. Let $M \in \mathcal{B}$ and $N \in \mathcal{F}_\sigma$ be a submodule of M. Suppose that $M/N \notin \mathcal{B}$, then there exists a nonzero $f \in \mathrm{Hom}_R(P, M/N)$. Then f extends $f' \in \mathrm{Hom}_R(P, M)$ such $gf' = f$, where g is a canonical epimorphism from M to M/N. This is a contradiction to the fact that $M \in \mathcal{B}$. Thus a torsion theory $(\mathcal{T},\mathcal{B})$ generated by $\mathcal{B}$ is σ-cohereditary by Proposition 9. □

5. Dualization of σ-Goldie torsion theory

A module N is called small if N is a small submodule of some module. It is well known that N is small if and only if N is small in $E(M)$. Now we consider dualizations of σ-Goldie torsion theory.

Theorem 11. *Let σ be an epi-preserving idempotent radical. We denote $\mathcal{B} = \{N \in \mathcal{F}_\sigma | N$ is small in some module $M\}(= \{M \in \mathcal{F}_\sigma | M$ is small in $E(M)\})$. Then a torsion theory $(\mathcal{T},\mathcal{F})$ cogenerated by $\mathcal{B}$ is cohereditary and σ-costable.*

Proof. It is easily verified that $\mathcal{B}$ is closed under taking direct sums, factor modules and submodules. Thus by Proposition 9($\sigma = 1$), $\mathcal{F}$ is closed under taking factor modules, and so $(\mathcal{T},\mathcal{F})$ is a cohereditary torsion theory. Next we show that $\mathcal{F}$ is closed under taking σ-projective covers. Let $M \in \mathcal{F}$. Suppose that $P_\sigma(M) \notin \mathcal{F}$, then there exists a module X in $\mathcal{T}$ and $\mathrm{Hom}_R(X, P_\sigma(M)) \neq 0$. Consider the following exact sequence. $0 \to \mathrm{Hom}_R(X, K_\sigma(M)) \to \mathrm{Hom}_R(X, P_\sigma(M)) \to \mathrm{Hom}_R(X, M)$. Since $X \in \mathcal{T}$ and $M \in \mathcal{F}$, $\mathrm{Hom}_R(X,M) = 0$. Since $K_\sigma(M)$ is σ-torsion free small submodule of $P_\sigma(M)$, $K_\sigma(M)$ is in $\mathcal{B}$. Thus $\mathrm{Hom}_R(X, K_\sigma(M)) = 0$, and so $\mathrm{Hom}_R(X, P_\sigma(M)) = 0$. But this is a contradiction, and so it follows that $P_\sigma(M) \in \mathcal{F}$. □

A module of $\mathcal{B}$ in Theorem 11 is a generalization of small module. Last we give another extension of small modules. We call a module N a σ-small module if there exists a module L and a small σ-dense submodule K of L such that N is isomorphic to K.

Proposition 12. *A module N is a σ-small module if and only if N is small in $E_\sigma(N)$.*

Proof. Let N be a σ-small module. Then there exists a module L and its σ-dense small submodule K such that $N \cong K$. Consider the following diagram.

$$\begin{array}{ccc} N & \overset{h|_N}{\to} & K \subseteq^\circ L \twoheadrightarrow L/K \in \mathcal{T}_\sigma \\ \cap & & \cap \\ E_\sigma(N) & \underset{h}{\to} & E_\sigma(K), \end{array}$$

where h is an isomorphism and $h|_N$ is an isomorphism and an restriction of h to N. Then there exists a $g : L \to E_\sigma(K)$ such that $g|_K = 1_K$. Then $K = g(K) \subseteq^\circ g(L) \subseteq E_\sigma(K)$, and so K is small in $E_\sigma(K)$. Thus $h^{-1}(K)$ is small in $h^{-1}(E_\sigma(K))$, and so N is small in $E_\sigma(N)$. The converse is clear. □

Proposition 13. *Let $\mathcal{B}$ be the class of all σ-small modules. Then a torsion theory generated by $\mathcal{B}$ is a σ-hereditary torsion theory. A torsion theory cogenerated by $\mathcal{B}$ is a cohereditary torsion theory.*

Proof. It is easiy verified that $\mathcal{B}$ is closed under taking factor modules and σ-dense submodules, then a torsion theory generated by $\mathcal{B}$ is a σ-hereditary torsion theory by Proposition 1 and Proposition12. A torsion theory cogenerated by $\mathcal{B}$ is a cohereditary torsion theory by Proposition 9. □

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General Education
Hakodate National College of Technology
14-1 Tokura-cho Hakodate Hokkaido, 042-8501 Japan
E-mail address: takehana@hakodate-ct.ac.jp