One point extension of a quiver algebra defined by two cycles and a quantum-like relation

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Notation

- k: field,
- A: finite dimensional k-algebra,
- $A^e := A \otimes_k A^{\mathrm{op}}$: enveloping algebra,
- $\mathrm{HH}^n(A) \cong \mathrm{Ext}^n_{A^e}(A, A)$: Hochschild cohomology group of A,
- $\mathrm{HH}^*(A) \cong \oplus_{n \geq 0} \mathrm{HH}^n(A)$: Hochschild cohomology ring of A with Yoneda product,
- $\mathcal{N}$: ideal of $\mathrm{HH}^*(A)$ generated by all homogeneous nilpotent elements.
- $\mathrm{HH}^*(A)/\mathcal{N}$: Hochschild cohomology ring of A modulo nilpotence.

The support variety of M

Definition [[SnSo(2004)], Definision 3.3]

The support variety of A-module M is given by

$$V(M) = \{m \in \mathrm{MaxSpec}\,\mathrm{HH}^*(A)/\mathcal{N} | \mathrm{AnnExt}^*_A(M, M) \subseteq m'\}$$

where $\mathrm{AnnExt}^*_A(M, M)$ is the annihilator of $\mathrm{Ext}^*_A(M, M)$ and m' is the preimage of m in $\mathrm{HH}^*(A)$.

Question [Sn(2009)]

Whether we can give necessary and sufficient conditions on a finite dimensional algebra for the Hochschild cohomology ring modulo nilpotence to be finitely generated as an algebra?

With respect to sufficient condition, it is shown that $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as an algebra for various classes of algebras by many authors as follows:

- Any block of a group ring of a finite group (See [Ev(1961)], [V(1959)])
- Any block of a finite dimensional cocommutative Hopf algebra (See [FSu(1997)])
- Finite dimensional algebras of finite global dimension (See [Ha(1989)])
- Finite dimensional self-injective algebras of finite representation type over an algebraically closed field (See [GSnSo(2003)])
- Finite dimensional monomial algebras (See [GSnSo(2006)])
- A class of special biserial algebras (See [SnT(2010)])
- A Hecke algebra (See [ScSn])

- Koenig and Nagase produced many examples of finite dimensional algebras A with a stratifying ideal for which $\mathrm{HH}^*(A)/\mathcal{N}$ is finitely generated as an algebra. (See [KN(2009)])

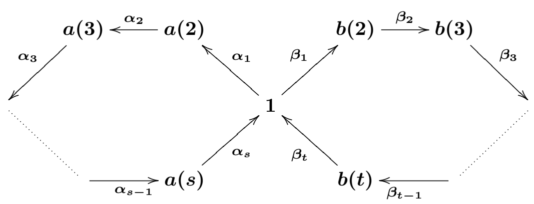
Stratifying ideal

Let e be an idempotent of A. If the two sided ideal AeA satisfies the following conditions, this ideal is called a stratifying ideal.

- The multiplication map $Ae \otimes_{eAe} eA \to AeA$ is an isomorphism.
- For all $n \geq 1$, $\mathrm{Tor}^{eAe}_n(Ae, eA) = 0$.

Quiver algebra defined by two cycles and a quantum-like relation

Let $s, t \geq 1$ be integers. We consider the quiver algebra $A_q = kQ/I_q$.
Q: the quiver with $s + t - 1$ vertices and $s + t$ arrows as follows:



I_q: the ideal of kQ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}$$

where $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$, $Y := \beta_1 + \beta_2 + \cdots + \beta_t$, integers $a, b \geq 2$ and q is non-zero element in k.

Quantum complete intersection

In the case $s = t = 1$, $A_q = k\langle x, y\rangle/\langle x^a, xy - qyx, y^b\rangle$ is a quantum complete intersection. This algebra is self-injective algebra.

- In the case $a = b = 2$, the Hochschild cohomology ring of A_q was determined by [BGMS(2005)] for any element q in k.
- In the case $a, b \geq 2$, the Hochschild cohomology ring of A_q was determined by [BE(2008)] where q is not a root of unity.
- In [BO(2008)], Bergh and Oppermann showed that A_q holds the finiteness conditions if and only if q is a root of unity.

Hochschild cohomology group

Let the following sequence be an A^e-projective resolution of A:

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} A \to 0.$$

Then we have the complex:

$$0 \to \operatorname{Hom}_{A^e}(P_0, A) \xrightarrow{d_1^*} \operatorname{Hom}_{A^e}(P_1, A) \xrightarrow{d_2^*} \cdots .$$

Hochschild cohomology group

The n-th Hochschild cohomology group of A is defined by

$$\mathrm{HH}^n(A) = \operatorname{Ext}^n_{A^e}(A, A) = \operatorname{Ker} d^*_{n+1}/\operatorname{Im} d^*_n$$

where $d^*_n = \operatorname{Hom}_{A^e}(d_n, A)$ for $n \geq 1$.

