NONCOMMUTATIVE GRADED GORENSTEIN ISOLATED SINGULARITIES

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Abstract. Gorenstein isolated singularities play an essential role in representation theory of Cohen-Macaulay modules. In this article, we define a notion of noncommutative graded isolated singularity and study AS-Gorenstein isolated singularities. For an AS-Gorenstein algebra A of dimension $d \geq 2$, we show that A is a graded isolated singularity if and only if the stable category of graded maximal Cohen-Macaulay modules over A has the Serre functor. Using this result, we also show the existence of cluster tilting modules over certain fixed subalgebras of AS-regular algebras.

Key Words: graded isolated singularity, graded maximal Cohen-Macaulay module, AS-Gorenstein algebra, Serre functor, cluster tilting.

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1. Introduction

Throughout this paper, k is an algebraically closed field of characteristic 0. In representation theory of orders, which generalize both finite dimensional algebras and Cohen-Macaulay rings, studying the categories of Cohen-Macaulay modules is active (see [5] for details). In particular, the following results play key roles in the theory (we present graded versions due to [6, Corollary 2.5, Theorem 3.2, Theorem 4.2]).

Theorem 1. *Let R be a noetherian commutative graded local Gorenstein ring of dimension d and of Gorenstein parameter ℓ. Assume that R is an isolated singularity. Then the stable category of graded maximal Cohen-Macaulay modules has the Serre functor $(-\ell)[d-1]$.*

Theorem 2. *Let $S = k[x_1, \dots, x_d]$ be a polynomial ring generated in degree 1, G a finite subgroup of $\mathrm{SL}_d(k)$, and S^G the fixed subring of S.*

1. *Then the skew group algebra $S * G$ is isomorphic to $\underline{\mathrm{End}}_{S^G}(S)$ as graded algebras.*
2. *Assume that S^G is an isolated singularity. Then S is a $(d-1)$-cluster tilting module in the categories of graded maximal Cohen-Macaulay modules over S^G.*

The proofs of these results rely on commutative ring theory. This paper tries to give a noncommutative (not necessarily order) version of them.

One of the noncommutative analogues of polynomial rings (resp. Gorenstein local rings) is AS-regular algebras (resp. AS-Gorenstein algebras). In this paper, we define a notion of noncommutative graded isolated singularity by the smoothness of the noncommutative projective scheme (see also [8]), and we focus on studying AS-Gorenstein

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isolated singularities. In particular, a noncommutative version of Theorem 1 will be given in Theorem 7, and a partial generalization of Theorem 2 for some fixed subalgebras of AS-regular algebras will be given in Theorem 11.

2. Preliminaries

Let A be a connected graded algebra and $\mathfrak{m} = \bigoplus_{i>0} A_i$ the maximal homogeneous two-sided ideal of A. The trivial A-module $A/\mathfrak{m}$ is denoted by k. We denote by $\mathsf{GrMod}\, A$ the category of graded right A-modules with degree zero A-module homomorphisms, and by $\mathsf{grmod}\, A$ the full subcategory consisting of finitely generated graded right A-modules. The group of graded k-algebra automorphisms of A is denoted by $\operatorname{GrAut} A$. Let M be a graded right A-module. For an integer $n \in \mathbb{Z}$, we define the truncation $M_{\geq n} := \bigoplus_{i \geq n} M_i \in \mathsf{GrMod}\, A$ and the shift $M(n) \in \mathsf{GrMod}\, A$ by $M(n)_i := M_{n+i}$ for $i \in \mathbb{Z}$. We write

$$\underline{\operatorname{Ext}}^i_A(M, N) = \bigoplus_{n \in \mathbb{Z}} \operatorname{Ext}^i_{\mathsf{GrMod}\, A}(M, N(n)).$$

For a graded algebra automorphism $\sigma \in \operatorname{GrAut} A$, we define a new graded right A-module $M_\sigma \in \mathsf{GrMod}\, A$ by $M_\sigma = M$ as graded vector spaces with the new right action $m * a = m\sigma(a)$ for $m \in M$ and $a \in A$. We denote by $(-)^* = \underline{\operatorname{Hom}}_k(-, k)$ the graded Matlis duality. If M is locally finite, then $M^{**} \cong M$ as graded A-modules. We define the functor $\underline{\Gamma}_\mathfrak{m} : \mathsf{GrMod}\, A \to \mathsf{GrMod}\, A$ by $\underline{\Gamma}_\mathfrak{m}(-) = \lim_{n \to \infty} \underline{\operatorname{Hom}}_A(A/A_{\geq n}, -)$. The derived functor of $\underline{\Gamma}_\mathfrak{m}$ is denoted by $\mathrm{R}\underline{\Gamma}_\mathfrak{m}(-)$, and its cohomologies are denoted by $\underline{\mathrm{H}}^i_\mathfrak{m}(-) = h^i(\mathrm{R}\underline{\Gamma}_\mathfrak{m}(-))$.

