ON THE RELATION OF THE UPPER BOUND OF GLOBAL DIMENSION AND THE LENGTH OF SERIAL ALGEBRA WHICH HAS FINITE GLOBAL DIMENSION

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ABSTRACT. The aim of this note is to study the relationship between the global dimension and the Loewy length of serial algebras that have finite global dimension. To compute global dimension, we define the associative quiver of an admissible sequence $(a_1, \cdots, a_n)$ of a serial algebra A. This note concludes the following result. For positive integer k with $k < n/2$, if the Loewy length $L(A)$ of A is minimal positive integer which greater than n/k, then the global dimension is less than or equal to $2n - 2k - 1$.

Key Words: serial algebra, global dimension

Let A be a finite dimensional basic connected serial algebra over an algebraically closed field, and n is the number of the non isomorphic simple left modules of A. If the global dimension gl.dim$A$ of A is finite, then gl.dim$A \leq 2n - 2$ and the Loewy length $L(A)$ of A is less than or equal to $2n - 1$[3]. In this note we consider the relationship $L(A)$ and gl.dim$A$.

1. Notation

The quiver of A is one of the following.

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n \qquad\qquad \begin{tikzcd} 1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n \arrow[lll, bend left] \end{tikzcd}$$

The algebra A whose quiver is the first one called chain type and the other called cyclic type. Let $P_i, S_i\,(1 \leq i \leq n)$ be the indecomposable left projective module and the simple left module of A corresponding to the vertex i. Then P_{i+1} is a projective cover of radP_i for $i = 1, 2, \cdots, n-1$, and P_1 is a projective cover of radP_n in case of cyclic type. The sequence of positive integers $(a_1, \cdots, a_n)$ where $a_i = L(P_i)\,(1 \leq i \leq n)$ is called the admissible sequence of A and has the property that $a_{i+1} \geq a_i - 1 \geq 1$ for all $i = 1, 2, \cdots, n-1$ and $a_1 \geq a_n - 1$. Conversely, for any sequence $(a_1, \cdots, a_n)$ of positive integers with this property, there is a serial algebra with this sequence as its admissible sequence.

Now, let A be a serial algebra with admissible sequence $(a_1, \cdots, a_n)$. Then P_i has unique composition series of following shape. Where i is the corresponding simple module of vertex i, and $[k]$ denotes the least positive residue of k modulo n for any positive integer k.

$$P_i = \begin{pmatrix} i \\ [i+1] \\ \vdots \\ [i + a_i - 1] \end{pmatrix}$$

The detailed version of this paper will be submitted for publication elsewhere.

2. Regular points of admissible sequence

In the paper [3], Gustafson introduced the notion of f-regular points and computed the global dimension of serial rings.

Definition 1. The function f on $\{1, 2, \cdots, n\}$ for admissible sequence $(a_1, \cdots, a_n)$ is defined by $f(i) = [i + a_i]$. The point $i \in \{1, \cdots, n\}$ is f-regular if $f^t(i) = i$ for some positive integer t.

Since $f^{n-1}(i)$ is f-regular for any i , the set of f-regular points is not empty.

Definition 2. For $i \in \{1, \cdots, n\}$, the distance $h(i)$ of i from f-regular points defined by $h(i) = 0$ for f-regular point i, and $h(i) = t$ if $f^{t-1}(i)$ is not f-regular but $f^t(i)$ is f-regular for positive integer t. The maximal distance d from f-regular points defined by $d = \max\{h(i) | i = 1, \cdots, n\}$.

The minimal projective resolution of S_i is following.

$$\cdots \to P_{f^2(i+1)} \to P_{f^2(i)} \to P_{f(i+1)} \to P_{f(i)} \to P_{i+1} \to P_i \to S_i \to 0$$

If the projective dimension proj.dimS_i of S_i is finite, then its left end is one of the following shape.

$$0 \to P_{f^k(i)} \to P_{f^{k-1}(i+1)} \to \cdots$$
$$0 \to P_{f^k(i+1)} \to P_{f^k(i)} \to \cdots$$

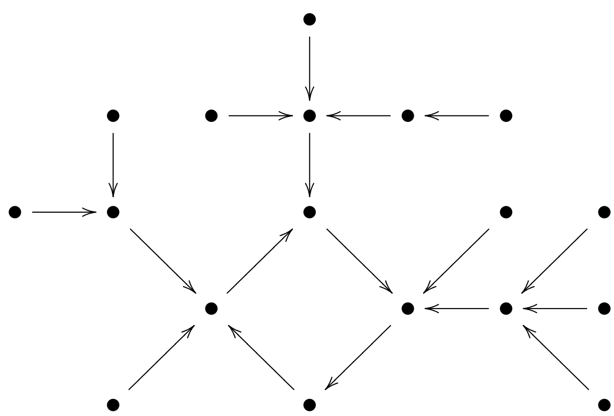
So $f^{k+1}(i) = f^k(i + 1)$ or $f^{k+1}(i) = f^{k+1}(i + 1)$ and this is f-regular. It follows that proj.dim$S_i \leq 2d$. Then we have following lemma.