Projective resolution of A_q

We have the A^e-projective resolution of A_q:

$$\cdots P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} A_q \to 0.$$

where

$$P_{2n} = \coprod_{l=0}^{2n} Ae_1 \otimes e_1 A \oplus \coprod_{i=2}^{s} Ae_{a(i)} \otimes e_{a(i)} A \oplus \coprod_{j=2}^{t} Ae_{b(j)} \otimes e_{b(j)} A,$$

$$P_{2n+1} = \coprod_{l=1}^{2n} Ae_1 \otimes e_1 A \oplus \coprod_{i=1}^{s} Ae_{a(i+1)} \otimes e_{a(i)} A \oplus \coprod_{j=1}^{t} Ae_{b(j+1)} \otimes e_{b(j)} A.$$

Projective resolution of A_q

And we have the following complex:

$$0 \to \operatorname{Hom}_{A^e}(P_0, A) \xrightarrow{d_1^*} \operatorname{Hom}_{A^e}(P_1, A) \xrightarrow{d_2^*} \operatorname{Hom}_{A^e}(P_2, A) \to \cdots ,$$

where

$$\operatorname{Hom}_{A^e}(P_{2n}, A) \cong \coprod_{l=0}^{2n} e_1 A e_1 \oplus \coprod_{i=2}^{s} e_{a(i)} A e_{a(i)} \oplus \coprod_{j=2}^{t} e_{b(j)} A e_{b(j)},$$

$$\operatorname{Hom}_{A^e}(P_{2n+1}, A) \cong \coprod_{l=1}^{2n} e_1 A e_1 \oplus \coprod_{i=1}^{s} e_{a(i+1)} A e_{a(i)} \oplus \coprod_{j=1}^{t} e_{b(j+1)} A e_{b(j)}.$$

Hochschild cohomology ring of A_q modulo nilpotence

Let q an r-th root of unity in k and $\overline{z}$ the remainder when we divide z by r for any integer z. Then we have $0 \leq \overline{z} \leq r-1$.

Theorem 1

Let $s, t \geq 2$. Then $\mathrm{HH}^*(A_q)/\mathcal{N}$ is isomorphic to the polynomial ring of two variables:

$$\mathrm{HH}^*(A_q)/\mathcal{N} \cong \begin{cases} k[x^{2r}, y^{2r}] & \text{if } \overline{a} \neq 0, \overline{b} \neq 0, \\ k[x^2, y^{2r}] & \text{if } \overline{a} \neq 0, \overline{b} = 0, \\ k[x^{2r}, y^2] & \text{if } \overline{a} = 0, \overline{b} \neq 0, \\ k[x^2, y^2] & \text{if } \overline{a} = \overline{b} = 0. \end{cases}$$

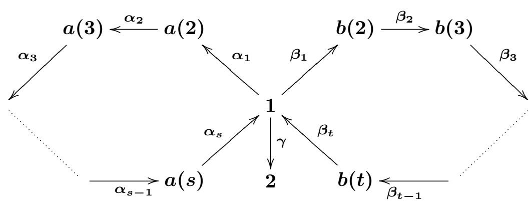
where $x^n = e_{1,0} + \sum_{j=2}^{t} e_{b(j)}$, $y^n = e_{1,n} + \sum_{i=2}^{s} e_{a(i)}$ in $\mathrm{HH}^n(A_q)$.

Theorem 2

In the case where q is not a root of unity, $\mathrm{HH}^*(A_q)/\mathcal{N} \cong k$.

One point extension of A_q

We consider the algebra $B = k\Gamma/I_{q,v,u}$. Γ: the quiver with $s+t$ vertices and $s+t+1$ arrows as follows:



$I_{q,v,u}$: the ideal of $k\Gamma$ generated by

$$X^{sa}, X^sY^t - qY^tX^s, Y^{tb}, \gamma X^{sv+u}$$

for $a, b \geq 2$, $0 \leq v \leq a-1$, $0 \leq u \leq s-1$ and $(v, u) \neq (0, 0)$ where we set $X := \alpha_1 + \alpha_2 + \cdots + \alpha_s$ and $Y := \beta_1 + \beta_2 + \cdots + \beta_t$.

Projective resolution of one point extension algebra

- $B = \begin{pmatrix} k & M \\ 0 & A \end{pmatrix}$: one point extension algebra of A by the A-module M.
- $\mathcal{F}$: $\mathrm{Mod}\, A^e \to \mathrm{Mod}\, B^e$: the natural functors given by $\mathcal{F}(Q) = \begin{pmatrix} 0 & M \\ 0 & A \end{pmatrix} \otimes_A Q$.
- $\mathcal{G}$: $\mathrm{Mod}\, A \to \mathrm{Mod}\, B^e$: the natural functors given by $\mathcal{G}(L) = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}$.
- A^e-projective resolution of A: $\cdots \to Q_n \xrightarrow{\delta_n} \cdots \xrightarrow{\delta_1} Q_0 \xrightarrow{\delta_0} A \to 0$.
- A-projective resolution of M: $\cdots \to L_n \xrightarrow{r_n} \cdots \xrightarrow{r_1} L_0 \xrightarrow{r_0} M \to 0$.