Definition 3. A connected graded algebra A is called a d-dimensional AS-Gorenstein algebra (resp. AS-regular algebra) of Gorenstein parameter ℓ if

- A is noetherian,
- $\operatorname{id}_A A = \operatorname{id}_{A^{\mathrm{op}}} A = d < \infty$ (resp. $\operatorname{gldim} A = d < \infty$) and
- $\underline{\operatorname{Ext}}^i_A(k, A) \cong \underline{\operatorname{Ext}}^i_{A^{\mathrm{op}}}(k, A) \cong \begin{cases} k(\ell) & \text{if } i = d, \\ 0 & \text{if } i \neq d. \end{cases}$

If A is a d-dimensional AS-Gorenstein algebra of Gorenstein parameter ℓ, then i-th local cohomology $\underline{\mathrm{H}}^i_\mathfrak{m}(A)$ of A is zero for all $i \neq d$. The graded A-A bimodule $\omega_A := \underline{\mathrm{H}}^d_\mathfrak{m}(A)^*$ is called the canonical module of A. It is known that there exists a graded algebra automorphism $\nu \in \operatorname{GrAut} A$ such that $\omega_A \cong A_\nu(-\ell)$ as graded A-A bimodules (cf. [7, Theorem 1.2]). We call this graded algebra automorphism $\nu \in \operatorname{GrAut} A$ the generalized Nakayama automorphism of A.

We denote by $\mathsf{tors}\, A$ the full subcategory of $\mathsf{grmod}\, A$ consisting of finite dimensional modules over k, and

$$\mathsf{tails}\, A := \mathsf{grmod}\, A / \mathsf{tors}\, A$$

the quotient category, which is called the noncommutative projective scheme associated to A in [1]. If A is a commutative graded algebra finitely generated in degree 1 over k, then $\mathsf{tails}\, A$ is equivalent to the category of coherent sheaves on $\operatorname{Proj} A$ by Serre, justifying the terminology. We usually denote by $\mathcal{M} \in \mathsf{tails}\, A$ the image of $M \in \mathsf{grmod}\, A$. If $M, N \in \mathsf{grmod}\, A$, then $\mathcal{M} \cong \mathcal{N}$ in $\mathsf{tails}\, A$ if and only if $M_{\geq n} \cong N_{\geq n}$ in $\mathsf{grmod}\, A$ for some n, explaining the word of "tails".

We define a notion of noncommutative graded isolated singularity by the smoothness of the noncommutative projective scheme. Recall that the global dimension of $\mathsf{tails}\, A$ is defined by

$$\operatorname{gldim}(\mathsf{tails}\, A) := \sup\{i \mid \operatorname{Ext}^i_{\mathsf{tails}\, A}(\mathcal{M}, \mathcal{N}) \neq 0 \text{ for some } \mathcal{M}, \mathcal{N} \in \mathsf{tails}\, A\}.$$

Definition 4. A noetherian connected graded algebra A is called a graded isolated singularity if $\mathsf{tails}\, A$ has finite global dimension.

If A is a graded quotient of a polynomial ring generated in degree 1, then A is a graded isolated singularity (in the above sense) if and only if $A_{(\mathfrak{p})}$ is regular for any homogeneous prime ideal $\mathfrak{p} \neq \mathfrak{m}$, justifying the definition. It is easy to see that if A has finite global dimension, then $\mathsf{tails}\, A$ has finite global dimension, so A is a graded isolated singularity. The purpose of this paper is to study AS-Gorenstein isolated singularities.

For the rest of this section, we recall that Jørgensen and Zhang [9] gave a noncommutative graded version of Watanabe's theorem. Let A be a graded algebra and let $\sigma \in \operatorname{GrAut} A$, $M, N \in \mathsf{GrMod}\, A$. A k-linear graded map $f : M \to N$ is called σ-linear if $f : M \to N_\sigma$ is a graded A-module homomorphism. If A is AS-Gorenstein, then by [9, Lemma 2.2], $\sigma : A \to A$ induces a σ-linear map $\underline{\mathrm{H}}^d_{\mathfrak{m}}(\sigma) : \underline{\mathrm{H}}^d_{\mathfrak{m}}(A) \to \underline{\mathrm{H}}^d_{\mathfrak{m}}(A)$. Moreover, there exists a constant $c \in k^\times$ such that $\underline{\mathrm{H}}^d_{\mathfrak{m}}(\sigma) : \underline{\mathrm{H}}^d_{\mathfrak{m}}(A) \to \underline{\mathrm{H}}^d_{\mathfrak{m}}(A)$ is equal to $c(\sigma^{-1})^* : A^*(\ell) \to A^*(\ell)$. The constant c^{-1} is called the homological determinant of σ, and we denote $\operatorname{hdet} \sigma = c^{-1}$ (see [9, Definition 2.3]).