Lemma 3 (Gustafson). *Let d be the maximal distance from f-regular points, then* gl.dim$A \leq 2d$.

3. Associative quiver

We define the associative quiver Q_A of A by $\{1, \cdots, n\}$ is set of vertices and an arrow i to j if $f(i) = j$. An associative quiver is a disjoint union of left serial quivers which are defined below.

Definition 4. A quiver called left serial if it has unique oriented cycle and when removing all arrows of this cycle, the remaining is a disjoint union of trees with unique sink which is a vertex of the cycle.



It follows that a vertex i is f-regular if and only if i belongs to the cycle of the associative quiver. We call "f-regular vertex" instead of "f-regular point" when we treat the point as the vertex of the associative quiver.

Example 5. Let A be a serial algebra with admissible sequence $(3,3,3,2,2)$. Its indecomposable left projective modules are following.

$$\begin{pmatrix}1\\2\\3\end{pmatrix}\begin{pmatrix}2\\3\\4\end{pmatrix}\begin{pmatrix}3\\4\\5\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix}\begin{pmatrix}5\\1\end{pmatrix}$$

The associative quiver of A is following. The vertices 1, 4, 2 and 5 are f-regular.

$$\begin{tikzcd} 3 \arrow[r] & 1 \arrow[r, bend left] & 4 \arrow[l, bend left] & 2 \arrow[r, bend left] & 5 \arrow[l, bend left] \end{tikzcd}$$

An associative quiver is a disjoint union of left serial quivers. But if gl.dim$A < \infty$, it is only one component. For, if gl.dim$A < \infty$, any morphism between indecomposable projective modules that corresponding to f-regular vertices is an isomorphism or 0. Because if there is a non isomorphic and non zero morphism $g : P_i \to P_j$, then its cokernel K has infinite chain of projective resolution.

$$\cdots \to P_{f^2(i)} \to P_{f^2(j)} \to P_{f(i)} \to P_{f(j)} \to P_i \to P_j \to K \to 0$$

If there exist two distinct cycles in the associative quiver, then it follows that there exist a non isomorphic and non zero morphism between the corresponding indecomposable projective modules on these cycles.

Proposition 6. *If* gl.dim$A < \infty$*, then Q_A is left serial (only one component).*

4. Serial algebra of chain type

Let A be a serial algebra of chain type with admissible sequence $(a_1, \cdots, a_n)$. It is well known that $L(A) \le n$ and gl.dim$A \le n-1$. Next theorem is generalization of this.

Theorem 7. *Suppose that A is a serial algebra of chain type with admissible sequence $(a_1, \cdots, a_n)$ and $l = L(A)$.*
Then gl.dim$A \le n - l + 1$.

The inequality of global dimension of above theorem is sharp.

Example 8. Let A be a serial algebra with admissible sequence $(l, l-1, \cdots, 3, 2, 2, \cdots, 2, 1)$. gl.dim$A = n - l + 1$. Indeed, proj.dim$S_{l-2} = n - l + 1$ and this is the maximal.

$$0 \to P_n \to P_{n-1} \to \cdots \to P_{l-2} \to S_{l-2} \to 0$$

5. Serial algebra of cyclic type

Let A be a serial algebra of cyclic type with admissible sequence $(a_1, \cdots, a_n)$. If gl.dim$A < \infty$, then $L(A) \le 2n-1$[3]. It is well known that if $L(A) = 2$ then gl.dim$A = \infty$. So, fixed $l\,(2 < l < 2n)$, we calculate upper bound of gl.dim$A$.

Definition 9. For $1 \le i \le n-1$, the vertex i is called a step vertex if $a_{i+1} = a_i - 1$ and vertex n is called a step vertex if $a_1 = a_n - 1$.

If i is a step vertex, then the associative quiver contains following graph as subquiver.

$$\begin{tikzcd} i \arrow[dr] & \\ & f(i) \\ i+1 \arrow[ur] & \end{tikzcd}$$

We don't avoid the case $i = f(i)$ and $i + 1 = f(i)$.

Lemma 10. *If Q_A has r f-regular vertices and s step vertices and if* gl.dim$A < \infty$*, then the maximal distance d from f-regular vertices is less than or equal to $n - r - s + 1$.*

Example 11. Let A be a serial algebra with admissible sequence $(4, 4, 4, 4, 3, 2, 2)$. The associative quiver is following. 1 and 5 are f-regular and 4 and 5 are step vertices. $d = h(3) = 4 = 7 - 2 - 2 + 1$ and gl.dim$A$ = proj.dim$S_2 = 7 \leq 8 = 2 \cdot d = 2 \cdot 4$.

$$\begin{tikzcd} 3 \arrow[r] & 7 \arrow[r] & 2 \arrow[r] & 6 \arrow[r] & 1 \arrow[r, bend left] & 5 \arrow[l, bend left] \\ & & & 4 \arrow[ur] & & \end{tikzcd}$$

If Q_A has only one step point, then d is maximal among the algebras which have fixed r regular points. In this case Q_A is following shape. There are r points that belong to the cycle and d is $n - r$. Lemma 3 shows that gl.dim$A \leq 2n - 2r$. In case of $n < l < 2n$, this bound is sharp. But the case $l \leq n$ is not.

$$\begin{tikzcd} & & & & \bullet \arrow[dr] & \\ \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[ur] & & \bullet \arrow[dl] \\ & & & & \cdots \arrow[ul] & \end{tikzcd}$$

For any positive real number x, let $\lceil x \rceil$ be the minimum positive integer that greater than x.