We give an explicit projective bimodule resolution of a one point extension algebra by using the following Theorem.

Projective resolution of one point extension algebra

Theorem [GMS(2003)]

We have a B^e-projective resolution of B:

$$\cdots \to P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} B \to 0.$$

- $P_0 = \mathcal{F}(Q_0) \oplus (Be' \otimes e'B)$ where $e' = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in B$.
  $P_n = \mathcal{F}(Q_n) \oplus \mathcal{G}(L_{n-1})$.
- $d_0 = (\mathcal{F}(\delta_0), id_{Be' \otimes e'B})$.
  $d_n = \begin{pmatrix} \mathcal{F}(\delta_n) & \sigma_n \\ 0 & -\mathcal{G}(r_{n-1}) \end{pmatrix}$ for $n \geq 1$.
- σ_n: $\mathcal{G}(L_{n-1}) \to \mathcal{F}(Q_{n-1})$ is a B^e-homomorphism such that $\mathcal{F}(\delta_n) \circ \sigma_{n+1} = \sigma_n \circ \mathcal{G}(\eta_n)$,
- σ_0 is the natural monomorphism.

Projective resolution of B

Remark

The following sequence is a minimal projective resolution of M.

$$\cdots \to L_{2n} \xrightarrow{r_{2n}} L_{2n-1} \xrightarrow{r_{2n-1}} \cdots \to L_1 \xrightarrow{r_1} L_0 \xrightarrow{r_0} M \to 0$$

where $L_{2n} = e_1 A_q$, $L_{2n+1} = e_{a(s+1-u)} A_q$ for $n \geq 0$, r_0 is a natural epimorphism and for $n \geq 1$,

$$r_{2n-1}(e_{a(s+1-u)}) = X^{sv+u} e_{a(s+1-u)},$$
$$r_{2n}(e_1) = X^{s(a-v-1)+s-u} e_1.$$

Then we have the following B^e-module:

$$\mathcal{G}(L_{2n}) = Be_2 \otimes e_1 B,$$
$$\mathcal{G}(L_{2n+1}) = Be_2 \otimes e_{a(s+1-u)} B.$$

Projective resolution of B

We have the B^e-projective resolution of B:

$$\cdots P_n \xrightarrow{d_n} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\pi} B \to 0.$$

where

$$d_1 : \varepsilon^{1\prime} \mapsto \gamma \varepsilon_0^0 - \varepsilon^{0\prime} \gamma,$$
$$d_{2n} : \varepsilon^{2n+1\prime} \mapsto \gamma \varepsilon_{2n}^{2n} - \varepsilon^{2n\prime} X^{s(a-1-v)+s-u},$$
$$d_{2n+1} : \varepsilon^{2n\prime} \mapsto \sum_{l=0}^{v-1} \sum_{l'=0}^{s-1} \gamma X^{sl+l'} \varepsilon_{a(s-l')}^{2n-1} X^{s(v-l)+u-1-l'} + \sum_{l'=0}^{u-1} \gamma X^{sv+l'} \varepsilon_{a(s-l')}^{2n-1} X^{u-1-l'} - \varepsilon^{2n-1\prime} X^{sv+u},$$

where we set $\varepsilon_0^0 = \varepsilon_{2n}^{2n} = e_1 \otimes e_1$, $\varepsilon_{a(i)}^{2n+1} = e_{a(i+1)} \otimes e_{a(i)}$, $\varepsilon^{0\prime} = e_2 \otimes e_2$, $\varepsilon^{2n\prime} = e_2 \otimes e_{a(s+1-u)}$ and $\varepsilon^{2n+1\prime} = e_2 \otimes e_1$ for $n \geq 0$.