Theorem 5. [9, Theorem 3.3] *If A is AS-Gorenstein of dimension d, and G is a finite subgroup of* $\operatorname{GrAut} A$ *such that* $\operatorname{hdet} \sigma = 1$ *for all $\sigma \in G$, then the fixed subalgebra A^G is AS-Gorenstein of dimension d.*

3. Serre Functors

Definition 6. Let C be a k-linear category such that $\dim_k \operatorname{Hom}_{\mathsf{C}}(\mathcal{M}, \mathcal{N}) < \infty$ for all $\mathcal{M}, \mathcal{N} \in \mathsf{C}$. An autoequivalence $S : \mathsf{C} \to \mathsf{C}$ is called the Serre functor for C if we have a functorial isomorphism

$$\operatorname{Hom}_{\mathsf{C}}(\mathcal{M}, \mathcal{N}) \cong \operatorname{Hom}_{\mathsf{C}}(\mathcal{N}, S(\mathcal{M}))^*$$

for all $\mathcal{M}, \mathcal{N} \in \mathsf{C}$.

Note that the Serre functor is unique if it exists. Let A be an AS-Gorenstein algebra of dimension d. We say that $M \in \mathsf{grmod}\, A$ is graded maximal Cohen-Macaulay if $\underline{\operatorname{Ext}}^i_A(M, A) = 0$ for any $i > 0$. We denote by $\mathsf{CM}^{\mathsf{gr}}(A)$ the full subcategory of $\mathsf{grmod}\, A$ consisting of graded maximal Cohen-Macaulay modules, and by $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ the stable category of $\mathsf{CM}^{\mathsf{gr}}(A)$. Thus $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ has the same objects as $\mathsf{CM}^{\mathsf{gr}}(A)$ and the morphism set is given by

$$\operatorname{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(M, N) = \operatorname{Hom}_{\mathsf{GrMod}\, A}(M, N)/P(M, N)$$

for any $M, N \in \mathsf{CM}^{\mathsf{gr}}(A)$, where $P(M, N)$ consists of the degree zero A-module homomorphisms that factor through a projective module in $\mathsf{GrMod}\, A$. The syzygy gives a functor $\Omega : \underline{\mathsf{CM}}^{\mathsf{gr}}(A) \to \underline{\mathsf{CM}}^{\mathsf{gr}}(A)$. By [2], we see that $\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ is a triangulated category with respect to the translation functor $M[-1] = \Omega M$.

We have the following main result in this section.

Theorem 7. *Let A be an AS-Gorenstein algebra of dimension $d \geq 2$. Then the following are equivalent.*

(1) *A is a graded isolated singularity.*
(2) *$\underline{\mathsf{CM}}^{\mathsf{gr}}(A)$ has the Serre functor $- \otimes_A \omega_A[d-1]$, that is, there exists a functorial isomorphism*

$$\mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(M, N) \cong \mathrm{Hom}_{\underline{\mathsf{CM}}^{\mathsf{gr}}(A)}(N, M \otimes_A \omega_A[d-1])^*$$

for any $M, N \in \mathsf{CM}^{\mathsf{gr}}(A)$.

In order to give an example of this result, we prepare a noncommutative graded version of a classical result by Auslander. Let A be an AS-Gorenstein algebra. We call A CM-representation-finite if there exist finitely many indecomposable graded maximal Cohen-Macaulay modules $X_1, \ldots, X_n$ so that, up to isomorphism, the indecomposable graded maximal Cohen-Macaulay modules in $\mathsf{grmod}\, A$ are precisely the degree shifts $X_i(s)$ for $1 \leq i \leq n$ and $s \in \mathbb{Z}$.

Proposition 8. *Let A be an AS-regular algebra of dimension 2, and let G be a finite subgroup of* $\mathrm{GrAut}\, A$ *such that* $\mathrm{hdet}\, \sigma = 1$ *for all $\sigma \in G$. Then A^G is CM-representation-finite. In fact, the indecomposable maximal Cohen-Macaulay modules over A^G are precisely the indecomposable summands of $A(s)$. Moreover, A^G is an AS-Gorenstein isolated singularity.*

Example 9. Let

$$A = k\langle x, y\rangle/(xy - \alpha yx) \quad 0 \neq \alpha \in k, \quad \deg x = \deg y = 1.$$