Theorem 12. *Let A be a a serial algebra of cyclic type with admissible sequence $(a_1, \cdots, a_n)$ which has finite global dimension and $l = L(A)$.*

(1) *If $n < l < 2n$, then* gl.dim$A \leq 4n - 2l$.
(2) *If $l = n$, then* gl.dim$A \leq 2n - 3$.
(3) *For any positive integer k with $n \geq 2k + 3$ and $\lceil \frac{n}{k+1} \rceil < \lceil \frac{n}{k} \rceil$,*
if $\lceil \frac{n}{k+1} \rceil \leq l < \lceil \frac{n}{k} \rceil$, then gl.dim$A \leq 2n - 2k - 3$.

These inequality of global dimension are sharp.

Example 13. For positive integer $t \, (1 \leq t < n)$, let A be a serial algebra with admissible sequence $(n + t, n + t - 1, \cdots, n + 1, n + 1, \cdots, n + 1, n)$.
This is the case (1) of the theorem which the equality holds. $h(i) = n - t \, (1 \leq i \leq$

t), $h(t+j) = n-t-j\,(1 \leq j \leq n-t-1)$, and $h(n) = 0$. So $d = n-t$, and gl.dim$A = 2n-2t$. Q_A is following.

$$\begin{tikzcd}
1 \arrow[drr] & & & & & & \\
2 \arrow[rr] & & t+1 \arrow[r] & t+2 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[r] & n \arrow[loop right] \\
& \ddots \arrow[ur] & & & & & \\
\ddots \arrow[ur] & & t \arrow[uu] & & & &
\end{tikzcd}$$

Above example is generalization of Gustafson's example.

Example 14 (Gustafson)**.** In example 13, the case of $t = 1$ is following.
Let A be a serial algebra with admissible sequence $(n+1, n+1, \cdots, n+1, n)$.
gl.dim$A = 2n-2$. Q_A is following.

$$\begin{tikzcd}
1 \arrow[r] & 2 \arrow[r] & \cdots \arrow[r] & n-1 \arrow[r] & n \arrow[loop right]
\end{tikzcd}$$

Example 15. Let A be a serial algebra with admissible sequence $(n, n-1, \cdots, n-1, n-1)$. This is the case (2) of the theorem which the equality holds. gl.dim$A = 2n-3$. Q_A is following.

$$\begin{tikzcd}
n \arrow[r] & n-1 \arrow[r] & \cdots \arrow[r] & 2 \arrow[r] & 1 \arrow[loop right]
\end{tikzcd}$$

In the case (3) of the theorem, the equality of gl.dim$A$ holds when Q_A has $k+1$ regular vertices and one step vertex. In this case $d = n-k-1$, and gl.dim$A = 2d-1 = 2n-2k-3$.

Example 16. Let A be a serial algebra with admissible sequence $(3, 3, 3, 3, 2, 2)$. In this case, $n = 6, l = \lceil \frac{6}{2} \rceil = 3 < \lceil \frac{6}{1} \rceil, k = 1$, and gl.dim$A = 7 = 2 \cdot 6 - 2 \cdot 1 - 3$. Q_A is following.

$$\begin{tikzcd}
3 \arrow[r] & 6 \arrow[r] & 2 \arrow[r] & 5 \arrow[r] & 1 \arrow[r, bend left] & 4 \arrow[l, bend left]
\end{tikzcd}$$

Example 17. Let A be a serial algebra with admissible sequence $(4, 4, 4, 4, 4, 3, 3, 3, 3, 3, 3, 3, 3, 3)$. In this case, $n = 14, l = \lceil \frac{14}{4} \rceil = 4 < \lceil \frac{14}{3} \rceil, k = 3$, and gl.dim$A = 19 = 2 \cdot 14 - 2 \cdot 3 - 3$. Q_A is following.

$$\begin{tikzcd}
& & & & & & & & & & & 12 \arrow[dr] & \\
4 \arrow[r] & 8 \arrow[r] & 11 \arrow[r] & 14 \arrow[r] & 3 \arrow[r] & 7 \arrow[r] & 10 \arrow[r] & 13 \arrow[r] & 2 \arrow[r] & 6 \arrow[r] & 9 \arrow[ur] & & 1 \arrow[dl] \\
& & & & & & & & & & & 5 \arrow[ul] &
\end{tikzcd}$$

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