Projective resolution of B

Let $P_n^* := \mathrm{Hom}_{B^e}(P_n, B)$. We have the following complex:

$$0 \to P_1^* \xrightarrow{d_1^*} P_2^* \xrightarrow{d_2^*} P_3^* \to \cdots \to P_{2n}^* \xrightarrow{d_{2n}^*} P_{2n+1}^* \to \cdots,$$

where

$$P_{2n}^* \cong \coprod_{l=0}^{2n} e_1 B e_1 \oplus \coprod_{i=2}^{s} e_{a(i)} B e_{a(i)} \oplus \coprod_{j=2}^{t} e_{b(j)} B e_{b(j)} \oplus e_2 B e_{a(s+1-u)},$$
$$P_{2n+1}^* \cong \coprod_{l=1}^{2n} e_1 B e_1 \oplus \coprod_{i=1}^{s} e_{a(i+1)} B e_{a(i)} \oplus \coprod_{j=1}^{t} e_{b(j+1)} B e_{b(j)} \oplus e_2 B e_1.$$

Main result

Theorem 3

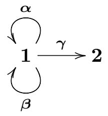
If $s, t \geq 2$ and q is an r-th root of unity then

$$\mathrm{HH}^*(B)/\mathcal{N} \cong \begin{cases} k \oplus k[x^{2r}, y^{2r}]x^{2r} & \text{if } \overline{a} \neq 0, \overline{b} \neq 0, \\ k \oplus k[x^2, y^{2r}]x^2 & \text{if } \overline{a} \neq 0, \overline{b} = 0, \\ k \oplus k[x^{2r}, y^2]x^{2r} & \text{if } \overline{a} = 0, \overline{b} \neq 0, \\ k \oplus k[x^2, y^2]x^2 & \text{if } \overline{a} = \overline{b} = 0. \end{cases}$$

where $x^n = e_{1,0} + \sum_{j=2}^{t} e_{b(j)}$ in $\mathrm{HH}^n(B)$ and $y^n = e_{1,n} + \sum_{i=2}^{s} e_{a(i)}$.
This is not finitely generated as an algebra.

Example 1

In the case $s = t = 1$, $\mathrm{HH}^*(B)/\mathcal{N}$ is determined in [Sn(2009)]. Γ is the quiver



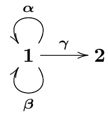
and $I = \langle \alpha^2, \beta^2, \alpha\beta - \beta\alpha, \alpha\gamma \rangle$. Snashall showed the following Theorem.

[Sn(2009), Theorem 4.5]

$$\mathrm{HH}^*(B)/\mathcal{N} \cong \begin{cases} k \oplus k[x,y]x & \text{if } \mathrm{char} k = 2, \\ k \oplus k[x^2,y^2]x^2 & \text{if } \mathrm{char} k \neq 2. \end{cases}$$

Example 2

Γ is the quiver



and $I = \langle \alpha^2, \beta^2, \alpha\beta - \beta\alpha \rangle$. Then we have the minimal A-projective resolution of A_A.

$$0 \to A \to A \to 0.$$

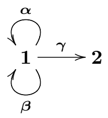
Theorem 4

Let $C = k\Gamma/I$.

$$\mathrm{HH}^*(C)/\mathcal{N} \cong \begin{cases} k[x,y] & \text{if } \mathrm{char} k = 2, \\ k[x^2,y^2] & \text{if } \mathrm{char} k \neq 2. \end{cases}$$

Example 3

Γ is the quiver



and $I_1 = \langle \alpha^2, \beta^2, \alpha\beta - \beta\alpha, \gamma\alpha, \gamma\beta \rangle$.

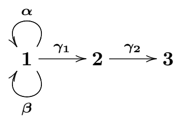
Remark

The minimal A-projective resolution of $S(1)$ given by

$$r_n : A^{n+1} \to A^n : \begin{cases} e_{1,1} \mapsto e_{1,1}\alpha, \\ e_{1,2l_1} \mapsto e_{1,2l_1-1}\beta - e_{1,2l_1}\alpha, \\ e_{1,2l_2+1} \mapsto e_{1,2l_2-1}\beta + e_{1,2l_2+1}\alpha, \\ e_{1,n+1} \mapsto e_{1,n}\beta. \end{cases}$$

Example 4

Let Δ be the quiver



and $I_2 = \langle \alpha^2, \beta^2, \alpha\beta - \beta\alpha, \gamma_1\alpha, \gamma_2\gamma_1 \rangle$.

Remark

The following sequence is a minimal A-projective resolution of $S(2)$.

$$\cdots \to e_1B \xrightarrow{r_n} e_1B \xrightarrow{r_{n-1}} \cdots \to e_1B \xrightarrow{r_1} e_2B \xrightarrow{r_0} S(2) \to 0$$

where r_0 is a natural epimorphism and

$$r_1(e_1) = \gamma_1,$$
$$r_n(e_1) = \alpha \text{ for } n \geq 2.$$

Theorem 5

Let $D = k\Gamma/I_1$.

$$\mathrm{HH}^*(D)/\mathcal{N} \cong k$$

Theorem 6

Let $E = k\Delta/I_2$.

$$\mathrm{HH}^*(E)/\mathcal{N} \cong \begin{cases} k \oplus k[x,y]x & \text{if } \mathrm{char} k = 2, \\ k \oplus k[x^2,y^2]x^2 & \text{if } \mathrm{char} k \neq 2. \end{cases}$$

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