Then A is an AS-regular algebra of dimension 2 and of Gorenstein parameter 2. We define a graded algebra automorphism $\sigma \in \mathrm{GrAut}\, A$ by $\sigma(x) = \xi x, \sigma(y) = \xi^2 y$ where ξ is a primitive 3-rd root of unity. One can check $\mathrm{hdet}\, \sigma = 1$. Let $G = \langle \sigma \rangle \leq \mathrm{GrAut}\, A$. Then A^G is AS-Gorenstein of dimension 2 and

$$H_{A^G}(t) = \frac{1 - t + t^2}{(1-t)^2(1 + t + t^2)}.$$

It follows from Proposition 8 that A^G is CM-representation-finite and a graded isolated singularity. But A^G is not AS-regular because $H_{A^G}(t)^{-1} \notin \mathbb{Z}[t]$. Theorem 7 shows that $\underline{\mathsf{CM}}^{\mathsf{gr}}(A^G)$ has the Serre functor.

4. n-cluster tilting modules

The notion of n-cluster tilting subcategories plays an important role from the viewpoint of higher analogue of Auslander-Reiten theory [3], [4]. It can be regarded as a natural generalization of the classical notion of CM-representation-finiteness.

Definition 10. Let A be a balanced Cohen-Macaulay algebra. A graded maximal Cohen-Macaulay module $X \in \mathsf{CM}^{\mathsf{gr}}(A)$ is called an n-cluster tilting module if

$$\begin{aligned}\mathrm{add}_A\{X(s) \mid s \in \mathbb{Z}\} &= \{M \in \mathsf{CM}^{\mathsf{gr}}(A) \mid \underline{\mathrm{Ext}}^i_A(M, X) = 0 \ (0 < i < n)\} \\ &= \{M \in \mathsf{CM}^{\mathsf{gr}}(A) \mid \underline{\mathrm{Ext}}^i_A(X, M) = 0 \ (0 < i < n)\}.\end{aligned}$$

Note that A is CM-representation-finite if and only if A has a 1-cluster tilting module. In fact if A and A^G are as in Proposition 8, then A^G has a 1-cluster tilting module $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$.

Let A be a connected graded algebra and G a finite subgroup of $\operatorname{GrAut} A$. Then the skew group algebra $A * G$ is an $\mathbb{N}$-graded algebra defined by $A * G = \bigoplus_{i \in \mathbb{N}}(A_i \otimes_k kG)$ as a graded vector space with the multiplication

$$(a \otimes \sigma)(a' \otimes \sigma') = a\sigma(a') \otimes \sigma\sigma'$$

for any $a, a' \in A$ and $\sigma, \sigma' \in G$. We have the following main result in this section.

Theorem 11. *Let A be a AS-regular domain of dimension $d \geq 2$ and of Gorenstein parameter ℓ generated in degree 1. Take $r \in \mathbb{N}^+$ such that $r \mid \ell$. We define a graded algebra automorphism σ_r of A by $\sigma_r(a) = \xi^{\deg a} a$ where ξ is a primitive r-th root of unity, and write $G = \langle \sigma_r \rangle$ for the finite cyclic subgroup of* $\operatorname{GrAut}(A)$ *generated by σ_r. Then*

(1) *the skew group algebra $A * G$ is isomorphic to $\underline{\operatorname{End}}_{A^G}(A)$ as graded algebras.*
(2) *A^G is a graded isolated singularity, and $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$ is a $(d-1)$-cluster tilting module.*

Moreover, it follows from the study of skew group algebras [10, Lemma 13] that $\underline{\operatorname{End}}_{A^G}(A)$ in the above theorem is a generalized AS-regular algebra of dimension d (ie, $\underline{\operatorname{End}}_{A^G}(A)$ has global dimension d and satisfies generalized Gorenstein condition).

Theorem 11 is a partial generalization of Theorem 2. Thanks to this result, we can obtain examples of $(d-1)$-cluster tilting modules over non-orders.

Example 12. Let

$$A = k\langle x, y\rangle/(\alpha xy^2 + \beta yxy + \alpha y^2x + \gamma x^3, \alpha yx^2 + \beta xyx + \alpha x^2y + \gamma y^3), \deg x = \deg y = 1$$

where $\alpha, \beta, \gamma \in k$ are generic scalars. Then A is an AS-regular algebra of dimension 3 and Gorenstein parameter 4. Let

$$G = \langle \sigma_4 \rangle = \left\langle \begin{pmatrix} \xi & 0 \\ 0 & \xi \end{pmatrix} \right\rangle \leq \operatorname{GrAut} A$$

where ξ is a primitive 4-th root of unity. Then A^G is an AS-Gorenstein isolated singularity, and $A \in \mathsf{CM}^{\mathsf{gr}}(A^G)$ is a 2-cluster tilting module. Moreover, we see that $\underline{\operatorname{End}}_{A^G}(A)$ is a generalized AS-regular algebra of dimension 3.

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